Chapter 8 introduces students to rewriting quadratic expressions and solving quadratic equations. Quadratic functions are any function which can be rewritten in the form $y = ax^2 + bx + c$ (where $a \neq 0$) and when graphed, produce a U-shaped curve called a parabola.

There are multiple methods that can be used to solve quadratic equations. One of them requires students to factor a quadratic expression first. In Lessons 8.1.1 through 8.1.4, students factor quadratic expressions.

In previous chapters, students used algebra tiles to build “generic rectangles” of quadratic expressions. In the figure below, the length and width of the rectangle are $(x + 2)$ and $(x + 4)$. Since the area of a rectangle is given by $(\text{base})(\text{height}) = \text{area}$, the area of the rectangle in the figure below can be expressed as a product, $(x + 2)(x + 4)$. But the small pieces of the rectangle also make up its area, so the area can be expressed as a sum, $4x + 8 + x^2 + 2x$, or simplified as, $x^2 + 6x + 8$. Thus students wrote $(x + 2)(x + 4) = x^2 + 6x + 8$.

In the figure at right, the length and width of the rectangle, which are $(x + 2)$ and $(x + 4)$, are factors of the quadratic expression $x^2 + 6x + 8$, since $(x + 2)$ and $(x + 4)$ multiply together to produce the quadratic $x^2 + 6x + 8$. Notice that the $4x$ and the $2x$ are located diagonally from each other. They are like terms and can be combined and written as $6x$.

The factors of $x^2 + 6x + 8$ are $(x + 2)$ and $(x + 4)$.

The $ax^2$ term and the $c$ term are always diagonal to one another in a generic rectangle. In this example, the $ax^2$ term is $(1x^2)$ and the $c$ term is the constant 8; the product of this diagonal is $1x^2 \cdot 8 = 8x^2$. The two $x$-terms make up the other diagonal and can be combined into a sum since they are like terms. The $b$ of a quadratic expression is the sum of the coefficients of these factors: $2x + 4x = 6x$, so $b = 8$. The product of this other diagonal is $(2x)(4x) = 8x^2$. Note that the products of the two diagonals are always equivalent. In the textbook, students may nickname this rule “Casey’s Rule,” after the fictional character Casey in problem 8-4.

To factor a quadratic expression, students need to find the coefficients of the two $x$-terms so that the products of the two diagonals are equivalent, and also the sum of the two $x$-terms is $b$. Students can use a “diamond problem” to help organize their sums and products. For more information on using a diamond problem to factor quadratic expressions, see the Math Notes box in Lesson 8.1.4.

For additional information, see the Math Notes boxes in Lessons 8.1.1 through 8.1.4. For additional examples and more practice, see the Checkpoint 10B materials at the back of the student textbook.
Example 1

Factor $x^2 + 7x + 12$.

Sketch a generic rectangle with 4 sections.

Write the $x^2$ and the 12 along one diagonal.

Find two terms whose product is $12x^2$ and whose sum is $7x$ (in this case, $3x$ and $4x$). Students are familiar with this situation as a “diamond problem” from Chapter 1.

Write these terms as the other diagonal. Either term can go in either diagonal space.

Find the base and height of the large outer rectangle by using the areas of the small pieces.

Write the complete equation, showing that the product form is equivalent to the sum (factored) form.

$\begin{array}{c|c} \hline 3x & 12 \\ \hline x^2 & 4x \\ \hline \hline \end{array}$

$\begin{array}{c|c} \hline 3 & 3x \\ \hline x & x^2 \\ \hline \hline \end{array}$

$\begin{array}{c|c} \hline 3 & 12 \\ \hline x & 4 \\ \hline \hline \end{array}$

$\begin{array}{c|c} \hline 3 & -30 \\ \hline x & -3x \\ \hline \hline \end{array}$

$\begin{array}{c|c} \hline 3 & -30 \\ \hline x & -3x \\ \hline \hline \end{array}$

Example 2

Factor $x^2 + 7x - 30$.

Sketch a generic rectangle with 4 sections.

