INVESTIGATIONS and EXPLORATIONS

1.1.1 – 1.1.5

By asking questions such as “What happens if…?” and “What if I change this…?” and answering them by trying different things, we can find out quite a lot of information about different shapes. In the first five sections of this first chapter, we explore symmetry, making predictions, perimeter, area, logical arguments, and angles by investigating each of them with interesting problems. These five sections are introductory and help the teacher determine students’ prior knowledge and preview some of the ideas that will be studied in this course. The following examples illustrate the geometry ideas in this section as well as some of the algebra review topics.

See the Math Notes Boxes on pages 5, 10, 15, 19, and 24.

Example 1

Suppose the rug in Figure 1 is enlarged as shown.

Fill in the table below to show how the perimeter and the area of the rug change as it is enlarged.

<table>
<thead>
<tr>
<th>Figure Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perimeter</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Area</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The perimeter of a figure is the distance (length) around the outside of the figure while the area measures the surface within the figure. The area is measured in square units while the perimeter is simply a unit of length, such as inches or centimeters. Counting the units around the outside of Figure 1, we get a perimeter of 16 units. By counting the number of square units within Figure 1, we find the area is 12 square units. We do the same for the next two figures and record the information in the table, then look for a pattern in the data.

<table>
<thead>
<tr>
<th>Figure Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perimeter</td>
<td>16</td>
<td>32</td>
<td>48</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Area</td>
<td>12</td>
<td>48</td>
<td>108</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Now comes the task of finding a pattern from these numbers. The perimeters seem to be connected to the number 16, while the areas seem connected to 12. Using this observation, we can rewrite the table and then extend the pattern to complete it as shown in the table below.

<table>
<thead>
<tr>
<th>Figure Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perimeter (in units)</td>
<td>1(16)</td>
<td>2(16)</td>
<td>3(16)</td>
<td>4(16)</td>
<td>5(16)</td>
<td>20(16)</td>
</tr>
<tr>
<td>Area (in square units)</td>
<td>1(12)</td>
<td>4(12)</td>
<td>9(12)</td>
<td>16(12)</td>
<td>25(12)</td>
<td>400(12)</td>
</tr>
</tbody>
</table>

Notice that the first multiplier for the area is the square of the figure number.

**Example 2**

By using a hinged mirror and a piece of paper, the students explored how a kaleidoscope works. Through this investigation, the students saw how angles are related to shapes. In particular, by placing the hinge of the mirror at a certain angle, the students could create shapes with a specific number of sides. The hinge represents the angle at the center (or central angle) of the shape. (See pages 21 and 22 in the student text.) How many sides would the resulting shape have if the mirror is placed (1) as an acute angle (less than 90°)? (2) as a right angle (exactly 90°)? (3) as an obtuse angle (between 90° and 180°)?

If the central angle is acute, the resulting figure is a triangle, so figures formed with this kind of angle are limited to three sides. As the hinge opens further and forms a right angle, the figure adds another side, creating a quadrilateral. If the two edges of the mirror have the same length on the paper, the quadrilateral is a square.

As the hinge opens even further, the angle it makes is now obtuse. This will create more and more sides on the shape as the angle increases in size. It is possible to create a pentagon (five sided figure), a hexagon (six sided figure), and, in fact, any number of sides using obtuse angles of increasing measures. If the hinge is straight across, it forms a straight angle (measuring 180°), and the figure is no longer a closed shape, but a line.
Example 3

Solve the equation for \( x \): 
\[
2(x - 4) + 3(x + 1) = 43 + x
\]

\[
2(x - 4) + 3(x + 1) = 43 + x \\
2x - 8 + 3x + 3 = 43 + x \\
5x - 5 = 43 + x \\
4x = 48 \\
\frac{4x}{4} = \frac{48}{4} \\
x = 12
\]

In solving equations such as the one above, we simplify, combine like terms, and collect the variables on one side of the equal sign and the numbers on the other side.

The Methods and Meanings Math Notes box on page 19 also explains this process.

Problems

Find the perimeter and area of each shape below.

1. 

2. 

3. 

4. 

5. If the perimeter for the rectangle for problem 4 is 34 units, write an equation and solve for \( x \).

6. Solve for \( x \). Show the steps leading to your solution. 
\(-2x + 6 = 5x - 8\)

7. Solve for \( x \). Show the steps leading to your solution. 
\(3(2x - 1) + 9 = 4(x + 3)\)
For problems 8-11, estimate the size of each angle to the nearest 10°. A right angle is shown for reference so you should not need a protractor. Then classify each angle as either acute, right, obtuse, straight, or circular.

