Students revisit parabolas to examine them from a geometric approach rather than from the equations. They learn that the special properties of a parabola are what make a satellite dish work. Students also learn where they can find a parabola on a cone, as well as find other curves there. These curves, known as the conic sections, are the circle, parabola, ellipse, and hyperbola. Starting from a geometric definition of each of these curves, students develop methods for writing their equations. For further information, see the Math Notes boxes following problems 11-55 on page 568, 11-68 on page 571, 11-86 on page 576, 11-98 on page 580, and 11-116 on page 585.

Example 1

Sketch the parabola that is the set of all points equidistant from the focus (2, 3) and the directrix \( y = -1 \).

Most students first saw a parabola when they graphed \( y = x^2 \), and have always thought of a parabola as the result of graphing a quadratic equation. But the actual definition of a parabola is the set of all points equidistant from a point (called the focus) and a line (called the directrix). Given a point and a line, we can find the equation of the parabola defined by them.

To do this, we choose a generic point \((x, y)\) that is equidistant from the point \((2, 3)\) and the line \( y = -1 \). The distance from the point \((x, y)\) to the focus \((2, 3)\) is:

\[
 d = \sqrt{(x - 2)^2 + (y - 3)^2}.
\]

The distance from the point \((x, y)\) and the line \( y = -1 \) is \( y - (-1) \), or simply \( y + 1 \) since this is a vertical segment. The parabola is the set of all points where these two distances are equal. Therefore, we can write:

\[
 y + 1 = \sqrt{(x - 2)^2 + (y - 3)^2}
\]

since we are looking for all the points \((x, y)\) that make these two distances equal. To solve this equation, we begin by squaring both sides of the equation.

\[
 (y + 1)^2 = \left( \sqrt{(x - 2)^2 + (y - 3)^2} \right)^2
\]

\[
 y^2 + 2y + 1 = (x - 2)^2 + (y - 3)^2
\]

\[
 y^2 + 2y + 1 = x^2 - 4x + 4 + y^2 - 6y + 9
\]

\[
 8y = x^2 - 4x + 12
\]

\[
 y = \frac{1}{8} (x^2 - 4x) + \frac{12}{8}
\]

\[
 y = \frac{1}{8} (x^2 - 4x + 4) + \frac{12}{8} - \frac{1}{2}
\]

\[
 y = \frac{1}{8} (x - 2)^2 + 1
\]

After completing the square, we can see that this parabola has a vertex of \((2, 1)\), and opens upward.
Example 2

Show how you can slice a cone to create each of the following curves.

a. A circle   
   b. An ellipse  
   c. A parabola  
   d. A hyperbola

For each of these curves we will slice a cone with a plane, intersecting the cone at different angles. Although we typically think of a cone as a “circular-based pyramid,” a complete cone actually has two halves. Each half touches at the tips. When we slice this cone, we have to consider what happens in both halves.

The circle is the easiest to find on the cone. Simply slice the cone anywhere other than at the intersecting tips, with a plane parallel to the bases. We can create circles of any size radius with this method. To produce an ellipse, angle the plane slightly. We can rotate the plane any number of degrees up to the point where it is parallel to the side of the cone.

Once the plane is parallel to the side of the cone, we no longer have an ellipse, but instead we have a parabola. In the figure at right, the plane is parallel to the side and the parabola is shown at the bottom of the second figure at right. Since the plane is parallel to the edge of the cone, the intersection will continue, forming the curve of the parabola. This will expand wider and wider as the cone does.

If we continued to tip the plane that created the parabola steeper and steeper, the plane would eventually interest the other half of the cone as well. When this happens, we get the hyperbola, which has two branches.

Example 3

Write the equation of an ellipse whose foci are (2, 3) and (2, –5), and the sum of the distance from a point on the ellipse to each focus point is 12.

The locus definition of an ellipse is the set of all points whose sum of the distances from two fixed points is a constant. In this example the two points are the foci (2, 3) and (2, –5) (foci is the plural of focus), and the constant sum is 12. To come up with the equation we will use this definition and the process we used to write the equation of the parabola.

