REAL NUMBERS

COMPARING AND ORDERING REAL NUMBERS

Any number that can be written as the ratio of two integers $\frac{a}{b}$ with $b \neq 0$, is called a rational number. Rational numbers can be matched to exactly one point on a number line. There are many points on the number line for which there are no rational numbers. These numbers are called irrational numbers. Numbers such as $\pi$, $\sqrt{2}$, and $-\sqrt{5}$ are irrational numbers. The rational numbers and the irrational numbers make up all of the numbers on the number line and together are called the real numbers.

Example 1

Indicate the approximate location of each of the following real numbers on a number line:

$-3, \frac{10}{3}, 1.2, \frac{1}{5}, \pi, \sqrt{17}, -\frac{4}{5}$

We know that $\frac{10}{3} = 3 \frac{1}{3}, \pi \approx 3.14$, and $\sqrt{17} \approx 4.1$. Approximating the locations of the other fractions and decimals yields the line below.

Example 2

Without using a calculator, order these numbers from least to greatest:

$2\pi, \sqrt{37}, 6, \frac{20}{3}, 5.9, 6.4 \times 10^0$

Since $\pi \approx 3.14$, $2\pi \approx 6.28$, $6 = \sqrt{36} < \sqrt{37} \approx 6.1$, $\frac{20}{3} = 6 \frac{2}{3} \approx 6.67$ and $6.4 \times 10^0 = 6.4 \times 1 = 6.4$, the order is: $5.9 < 6 < \sqrt{37} < 2\pi < 6.4 \times 10^0 < \frac{20}{3}$
Problems

Indicate the approximate location of each of the following real numbers on a number line.

1. $\frac{2}{3}$, $-0.75$, $\sqrt{8}$, $-\frac{9}{5}$, $\frac{\pi}{3}$, $2\frac{1}{4}$

2. $-\sqrt{10}$, $\frac{\pi}{2}$, $1\frac{7}{8}$, $2.8$, $-1.6$, $\frac{15}{31}$

Without using a calculator, order the numbers from least to greatest.

3. $\sqrt{102}$, $10$, $3\pi$, $\sqrt{99}$, $1.1 \times 10^1$, $9.099$

4. $\frac{17}{4}$, $3.9$, $4$, $\sqrt{17}$, $(\pi + 0.5)$, $3.899$

5. $-\sqrt{10}$, $-\frac{10}{3}$, $-3$, $-2.95$, $-3\frac{1}{4}$, $-3.5 \times 10^0$

6. $-5$, $-\sqrt{24}$, $-\frac{26}{5}$, $-5.5$, $-2\pi$, $-4\frac{3}{4}$

7. $8.1 \times 10^3$, $2.1 \times 10^3$, $7.25 \times 10^3$, $1.05 \times 10^4$, $7.4 \times 10^3$, $1.75 \times 10^4$

8. $1.62 \times 10^{-2}$, $9.3 \times 10^{-3}$, $1.97 \times 10^{-1}$, $4.5 \times 10^{-2}$, $6.1 \times 10^{-3}$, $8.3 \times 10^{-2}$

Answers

1. $\frac{9}{5}$, $-0.75$, $\frac{2}{3}$, $\frac{\pi}{3}$, $2\frac{1}{4}$, $\sqrt{8}$

2. $-\sqrt{10}$, $-1.6$, $\frac{15}{31}$, $\frac{\pi}{2}$, $1\frac{7}{8}$, $2.8$

3. $9.099 < 3\pi < \sqrt{99} < 10 < \sqrt{102} < 1.1 \times 10^1$

4. $(\pi + 0.5) < 3.899 < 3.9 < 4 < \sqrt{17} < \frac{17}{4}$

5. $-3.5 \times 10^0 < -\frac{10}{3} < -3\frac{1}{4} < -\sqrt{10} < -3 < -2.95$

6. $-2\pi < -5.5 < -\frac{26}{5} < -5 < -\sqrt{24} < -4\frac{3}{4}$

7. $7.25 \times 10^3 < 7.4 \times 10^3 < 8.1 \times 10^3 < 1.05 \times 10^4 < 1.75 \times 10^4 < 2.1 \times 10^5$