Write the $x^2$ and the $-30$ along one diagonal.

Find two terms whose product is $-30x^2$ and whose sum is $7x$. In this case, $-3x$ and $10x$.

Write these terms as the other diagonal. Either term can go in either diagonal space.

Find the base and height of the outer large rectangle.

Write the complete equation showing the factored form.

$\begin{array}{c|c} \hline -3x & -30 \\ \hline x^2 & 10x \\ \hline \hline \end{array}$

$\begin{array}{c|c} \hline 3 & -30 \\ \hline x & -3x \\ \hline \hline \end{array}$

$\begin{array}{c|c} \hline 3 & -30 \\ \hline x & -3x \\ \hline \hline \end{array}$

$\begin{array}{c|c} \hline 3 & -30 \\ \hline x & -3x \\ \hline \hline \end{array}$

$x^2 + 7x - 30 = (x - 3)(x + 10)$
Example 3

Factor \( x^2 - 15x + 56 \).

Sketch a generic rectangle with 4 sections.
Write the \( x^2 \) and the 56 along one diagonal.
Find two terms whose product is \( 56x^2 \) and whose sum is \( -15x \). Write these terms as the other diagonal.

Find the base and height of the outer rectangle.
Write the complete equation. \( x^2 - 15x + 56 = (x - 7)(x - 8) \)

Example 4

Factor \( 12x^2 - 19x + 5 \).

Sketch a generic rectangle with 4 sections.
Write the \( 12x^2 \) and the 5 along one diagonal.
Find two terms whose product is \( 60x^2 \) and whose sum is \( -19x \).
Write these terms as the other diagonal.
Find the base and height of the rectangle. Check the signs of the factors.
Write the complete equation. \( (3x - 1)(4x - 5) = 12x^2 - 19x + 5 \)
**Example 5**

Factor $3x^2 + 21x + 36$.

Note: If a common factor appears in all the terms, it should be factored out first. For example: $3x^2 + 21x + 36 = 3(x^2 + 7x + 12)$.

Then $x^2 + 7x + 12$ can be factored in the usual way, as in Example 1 above:

$$x^2 + 7x + 12 = (x + 3)(x + 4).$$

Then, since the expression $3x^2 + 21x + 36$ had a factor of three,

$$3x^2 + 21x + 36 = 3(x^2 + 7x + 12) = 3(x + 3)(x + 4).$$

**Problems**

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$x^2 + 5x + 6$</td>
<td>2.</td>
<td>$2x^2 + 5x + 3$</td>
<td>3.</td>
</tr>
<tr>
<td>5.</td>
<td>$x^2 + 15x + 44$</td>
<td>6.</td>
<td>$x^2 + 7x + 6$</td>
<td>7.</td>
</tr>
<tr>
<td>9.</td>
<td>$4x^2 + 12x + 9$</td>
<td>10.</td>
<td>$24x^2 + 22x - 10$</td>
<td>11.</td>
</tr>
<tr>
<td>13.</td>
<td>$x^3 - 11x^2 + 28x$</td>
<td>14.</td>
<td>$2x^2 + 11x - 6$</td>
<td>15.</td>
</tr>
<tr>
<td>17.</td>
<td>$4x^2 - 12x + 9$</td>
<td>18.</td>
<td>$3x^2 + 2x - 5$</td>
<td>19.</td>
</tr>
</tbody>
</table>

**Answers**

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$(x + 2)(x + 3)$</td>
<td>2.</td>
<td>$(x + 1)(2x + 3)$</td>
<td>3.</td>
</tr>
<tr>
<td>5.</td>
<td>$(x + 11)(x + 4)$</td>
<td>6.</td>
<td>$(x + 6)(x + 1)$</td>
<td>7.</td>
</tr>
<tr>
<td>9.</td>
<td>$(2x + 3)(2x + 3)$</td>
<td>10.</td>
<td>$2(3x - 1)(4x + 5)$</td>
<td>11.</td>
</tr>
<tr>
<td>13.</td>
<td>$x(x - 4)(x - 7)$</td>
<td>14.</td>
<td>$(x + 6)(2x - 1)$</td>
<td>15.</td>
</tr>
<tr>
<td>17.</td>
<td>$(2x - 3)(2x - 3)$</td>
<td>18.</td>
<td>$(3x + 5)(x - 1)$</td>
<td>19.</td>
</tr>
</tbody>
</table>
Although most factoring problems can be done with generic rectangles, there are two special factoring patterns that, if recognized, can be done by sight. The two patterns are known as the **Difference of Squares** and **Perfect Square Trinomials**. The general patterns are as follows:

