Geometric sequences are examples of exponential functions. In these sections, students generalize what they have learned about geometric sequences, and investigate functions of the form $y = km^x$ ($m > 0$). Students look at multiple representations of exponential functions, including graphs, tables, equations, and context. They learn how to easily move from one representation to another. Exponential functions are used for calculating growth and decay, such as interest on a loan or the age of fossils. While working on these problems students also review the laws of exponents. For further information see the Math Notes boxes following problems 3-6 on page 118 and 3-52 on page 132.

Example 1

LuAnn has $500 with which to open a savings account. She can open an account at Fredrico’s Bank, which pays 7% interest, compounded monthly, or Money First Bank, which pays 7.25%, compounded quarterly. LuAnn plans to leave the money in the account, untouched, for ten years. In which account should she place the money? Justify your answer.

The obvious answer is that she should put the money in the account that will pay her the most interest over the ten years, but which bank is that? At both banks the principle (the initial value) is $500. Fredrico’s Bank pays 7% compounded monthly, which means the interest is $\frac{0.07}{12}$ each month, or approximately 0.00583 in interest each month. If LuAnn puts her money into Fredrico’s Bank, after one month she will have:

$$500 + 500(0.00583) = 500(1.00583) \approx 502.92.$$

To calculate the amount at the end of the second month, we must multiply by 1.00583 again, making the amount:

$$500(1.00583)^2 \approx 505.85.$$

At the end of three months, the balance is:

$$500(1.00583)^3 \approx 508.80.$$

This will happen every month for ten years, which is 120 months. At the end of 120 months, the balance will be:

$$500(1.00583)^{120} \approx 1004.43.$$
We perform a similar calculation for Money First Bank. Its interest rate is higher, 7.25%, but it is only compounded and calculated quarterly, (quarterly means four times each year, or every three months). Hence, every quarter the bank calculates \( \frac{0.0725}{4} = 0.018125 \) interest. At the end of the first quarter, LuAnn would have:

\[
500(1.018125) \approx 509.06.
\]

At the end of ten years (40 quarters) LuAnn would have:

\[
500(1.018125)^{40} \approx 1025.69.
\]

Since Money First would pay her approximately $21 more in interest than Fredrico’s Bank, she should put her money in Money First Bank.

**Example 2**

Most homes appreciate in value, at varying rates, depending on the home’s location, size, and other factors. But, if the home is used as a rental, the Internal Revenue Service allows the owner to assume that it will depreciate in value. Suppose a house that costs $150,000 is used as a rental property, and depreciates at a rate of 8% per year. What is the multiplier that will give the value of the house after one year? What is that value? What is the value after ten years? What is the “half-life” of the house (that is, when has the house lost half of its value)? Draw the graph of this situation.

Unlike interest, which increases the value of the house, depreciation takes value away. After one year, the value of the house is $150000 – 0.08(150000) which is the same as $150000(0.92). (Check this!) Therefore the multiplier is 0.92. After one year, the value of the house is $150000(0.92) = $138,000. After ten years, the value of the house will be $150000(0.92)^{10} = $65,158.27. To find the “half-life,” we need to determine when the value of the house reaches $75,000. We just found that at ten years, the value is below $75,000, so the half-life occurs in less than ten years. To help answer this question, list the house’s values in a table to see the depreciation.

<table>
<thead>
<tr>
<th># Years</th>
<th>House’s value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>138000</td>
</tr>
<tr>
<td>2</td>
<td>126960</td>
</tr>
<tr>
<td>3</td>
<td>116803.20</td>
</tr>
<tr>
<td>4</td>
<td>107458.94</td>
</tr>
<tr>
<td>5</td>
<td>98862.23</td>
</tr>
<tr>
<td>6</td>
<td>90953.25</td>
</tr>
<tr>
<td>7</td>
<td>83676.99</td>
</tr>
<tr>
<td>8</td>
<td>76982.83</td>
</tr>
<tr>
<td>9</td>
<td>70824.20</td>
</tr>
</tbody>
</table>

Shortly after the eighth year the house will be half its value. Note: If you try to write an equation and solve, you will have to use guess and check to solve at this point. The equation, $75000 = 150000(0.92)^x$, requires some new mathematics to solve it. We will learn how to solve such equations later in the year. If you wish to try to solve this by guess and check, the answer you are looking for is approximately 8.313.
Example 3

Solve the following equations for $x$.

a. $x^7 = 42$

As with many equations, we need to isolate the variable (get the variable by itself), and then eliminate the exponent. This will require one of the Laws of Exponents, namely $(x^a)^b = x^{ab}$.