Begin by letting \((x, y)\) be any point on the ellipse. The distance from this point to \((2, 3)\) we will call \(d_1\),

\[
d_1 = \sqrt{(x - 2)^2 + (y - 3)^2}.
\]

The distance from \((x, y)\) to the other focus point, \((2, –5)\) will be \(d_2\), and

\[
d_2 = \sqrt{(x - 2)^2 + (y + 5)^2}.
\]
Since the sum of these two distances is a constant of 12, we can write:

\[ 12 = d_1 + d_2 \]

\[ 12 = \sqrt{(x-2)^2 + (y-3)^2} + \sqrt{(x-2)^2 + (y+5)^2} \]

The algebra to simplify this equation will not be as easy as the algebra used when finding the parabola. We will square both sides, but squaring will not completely eliminate the radical. In fact, we will have to square twice to completely eliminate it.

\[ 12 = \sqrt{(x-2)^2 + (y-3)^2} + \sqrt{(x-2)^2 + (y+5)^2} \]

\[ 12 - \sqrt{(x-2)^2 + (y-3)^2} = \sqrt{(x-2)^2 + (y+5)^2} \]

\[ (12 - \sqrt{(x-2)^2 + (y-3)^2})^2 = \left(\sqrt{(x-2)^2 + (y+5)^2}\right)^2 \]

\[ 144 - 24\sqrt{(x-2)^2 + (y-3)^2} + (x-2)^2 + (y-3)^2 = (x-2)^2 + (y+5)^2 \]

\[ 144 + (y-3)^2 - (y+5)^2 = 24\sqrt{(x-2)^2 + (y-3)^2} \]

\[ 144 + y^2 - 6y + 9 - y^2 - 10y - 25 = 24\sqrt{(x-2)^2 + (y-3)^2} \]

\[ 128 - 16y = 24\sqrt{(x-2)^2 + (y-3)^2} \]

\[ (128 - 16y)^2 = \left(24\sqrt{(x-2)^2 + (y-3)^2}\right)^2 \]

\[ 16384 - 4096y + 256y^2 = 576\left(\sqrt{(x-2)^2 + (y-3)^2}\right)^2 \]

\[ 16384 - 4096y + 256y^2 = 576(x^2 - 4x + 4 + y^2 - 6y + 9) \]

\[ 16384 - 4096y + 256y^2 = 576x^2 - 2304x + 2304 + 576y^2 - 3456y + 5184 \]

\[ 0 = 576x^2 - 2304x + 320y^2 + 640y - 8896 \]

\[ 0 = 576(x^2 - 4x) + 320(y^2 + 2y) - 8896 \]

\[ 0 = 576(x^2 - 4x + 4) - 2304 + 320(y^2 + 2y + 1) - 320 - 8896 \]

\[ 0 = 576(x-2)^2 + 320(y+1)^2 - 11520 \]

\[ 11520 = 576(x-2)^2 + \frac{320(y+1)^2}{11520} \]

\[ 1 = \frac{(x-2)^2}{20} + \frac{(y+1)^2}{36} \]

An equation of an ellipse and a hyperbola is written so that it is equal to one. An ellipse is formed when the sum of the distances is constant; a hyperbola is formed when the difference of the distances is a constant. Read the Method and Meanings box following problem 11-116 for further information about hyperbolas.
Problems

For each of the following sets of data, write the graphing form of the equation of the parabola with the given focus and directrix.

1. Focus: \((0, \frac{1}{4})\)  
Directrix: \(y = -\frac{1}{4}\)

2. Focus: \((0, -\frac{1}{12})\)  
Directrix: \(y = \frac{1}{12}\)

3. Focus: \((0, 4)\)  
Directrix: \(y = -4\)

4. Focus: \((5,1)\)  
Directrix: \(y = -3\)

5. Focus: \((-4,-2)\)  
Directrix: \(y = 3\)

6. Focus: \((-1,6)\)  
Directrix: \(x = 3\)

Write each of the following equations in graphing form, and state which conic section it is.