**Difference of Squares:**  
\[
\begin{align*}
  a^2x^2 - b^2y^2 &= (ax + by)(ax - by)
\end{align*}
\]

**Perfect Square Trinomial:**  
\[
\begin{align*}
  a^2x^2 + 2abxy + b^2y^2 &= (ax + by)^2
\end{align*}
\]

### Example 1

<table>
<thead>
<tr>
<th>Difference of Squares</th>
<th>Perfect Square Trinomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 - 49 = (x + 7)(x - 7) )</td>
<td>( x^2 - 10x + 25 = (x - 5)^2 )</td>
</tr>
<tr>
<td>( 4x^2 - 25 = (2x - 5)(2x + 5) )</td>
<td>( 9x^2 + 12x + 4 = (3x + 2)^2 )</td>
</tr>
<tr>
<td>( x^2 - 36 = (x + 6)(x - 6) )</td>
<td>( x^2 - 6x + 9 = (x - 3)^2 )</td>
</tr>
<tr>
<td>( 9x^2 - 1 = (3x - 1)(3x + 1) )</td>
<td>( 4x^2 + 20x + 25 = (2x + 5)^2 )</td>
</tr>
</tbody>
</table>

### Example 2

Sometimes removing a common factor reveals one of the special patterns:

\( 8x^2 - 50y^2 \)  \( \Rightarrow \)  \( 2(4x^2 - 25y^2) \)  \( \Rightarrow \)  \( 2(2x + 5y)(2x - 5y) \)

\( 12x^2 + 12x + 3 \)  \( \Rightarrow \)  \( 3(4x^2 + 4x + 1) \)  \( \Rightarrow \)  \( 3(2x + 1)^2 \)
Problems

Factor each difference of squares.
1. \(x^2 - 16\)  
2. \(x^2 - 25\)  
3. \(64m^2 - 25\)  
4. \(4p^2 - 9q^2\)  
5. \(9x^2y^2 - 49\)  
6. \(x^4 - 25\)  
7. \(64 - y^2\)  
8. \(144 - 25p^2\)  
9. \(9x^4 - 4y^2\)

Factor each perfect square trinomial.
10. \(x^2 + 4x + 4\)  
11. \(y^2 + 8y + 16\)  
12. \(m^2 - 10m + 25\)  
13. \(x^2 - 4x + 16\)  
14. \(a^2 + 8ab + 16b^2\)  
15. \(36x^2 + 12x + 1\)  
16. \(25x^2 - 30xy + 9y^2\)  
17. \(9x^2y^2 - 6xy + 1\)  
18. \(49x^2 + 1 + 14x\)

Factor completely.
19. \(9x^2 - 16\)  
20. \(9x^2 + 24x + 16\)  
21. \(9x^2 - 36\)  
22. \(2x^2 + 8xy + 8y^2\)  
23. \(x^2y + 10xy + 25y\)  
24. \(8x^2 - 72\)  
25. \(4x^3 - 9x\)  
26. \(4x^2 - 8x + 4\)  
27. \(2x^2 + 8\)