$$\left(x^7\right)^{\frac{1}{7}} = (42)^{\frac{1}{7}}$$

$$x^1 = (42)^{\frac{1}{7}}$$

$$x = 1.706$$

b. $3x^{12} = 132$

$$\frac{3x^{12}}{3} = \frac{132}{3}$$

$$x^{12} = 44$$

$$\left(x^{12}\right)^{\frac{1}{12}} = (44)^{\frac{1}{12}}$$

$$x = (44)^{\frac{1}{12}}$$

$$x \approx \pm 1.371$$

The final calculation takes the seventh root of 42 in part (a) and the twelfth root of 44 in part (b). Notice that there is only one answer for part (a), where the exponent is odd, but there are two answers ($\pm$) in part (b) where the exponent is even. Even roots always produce two answers, a positive and a negative. Be sure that if the problem is a real-world application that both the positive and the negative results make sense before stating both as solutions. You may have to disregard one solution so that the answer is feasible.
Problems

Simplify the following expressions as much as possible.

1. \((16a^8b^{12})^{3/4}\)
2. \(\frac{144^{1/2}x^{-3}}{(16^{3/4}x^{7})^0}\)
3. \(\frac{a^{2/3}b^{-3/4}c^{7/8}}{a^{-1/3}b^{1/4}c^{1/8}}\)

Find each of the indicated function values.

4. \(f(x) = 3\left(\frac{2}{3}\right)^x\), find \(f(4)\).
5. \(g(x) = -6x^7\), find \(g(-3)\).
6. \(h(x) = \frac{1}{4^x}\), find \(h(-2)\).

Solve the following equations for \(x\).

7. \(x^8 = 65,536\)
8. \(-5x^{-3} = \frac{25}{40}\)
9. \(3^5x = 9^{x-1}\)
10. \(\left(\frac{2x-1}{3^{4x+3}}\right)^x = 1\)
11. \(2(3x-5)^4 = 392\)
12. \(\frac{2^{4x-1}}{2^{3x+2}} = 4\)

Find the error in each of the following solutions. Then give the correct solution.

13. \(4(x + 7)^6 = 1392\)
\(x + 7 = 348\)
\(x = 51\)
14. \(5^{4x+2} = 10^{3x-1}\)
\(5^{4x+2} = 2 \cdot 5^{3x-1}\)
\(4x + 2 = 6x - 1\)
\(x = 1.5\)

15. In seven years, Seta’s son Stu is leaving home for college. Seta hopes to save $8000 to pay for his first year. She has $5000 now and has found a bank that pays 7.75% interest, compounded daily. At this rate, will she have the money she needs for Stu’s first year of college? If not, how much more does she need?

16. Eight years ago, Rudi thought that he was making a sound investment by buying $1000 worth of Pro Sports Management stock. Unfortunately, his investment depreciated steadily, losing 15% of its value each year. How much is the stock worth now? Justify your answer.

17. The new Bamo Super Ball has a rebound ratio of 0.97. If you dropped the ball from a height of 125 feet, how high will it bounce on the tenth bounce?
18. Fredrico’s Bank will let you decide how often your interest will be computed, but with certain restrictions. If your interest is compounded yearly you can earn 8%. If your interest is compounded quarterly, you earn 7.875%. Monthly compounding yields a 7.75% interest rate, while weekly compounding yields a 7.625% interest rate. If your interest is compounded daily, you earn 7.5%. What is the best deal? Justify your answer.

19. Investigate the equation \( y = \left( \frac{3}{4} \right)^x + 4 \).

**Answers**

1. \( 8a^6b^9 \)  
2. \( \frac{12}{x^3} \)  
3. \( \frac{ac^{3/4}}{b} \)

4. \( \frac{16}{27} \)  
5. 13,122  
6. 16

7. \( x = 8 \)  
8. \( x = -2 \)  
9. \( x = \frac{-2}{3} \)

10. \( x = 0 \)  
11. \( x = 2.91 \)  
12. \( x = 5 \)

13. Both sides need to be raised to the \( \frac{1}{6} \) (or the 6\textsuperscript{th} root taken), not divided by six. \( x \approx -4.35 \).

14. Since 5 and 10 cannot be written as the power of the same number, the only way to solve the equation now is by guess and check. \( x \approx 11.75 \). If you did not get this answer, do not worry about it now. The point of the problem is to spot the error. There is a second error: the 2 was not distributed in the fourth line.

15. Yes, she will have about \$8601.02 by then. The daily rate is \( \frac{0.0775}{365} \approx 0.000212329 \). Seven years is 2555 days, so we have \$5000(1.000212329)^{2555} \approx \$8601.02 .