7. \(2x^2 + 5y^2 = 98\)

8. \(2x^2 - y^2 = 2\)

9. \(y - x^2 = -5\)

10. \(3x^2 - 7y = 47\)

11. \(x^2 + y^2 + 2x = 17\)

12. \(9x^2 + 25y^2 = 225\)

13. \(xy = -12\)

14. \(16x^2 - 3y^2 = -11\)

15. \(x^2 + 9y^2 - 10x + 36y = -43\)

16. \(y = x^2 + 8x + 9\)

17. \(x^2 - y^2 - 2y = 17\)

18. \(18x - y^2 = -144\)
Answers

1. \( y = x^2 \)
2. \( y = -3x^2 \)
3. \( y = \frac{1}{16} x^2 \)
4. \( y = \frac{1}{8} (x - 5)^2 - 1 \)
5. \( y = -\frac{1}{10} (x + 4)^2 + \frac{1}{2} \)
6. \( x = -\frac{1}{8} (y - 6)^2 + 1 \)
7. \( \frac{x^2}{49} + \frac{y^2}{98} = 1 \), ellipse
8. \( x^2 - \frac{y^2}{2} = 1 \), hyperbola
9. \( y = x^2 - 5 \), parabola
10. \( y = \frac{3}{7} x^2 - \frac{47}{7} \), parabola
11. \( (x + 1)^2 + y^2 = 18 \), circle
12. \( \frac{x^2}{25} + \frac{y^2}{9} = 1 \), ellipse
13. \( y = -\frac{12}{x} \), hyperbola
14. \( \frac{y^2}{\frac{13}{3}} - \frac{x^2}{\frac{11}{16}} = 1 \), hyperbola
15. \( \frac{(x-5)^2}{18} + \frac{(y+2)^2}{2} = 1 \), ellipse
16. \( y = (x + 4)^2 - 7 \), parabola
17. \( \frac{x^2}{16} - \frac{(y+1)^2}{16} = 1 \), hyperbola
18. \( x = \frac{1}{18} y^2 - 8 \), parabola
**SAT Prep**

1. If \( k = 5t \) and \( t = 6 \), what is the value of \( 5k \)?
   a. 5  
   b. 6  
   c. 30  
   d. 36  
   e. 150

2. **Some integers in set \( U \) are negative.**
   If the statement above is true, which of the following must also be true?
   a. All integers in set \( U \) are negative.
   b. All integers in set \( U \) are positive.
   c. If an integer is negative, then it is in set \( U \).
   d. If an integer is positive, then it is in set \( U \).
   e. Not all integers in set \( U \) are positive.

3. Squaring the product of \( x \) and 4 gives the same result as squaring the difference of \( x \) and 4. Which of the following equations could be used to find all the possible values of \( x \)?
   a. \( 4x^2 = x^2 + 4^2 \)
   b. \( (4x)^2 = (x - 4)^2 \)
   c. \( 4^2x = x^2 - 4^2 \)
   d. \( (4x)^2 = x^2 + 4^2 \)
   e. \( 4x^2 = (x + 4)^2 \)

4. To make Tam’s tantalizingly tasty toffee cookies, flour, sugar and salt are mixed by weight in the ratio 7:3:1, respectively. In order to make 11 pounds of the dough for this cookie, what weight of sugar, in pound, is needed?
   a. 7  
   b. 3  
   c. 1  
   d. \( \frac{1}{3} \)  
   e. \( \frac{1}{7} \)

5. The local donut shop donated some donuts to Professor Galactic’s astronomy class. If each student takes two donuts, there will be 16 donuts left. If four students do not take any donuts and the rest of the students take six donuts, there will be no donuts left. How many donuts were donated to the class?
   a. 24  
   b. 30  
   c. 36  
   d. 40  
   e. 48
6. Tran can mow the lawn in three hours while it takes Collin four hours to mow the same lawn. How long will it take if they mow the lawn together? (They start at opposite sides and work toward each other.)
   a. 12 hours  b. 7 hours  c. \( \approx 1.7 \) hours  d. \( \approx 42 \) minutes  e. \( \approx 9 \) minutes

7. On a certain map, 15 miles are represented by one-half inch. On the same map, how many miles are represented by 3.75 inches?

8. How many of the first one hundred positive integers contain the digit 1?

9. The sum of two consecutive integers is greater than three but less than 13. What is one possible integer fitting these conditions?

10. Give three values for \( k \) for which the trinomial \( 3x^2 + kx + 6 \) factors.

**Answers**

1. E  
2. E  
3. B  
4. B  
5. C  
6. C  
7. 112.5 miles  
8. 20  
9. Any of 2, 3, 4, or 5 will work.  
10. 19, 9, 11, \(-19\), \(-9\), \(-11\)

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