Answers
1. \((x + 4)(x - 4)\)  
2. \((x + 5)(x - 5)\)  
3. \((8m + 5)(8m - 5)\)  
4. \((2p + 3q)(2p - 3q)\)  
5. \((3xy + 7)(3xy - 7)\)  
6. \((x^2 + 5)(x^2 - 5)\)  
7. \((8 + y)(8 - y)\)  
8. \((12 + 5p)(12 - 5p)\)  
9. \((3x^2 + 2y)(3x^2 - 2y)\)  
10. \((x + 2)^2\)  
11. \((y + 4)^2\)  
12. \((m - 5)^2\)  
13. not factorable (prime)  
14. \((a + 4b)^2\)  
15. \((6x + 1)^2\)  
16. \((5x - 3y)^2\)  
17. \((3xy - 1)^2\)  
18. \((7x + 1)^2\)  
19. \((3x + 4)(3x - 4)\)  
20. \((3x + 4)^2\)  
21. \(9(x + 2)(x - 2)\)  
22. \(2(x + 2y)^2\)  
23. \(y(x + 5)^2\)  
24. \(8(x + 3)(x - 3)\)  
25. \(x(2x + 3)(2x - 3)\)  
26. \(4(x - 1)^2\)  
27. \(2(x^2 + 4)\)
The graph of a quadratic function, a parabola, is a symmetrical curve. Its highest or lowest point is called the vertex. The graph is formed using the equation \( y = ax^2 + bx + c \). Students have been graphing parabolas by substituting values for \( x \) and solving for \( y \). This can be a tedious process, especially if an appropriate range of \( x \)-values is not known. One possible shortcut if only a quick sketch of the parabola is needed, is to find the \( x \)-intercepts first, then find the vertex and/or the \( y \)-intercept. To find the \( x \)-intercepts, substitute 0 for \( y \) and solve the quadratic equation, \( 0 = ax^2 + bx + c \). Students will learn multiple methods to solve quadratic equations in this chapter and in Chapter 9. One method to solve quadratic equations uses the Zero Product Property, that is, solving by factoring. This method utilizes two ideas:

1. When the product of two or more numbers is zero, then one of the numbers must be zero.
2. Some quadratics can be factored into the product of two binomials.

For additional information see the Math Notes box in Lesson 8.2.2.

**Example 1**

Find the \( x \)-intercepts of \( y = x^2 + 6x + 8 \).

The \( x \)-intercepts are located on the graph where \( y = 0 \), so write the quadratic expression equal to zero, then solve for \( x \).

\[
x^2 + 6x + 8 = 0
\]

Factor the quadratic.

\[(x + 4)(x + 2) = 0\]

Set each factor equal to 0.

\[(x + 4) = 0 \quad \text{or} \quad (x + 2) = 0\]

Solve each equation for \( x \).

\[x = -4 \quad \text{or} \quad x = -2\]

The \( x \)-intercepts are \((-4, 0)\) and \((-2, 0)\).

You can check your answers by substituting them into the original equation.

\[
(-4)^2 + 6(-4) + 8 \Rightarrow 16 - 24 + 8 \Rightarrow 0
\]

\[
(-2)^2 + 6(-2) + 8 \Rightarrow 4 - 12 + 8 \Rightarrow 0
\]
Example 2

Solve \( 2x^2 + 7x - 15 = 0 \).

Factor the quadratic. \((2x - 3)(x + 5) = 0\)

Set each factor equal to 0. \((2x - 3) = 0 \quad \text{or} \quad (x + 5) = 0\)

Solve for each \(x\). \(2x = 3\)
\[x = \frac{3}{2}\quad \text{or} \quad x = -5\]

Example 3

If the quadratic does not equal 0, rewrite it algebraically so that it does, then use the zero product property.

Solve \( 2 = 6x^2 - x \).

Set the equation equal to 0. \(2 = 6x^2 - x\)
\[0 = 6x^2 - x - 2\]

Factor the quadratic. \(0 = (2x + 1)(3x - 2)\)

Solve each equation for \(x\). \((2x + 1) = 0 \quad \text{or} \quad (3x - 2) = 0\)
\[2x = -1 \quad \text{or} \quad 3x = 2\]
\[x = -\frac{1}{2} \quad \text{or} \quad x = \frac{2}{3}\]

Example 4

Solve \( 9x^2 - 6x + 1 = 0 \).