16. It is now only worth about \$272.49.

17. About 92.18 feet.

18. The best way to do this problem is to choose any amount, and see how it grows over the course of one year. Taking \$100, after one year compounded yearly will yield \$108. Compounded quarterly, \$108.11. Compounded monthly, \$108.03. Compounded daily, \$107.79. Quarterly is the best.

19. This is a function that it is continuous and nonlinear (curved). It has a y-intercept of (0, 5), and no x-intercepts. The domain is all real values of \( x \), and the range is all real values of \( y > 4 \). This has a horizontal asymptote of \( y = 4 \), and no vertical asymptotes. It is an exponential function.
Part of developing skill when investigating a function is the ability to fit a curve to a set of data points. This means the students will use the data to write an equation that can be used to model a particular set of points. Students have seen general forms of various types of equations. They will learn how to determine which part of these general equations are parameters and which parts are variables and use this information to write equations for the data. For further information see the Math Notes box following problem 3-108 on page 148.

Example 1

A line passes through the points \((3, 6)\) and \((-2, 9)\). Find the equation of the line using the form \(y = mx + b\).

In the general form of the equation of a line listed above, the \(x\) and the \(y\) are the variables, while the \(m\) and the \(b\) are the parameters. When we write the equation of the line that passes through these points, the variables \(x\) and \(y\) will still be present. The parameters, however, will have specific numerical values that fit the data for this unique equation.

To do this, we substitute our values from the coordinate points above into the general equation. The \(x\)-coordinate replaces \(x\), and the \(y\)-coordinate replaces the \(y\). This gives us two equations with two unknowns. We can solve these equations for \(m\) and \(b\).

\[
\begin{align*}
6 &= m(3) + b \\
9 &= m(-2) + b
\end{align*}
\]

\[
\begin{align*}
6 &= 3m + b \\
9 &= -2m + b
\end{align*}
\]

\[
\begin{align*}
-3 &= 5m \\
\frac{3}{5} &= m
\end{align*}
\]

Now that we know \(m\), we can substitute it back into either equation to find \(b\). Once we know both values of these parameters, we can replace them in the general form to write the equation of the line passing through the two points.

\[
y = -\frac{3}{5}x + \frac{39}{5}
\]
Example 2

The general form of an exponential function with an asymptote of \( y = 0 \) is \( y = ab^x \). If an exponential function with an asymptote of \( y = 0 \) passes through the points (0, 8) and (4, 0.5), what is the equation of this function?

As we did in the last example, we will substitute the \( x \) - and \( y \) -coordinates of each pair of points into the general equation, then solve the two equations to determine the specific numerical values of the parameters \( a \) and \( b \).

\[
\begin{align*}
y &= ab^x \\
(0, \ 8) &:\quad 8 = ab^0 \\
(4, \ 0.5) &:\quad 0.5 = ab^4
\end{align*}
\]

Since \( b^0 = 1 \), the first equation simplifies to tell us that \( a = 8 \).

We substitute this values for \( a \) into the second equation in order to solve for \( b \).

\[
\begin{align*}
0.5 &= 8b^4 \\
\frac{1}{2} &= 8b^4 \\
\frac{1}{16} &= b^4 \\
\sqrt[4]{\frac{1}{16}} &= \sqrt[4]{b^4} \\
\frac{1}{2} &= b
\end{align*}
\]

With both parameters, we can now write the equation that represents these data points.

\[
y = 8 \left( \frac{1}{2} \right)^x
\]
Example 3

In the year 2000 when Club Leopard was first introduced on the Internet, it had 8,500 “leopards” (members). In 2007, the leopard population had risen to 22,610.

a. Model this data with a linear equation.
b. Model this data with an exponential function.
c. Use each model to predict the leopard population in the year 2012.
d. Which model do you believe is the better predictor? Why? Explain completely.

We can call the year 2000 our time zero, or \( x = 0 \), and the year 2007 will be \( x = 7 \). This gives us two data points, \((0, 8,500)\) and \((7, 22610)\). For part (a), we will use these data points with the general equation \( y = mx + b \) since we are finding a linear model first.

\[
8500 = m(0) + b \quad \Rightarrow \quad b = 8500
\]

\[
22610 = m(7) + b
\]

\[
22610 = 7m + 8500
\]

\[
14110 = 7m
\]

\[
m = 2015.7
\]

This gives the linear equation \( y = 2015.7x + 8500 \).