8. $6.1 \times 10^{-3} < 9.3 \times 10^{-3} < 1.62 \times 10^{-2} < 4.5 \times 10^{-2} < 8.3 \times 10^{-2} < 1.97 \times 10^{-1}$
OPERATIONS WITH RATIONAL AND IRRATIONAL NUMBERS

The sum (or difference) of any two rational numbers is rational because if \( \frac{a}{b} \) and \( \frac{c}{d} \) are rational then \( \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \) which is rational. The product (or quotient) of any two rational number is rational because if \( \frac{a}{b} \) and \( \frac{c}{d} \) are rational then \( \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \) which is rational.

The sum of a rational number and an irrational number is irrational. This can be shown using a “proof by contradiction.” Assume that \( a \) is rational and \( b \) is irrational and we want to show that \( a + b \) is irrational. Start by assuming \( a + b = c \) a rational number. This will lead to a contradiction. Subtracting \( a \) from both sides yields \( b = c - a \). The left side is irrational but the right side is \( c - a \), the difference of two rational numbers and is rational. Since it is impossible for a number to be both irrational and rational we have a contradiction. Therefore \( a + b \), the sum of a rational number and an irrational number is irrational. In the problems below, using a similar method, the product of a rational number and an irrational number can be shown to be irrational.

Mixed computation with rational and irrational numbers is usually done using decimal approximations and rounding off to the desired level of accuracy.

Example 1

The sum \( \frac{3}{7} + \sqrt{2} \) is irrational. It cannot be expressed in rational form. To approximate the sum to the nearest hundredth, use the decimal approximations to the nearest thousandth and then round. \( \frac{3}{7} \approx 0.429 \) and \( \sqrt{2} \approx 1.414 \) so \( \frac{3}{7} + \sqrt{2} \approx 0.429 + 1.414 = 1.843 \Rightarrow 1.84 \)

Example 2

The product \( 1 \frac{1}{4} \cdot \pi \) is irrational. It cannot be expressed in rational form. To approximate the product to the nearest tenth, use decimal approximations to the nearest hundredth and then round. \( 1 \frac{1}{4} = 1.25 \) and \( \pi \approx 3.14 \) so \( 1 \frac{1}{4} \cdot \pi \approx 1.25 \cdot 3.14 = 3.925 \Rightarrow 3.9 \)

Example 3

The difference \( \frac{7}{8} - \frac{1}{3} \) is rational and can be expressed as \( \frac{21}{24} - \frac{8}{24} = \frac{13}{24} \). It is usually left in this form but could also be approximated as a decimal to any desired place. For example \( \frac{13}{24} = 0.541666... \) and to the nearest thousandth \( \frac{13}{24} \approx 0.542 \).
Problems

In problems 1 through 9, tell if the answer will be rational or irrational. If it is rational, give the answer in rational form. If it is irrational, give the decimal approximation to the nearest hundredth.

1. \( \frac{1}{5} + \sqrt{7} \)
2. \( \frac{2}{5} + \frac{1}{3} \)
3. \( \frac{1}{4} + \sqrt{5} \)

4. \( \frac{2}{3} \cdot \sqrt{15} \)
5. \( \frac{2}{7} + \frac{4}{9} \)
6. \( 3 \frac{4}{7} - \sqrt{5} \)

7. \( \pi + \frac{2}{3} \)
8. \( \frac{4}{3} \cdot \pi \)
9. \( -2 - \sqrt{3} \)

10. Prove that the product of a rational number and an irrational is irrational. Hint: Use the same method as was used to show the sum is irrational.

Answers

1. I, 2.85
2. R, \( \frac{11}{15} \)
3. R, \( \frac{13}{4} \)

4. I, 2.58
5. R, \( \frac{9}{14} \)
6. I, 1.34

7. I, 3.81
8. I, 4.19
9. I, -3.73

10. Start by assuming that if \( a \) is rational and \( b \) is irrational, then \( a \cdot b = c \) where \( c \) is rational. Divide both sides by \( a \) and show a contradiction as was done in the box at the top of this section.