Factor the quadratic. \(9x^2 - 6x + 1 = 0\)
\[(3x - 1)(3x - 1) = 0\]

Solve each equation for \(x\). Notice the factors are the same so there will be only one solution. \((3x - 1) = 0\)
\[3x = 1\]
\[x = \frac{1}{3}\]
Problems

Solve for $x$.

1. $x^2 - x - 12 = 0$
2. $3x^2 - 7x - 6 = 0$
3. $x^2 + x - 20 = 0$
4. $3x^2 + 11x + 10 = 0$
5. $x^2 + 5x = -4$
6. $6x - 9 = x^2$
7. $6x^2 + 5x - 4 = 0$
8. $x^2 - 6x + 8 = 0$
9. $6x^2 - x - 15 = 0$
10. $4x^2 + 12x + 9 = 0$
11. $x^2 - 12x = 28$
12. $2x^2 + 8x + 6 = 0$
13. $2 + 9x = 5x^2$
14. $2x^2 - 5x = 3$
15. $x^2 = 45 - 4x$

Answers

1. $x = 4$ or $-3$
2. $x = -\frac{2}{3}$ or $3$
3. $x = -5$ or $4$
4. $x = -\frac{5}{3}$ or $-2$
5. $x = -4$ or $-1$
6. $x = 3$
7. $x = -\frac{4}{3}$ or $\frac{1}{2}$
8. $x = 4$ or $2$
9. $x = -\frac{3}{2}$ or $\frac{5}{3}$
10. $x = -\frac{3}{2}$
11. $x = 14$ or $-2$
12. $x = -1$ or $-3$
13. $x = -\frac{1}{2}$ or $2$
14. $x = -\frac{1}{2}$ or $3$
15. $x = 5$ or $-9$
In Lesson 8.2.3, students found that if the equation of a parabola is written in graphing form: \( f(x) = (x - h)^2 + k \) then the vertex can easily be seen as \((h, k)\). For example, for the parabola \( f(x) = (x + 3)^2 - 1 \) the vertex is \((-3, -1)\). Students can then set the function equal to zero to find the \(x\)-intercepts: solve \(0 = (x + 3)^2 - 1\) to find the \(x\)-intercepts. For help in solving this type of equation, see the Lesson 8.2.3 Resource Page, available at www.cpm.org. Students can set \(x = 0\) to find the \(y\)-intercepts: \(y = (0 + 3)^2 - 1\).

When the equation of the parabola is given in standard form: \( f(x) = x^2 + bx + c \), then using the process of completing the square can be used to convert standard form into graphing form. Algebra tiles are used to help visualize the process.

For additional examples and practice with graphing quadratic functions, see the Checkpoint 11 materials at the back of the student textbook.

**Example 1 (Using algebra tiles)**

Complete the square to change \( f(x) = x^2 + 8x + 10 \) into graphing form, identify the vertex and \(y\)-intercept, and draw a graph.

\( f(x) = x^2 + 8x + 10 \) would look like this:

![Example 1 Tiles](image)

Arrange the tiles as shown in the picture at right to make a square.

16 small unit tiles are needed to fill in the corner, but only ten unit tiles are available. Show the 10 small square tiles and draw the outline of the whole square.

The complete square would have length and width both equal to \((x + 4)\), so the complete square can be represented by the quadratic expression \((x + 4)^2\). But the tiles from \(x^2 + 8x + 10\) do not form a complete square—the expression \(x^2 + 8x + 10\) has six fewer tiles than a complete square. So \(x^2 + 8x + 10\) is a complete square minus 6, or \((x + 4)^2\) minus 6. That is,

\[x^2 + 8x + 10 = (x + 4)^2 - 6.\]

From graphing form, the vertex is at \((h, k) = (-4, -6)\). The \(y\)-intercept is where \(x = 0\), so \(y = (0 + 4)^2 - 6 = 10\). The \(y\)-intercept is \((0, 10)\), and the graph is shown at right.
**Example 2 (Using the general process)**

Complete the square to change \( f(x) = x^2 + 5x + 2 \) into graphing form, identify the vertex and \( y \)-intercept, and draw a graph.