To model with an exponential function we will use the equation \( y = ab^x \) with the same data points.

\[
8500 = ab^0 \quad \Rightarrow \quad a = 8500
\]

\[
22610 = 8500b^7
\]

\[
2.66 = b^7
\]

\[
\sqrt[7]{2.66} = \sqrt[7]{b^7}
\]

\[
1.15 = b
\]

This gives the exponential equation \( y = 8500(1.15)^x \).

We will use each equation with \( x = 12 \) to predict the population in 2012 as requested in part (c).

\[
y = 2015.7x + 8500 \quad \quad y = 8500(1.15)^x
\]

\[
y = 2015.7(12) + 8500 \quad \quad y = 8500(1.15)^{12}
\]

\[
y = 32688 \quad \quad y = 45477
\]

Part (d) asks which is the better predictor of the population. Population growth is typically exponential demonstrated by the rabbit problem the students did in Chapter 2. Based on this assumption, the exponential function would be the better model to predict the population.
Problems

For each of the following pairs of points, find the equation of the line that passes through them.
1. (0, 7) and (3, −2)  
2. (5, 10) and (10, 12)
3. (−4, 12) and (4, 0)  
4. (−3, −17) and (4, 25)

For each of the following pairs of points, find the equation of an exponential function with an asymptote \( y = 0 \) that passes through them.
5. (0, 6) and (3, 48)  
6. (1, 21) and (2, 147)
7. (−1, 72.73) and (3, 106.48)  
8. (−2, 351.5625) and (3, 115.2)

9. On a cold wintry day the temperature outside hovered at 0°F. Karen made herself a cup of cocoa, and took it outside where she would be chopping some wood. However, she decided to conduct a mini science experiment instead of drinking her cocoa, so she placed a thermometer in the cocoa and left it sitting next to her as she worked. She wrote down the time and the reading on the thermometer as shown in the table below.

<table>
<thead>
<tr>
<th>Time since 1st reading</th>
<th>0</th>
<th>1</th>
<th>5</th>
<th>7</th>
<th>10</th>
<th>12</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temp (°C)</td>
<td>70°</td>
<td>56.7°</td>
<td>24.4°</td>
<td>16.01°</td>
<td>8.5°</td>
<td>5.6°</td>
<td>3.0°</td>
</tr>
</tbody>
</table>

a. Plot the points. Describe any trends you see.

b. What type of function would best model/fit this data? Explain.

c. Find the equation of an exponential function with an asymptote at \( y = 0 \) that models this data.

Answers

1. \( y = −3x + 7 \)  
2. \( y = \frac{2}{5} x + 8 \)  
3. \( y = −\frac{3}{2} x + 6 \)
4. \( y = 6x + 1 \)  
5. \( y = 6(2)^x \)  
6. \( y = 3(7)^x \)
7. \( y = 80(1.1)^x \)  
8. \( y = 225(0.8)^x \)
9. The data appears to be a decreasing exponential function, with an asymptote at \( y = 0 \). One possible equation that will model this is \( y = 70(0.81)^x \).
CHECKPOINT PRACTICE PROBLEMS

Starting in Chapter 2, several problems are marked with the icon shown at right. This icon indicates a “Checkpoint” problem for an Algebra 1 topic that students should be able to solve correctly at this point in the course. If the student needs help to solve this kind of problem, or cannot consistently solve them correctly, then the student needs additional practice with this type of problem. After each Checkpoint, the student will be expected to solve that type of problem easily and accurately.

The Checkpoint problems for Chapter 3 are problems 3-55 (simplifying expressions with positive integral exponents) and 3-111 (factoring quadratic expressions). The practice problems below cover only these two topics.

Simplify each expression. (problem 3-55)

1. \((3x^2y)^5\) 2. \(-2(-3ab^7)^3\) 3. \(4(2x^3yz^4)(xy^4z^4)\)

4. \(\frac{(2x^3)^4}{(-4)^2}\) 5. \(\frac{-6a^7b^2}{(3ab)^2}\) 6. \(\frac{-(x^6z^6)^3}{x^{12}}\)

Factor each expression below. (problem 3-111)

7. \(x^2 + 24x + 144\) 8. \(x^2 - 36\) 9. \(8x^2 + 8x + 8\)
10. \(x^2 + 16x - 36\) 11. \(3x^2 - x - 4\) 12. \(18x^2 - 27x - 5\)

Answers

1. \(243x^{10}y^5\) 2. \(54a^3b^{21}\) 3. \(8x^4y^5z^8\)

4. \(x^{12}\) 5. \(-\frac{2a^5}{3}\) 6. \(-z^{18}\)

7. \((x + 12)^2\) 8. \((x + 6)(x - 6)\) 9. \(8(x^2 + x + 1)\)
10. \((x - 2)(x + 18)\) 11. \((3x - 4)(x + 1)\) 12. \((3x - 5)(6x + 1)\)
1. In the correctly worked addition problem at right, each X represents the same digit. What is the value of X?

   X 6
   X 7
   X 8
   + X 9
   190

   a. 3  b. 4  c. 6  d. 9  e. 10

2. The figure at right is not drawn to scale. The cone has a height of 22 inches and a radius of 12 inches. The cone is cut parallel to the circular base creating two shapes, one of which is a smaller cone. If the radius of the base of the small cone is 4 inches, what is the height of the small cone?

   a. $6\frac{1}{2}$  b. 7  c. $7\frac{1}{3}$  d. 8.05  e. 8.99

3. The table at right shows the distances required to stop a car traveling at a given speed. The car will travel a certain distance while the driver is "thinking" about what to do. Once the driver applies the brake, the car travels some more before coming to a stop. How many more feet does it take to stop a car traveling at 60 miles per hour than at 20 miles per hour?

<table>
<thead>
<tr>
<th>Speed (in miles per hour)</th>
<th>Thinking Distance (in feet)</th>
<th>Braking Distance (in feet)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>30</td>
<td>30</td>
<td>45</td>
</tr>
<tr>
<td>40</td>
<td>40</td>
<td>80</td>
</tr>
<tr>
<td>50</td>
<td>50</td>
<td>125</td>
</tr>
<tr>
<td>60</td>
<td>60</td>
<td>180</td>
</tr>
</tbody>
</table>

   a. 75  b. 105  c. 150  d. 175  e. 200
4. If the arrangement above represents the fact that \( a < b \) and \( c < d \), which of the following is true?

   a. \[
   \begin{array}{c}
   1 \\
   4 \\
   2 \\
   6
   \end{array}
   \]
   b. \[
   \begin{array}{c}
   1 \\
   2 \\
   6 \\
   4
   \end{array}
   \]
   c. \[
   \begin{array}{c}
   6 \\
   4 \\
   2 \\
   1
   \end{array}
   \]
   d. \[
   \begin{array}{c}
   2 \\
   1 \\
   4 \\
   6
   \end{array}
   \]
   e. \[
   \begin{array}{c}
   6 \\
   2 \\
   1 \\
   4
   \end{array}
   \]

5. The graph at right represents data taken at a local company. Which of the following is the closest approximation to the percent of employees of this company who travel for at least 16 minutes to get to work?

   a. 25%  
   b. 30%  
   c. 40%  
   d. 50%  
   e. 60%

6. Tia has \( d \) to spend on some songs at Downloads R Us. If she is a member, she can buy any song for \( m \) each, but to be a member she must pay a one time fee of $11. Which expression represents the number of songs Tia can download from Downloads R Us?

   a. \( d - 11m \)  
   b. \( 11 + dm \)  
   c. \( \frac{dm}{11} \)  
   d. \( \frac{d-11}{m} \)  
   e. \( 11 - d \)

7. On the widget assembly line, every 10\textsuperscript{th} widget is inspected for cracks, and every 5\textsuperscript{th} widget is inspected for dents. In a box of 100 widgets, what is the probability that a widget will have been inspected for both cracks and dents?

   a. \( \frac{1}{50} \)  
   b. \( \frac{1}{15} \)  
   c. \( \frac{3}{10} \)  
   d. \( \frac{1}{2} \)  
   e. \( \frac{1}{5} \)
8. The length of rectangle A is 40% greater than the length of rectangle B. The width of rectangle A is 10% less than the width of rectangle B. The area of rectangle A is:

a. 50 percent greater than the area of rectangle B.
b. 126 percent greater than the area of rectangle B.
c. is equal to the area of rectangle B.
d. 50 percent less than the area of rectangle B.
e. 26 percent less than the area of rectangle B.

Questions 9 – 11 refer to the following sequence of steps.

1. Choose any number.
2. Multiply the number by 8.
3. Add 7 to the result.
4. Subtract 1 from that answer.
5. Divide the result by 2.
6. Add 1.
8. Subtract the original number and print the result.

9. If 32 is the original number chosen, what number is printed in Step 8?
   a. 478   b. 361   c. 82   d. 4   e. 1

10. Which of the following could be a number printed in Step 8 when someone performs all the steps correctly?
    a. 1   b. 73   c. 361   d. 478   e. 854

11. Which of the following changes could be made to the procedure without changing the results printed in Step 8?
    a. Switch Steps 2 and 3.
    b. Switch Steps 3 and 4.
    c. Switch Steps 4 and 5.
    d. Switch Steps 5 and 6.
    e. No changes can be made without affecting the final results.

Answers

11. B