Rewrite the expression as:  
\[
 f(x) = x^2 + 5x + 2 
\]

\[
 f(x) = (x^2 + 5x + ?) + 2 
\]

Make \((x^2 + 5x + ?)\) into a perfect square by taking half of the \(x\)-term coefficient and squaring it:  
\[
 (\frac{5}{2})^2 = \frac{25}{4} 
\]

Then \((x^2 + 5x + \frac{25}{4})\) is a perfect square.

We need too write an equivalent function with \(+\frac{25}{4}\), but if we add \(\frac{25}{4}\), we also need to subtract it:  
\[
 f(x) = x^2 + 5x + 2 \\ f(x) = (x^2 + 5x + \frac{25}{4}) + 2 - \frac{25}{4} 
\]

The function is now in graphing form. The vertex is at \((-\frac{5}{2}, -\frac{17}{4})\) or \((-2.5, -4.25)\).

Alternatively, if the process above is unclear, draw a generic rectangle of \(x^2 + 5x + 2\) and imagine algebra tiles, as in Example 1 above.

```
<table>
<thead>
<tr>
<th>2.5x</th>
</tr>
</thead>
<tbody>
<tr>
<td>x^2</td>
</tr>
<tr>
<td>2.5x</td>
</tr>
</tbody>
</table>
```

There should be \((-2.5)^2 = 6.25\) tiles in the upper right corner to complete the square. But the expression \(x^2 + 5x + 2\) only provides 2 tiles. So there are 4.25 tiles missing. Thus, there are 4.25 tiles missing from the rectangle below:

```
<table>
<thead>
<tr>
<th>2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
</tr>
<tr>
<td>x^2</td>
</tr>
<tr>
<td>2.5x</td>
</tr>
<tr>
<td>x</td>
</tr>
<tr>
<td>2.5</td>
</tr>
</tbody>
</table>
```

That is, \((x + 2.5)^2 \) minus 4.25 tiles is the equivalent of \(x^2 + 5x + 2\), or,  
\[
x^2 + 5x + 2 = (x + 2.5)^2 - 4.25 
\]

The \(y\)-intercept is where \(x = 0\).  
Thus, \(y = (0 + 2.5)^2 - 4.25 = 2\) and the \(y\)-intercept is at \((0, 2)\).
Problems

Complete the square to write each equation in graphing form. Then state the vertex.

1. \( f(x) = x^2 + 6x + 7 \) 
2. \( f(x) = x^2 + 4x + 11 \)

3. \( f(x) = x^2 + 10x \) 
4. \( f(x) = x^2 + 7x + 2 \)

5. \( f(x) = x^2 - 6x + 9 \) 
6. \( f(x) = x^2 + 3 \)

7. \( f(x) = x^2 - 4x \) 
8. \( f(x) = x^2 + 2x - 3 \)

9. \( f(x) = x^2 + 5x + 1 \) 
10. \( f(x) = x^2 - \frac{1}{3}x \)

Answers

1. \( f(x) = (x + 3)^2 - 2; (-3, -2) \) 
2. \( f(x) = (x + 2)^2 + 7; (-2, 7) \)

3. \( f(x) = (x + 5)^2 - 25; (-5, -25) \) 
4. \( f(x) = (x + 3.5)^2 - 10.25; (-3.5, -10.25) \)

5. \( f(x) = (x - 3)^2; (3, 0) \) 
6. \( f(x) = x^2 + 3; (0, 3) \)

7. \( f(x) = (x - 2)^2 - 4; (2, -4) \) 
8. \( f(x) = (x + 1)^2 - 4; (-1, -4) \)

9. \( f(x) = \left( x + \frac{5}{2}\right)^2 - \frac{21}{4}; \left( -\frac{5}{2}, -\frac{21}{4}\right) \) 
10. \( f(x) = \left( x - \frac{1}{6}\right)^2 - \frac{1}{36}; \left( \frac{1}{6}, -\frac{1}{36}\right) \)