8. \[ \approx 160^\circ, \text{ obtuse} \]
9. \[ \approx 40^\circ, \text{ acute} \]
10. \[ 180^\circ, \text{ straight} \]
11. \[ 90^\circ, \text{ right} \]

Answers

1. Perimeter = 34 cm, Area = 42 square cm
2. Perimeter = 38 in., Area = 76 sq in.
3. Perimeter = 32 cm, Area = 38 square cm
4. Perimeter = 16x + 2 un, Area = 2x(6x + 1) or 12x^2 + 2x un^2
5. \[ 2(2x) + 2(6x + 1) = 34, \ x = 2 \]
6. \[ x = 2 \]
7. \[ x = 3 \]
8. \[ \approx 160^\circ, \text{ obtuse} \]
9. \[ \approx 40^\circ, \text{ acute} \]
10. \[ 180^\circ, \text{ straight} \]
11. \[ 90^\circ, \text{ right} \]
Studying transformations of geometric shapes builds a foundation for a key idea in geometry: congruence. In this introduction to transformations, the students explore three rigid motions: translation, reflection, and rotation. This exploration is done with simple tools that can be found at home (tracing paper) as well as with computer software. Students create a new shape by applying one or more of these motions to the original figure to create its image in a new position without changing its size or shape. Transformations also lead directly to studying symmetry in shapes. These ideas will help with describing and classifying geometric shapes later in the chapter.

See the Math Notes boxes on pages 5, 34, 38, and 42.

Example 1

Decide which transformation was used on each pair of shapes below. Some may be a combination of transformations.

a.  

b.  

c.  

d.  

e.  

f.  

Identifying a single transformation is usually easy for students. In (a), the parallelogram is reflected (flipped) across an invisible vertical line. (Imagine a mirror running vertically between the two figures. One figure would be the reflection of the other.) Reflecting a shape once changes its orientation, that is, how its parts “sit” on the flat surface. For example, in (a), the two sides of the figure at left slant upwards to the right, whereas in its reflection at right, they slant upwards to the left. Likewise, the angles in the figure at left “switch positions” in the figure at right. In (b), the shape is translated (or slid) to the right and down. The orientation is the same.
Part (c) shows a combination of transformations. First the triangle is reflected (flipped) across an invisible horizontal line. Then it is translated (slid) to the right. The pentagon in part (d) has been rotated (turned) clockwise to create the second figure. Imagine tracing the first figure on tracing paper, then holding the tracing paper with a pin at one point below the first pentagon, then turning the paper to the right 90°. The second pentagon would be the result. Some students might see this as a reflection across a diagonal line. The pentagon itself could be, but with the added dot (small circle), the entire shape cannot be a reflection. If it had been reflected, the dot would have to be on the corner below the one shown in the rotated figure. The triangles in part (e) are rotations of each other (90° again). Part (f) shows another combination. The triangle is rotated (the horizontal side becomes vertical) but also reflected since the hypotenuse of the triangle (the longest side) points in the opposite direction from the first figure.

Example 2

What will the figure at right look like if it is first reflected across line $l$ and then the result is reflected across line $m$?

Just as the name implies, a reflection is the image of the shape. First we want to imagine that line $l$ is a mirror. What would the image of the shape look like through this mirror?

The reflection is the new figure shown between the two lines. If we were to join each vertex (corner) of the original figure to its corresponding vertex on the second figure, those line segments would be perpendicular to line $l$ and the vertices of (and all the other points in) the reflection would be the same distance away from $l$ as they are in the original figure. One way to draw the reflection is to use tracing paper to trace the figure and the line $l$. Then turn the tracing paper over, so that line $l$ is on top of itself. This will show the position of the reflection. Transfer the figure to your paper by tracing it. Repeat this process with line $m$ to trace the third figure.

As we discovered in class, reflecting twice like this across two intersecting lines produces a rotation of the figure about the point $P$. Put the tracing paper back over the original figure to line $l$. Put a pin or the point of a pen or pencil on the tracing paper at point $P$ and rotate the tracing paper until the original figure will fit perfectly on top of the last figure.
Example 3

The shape at right is trapezoid ABCD. Find the coordinates of each vertex of this trapezoid. Translate the trapezoid seven units to the right and four units up. Label the new trapezoid A'B'C'D' and give the coordinates of the four vertices. Is it possible to translate the original trapezoid in such a way to create A”B”C”D” so that it is a reflection of ABCD? If so, what would be the reflecting line? Will this always be possible?

The coordinates of the trapezoid are A(-5, -2), B(-3, -2), C(-2, -4), and D(-6, -4). Translating (or sliding) the trapezoid seven units to the right and four units up gives a new trapezoid A'(2, 2), B'(4, 2), C'(5, 0) and D'(1, 0). If we forget about trapezoid A'B'C'D' for a moment and go back to ABCD, we now wonder if we can translate it in such a way that we can make it look as if it were a reflection rather than a translation. Since the trapezoid is symmetrical, it is possible to do so. We can slide the trapezoid horizontally left or right. In either case, the resulting figure would look like a reflection. This will not always work. It works here because the trapezoid we started with has a line of symmetry itself. The students explored which polygons have lines of symmetry, and which have rotational symmetry as well. Again they used tracing paper as well as computer software to investigate these properties.

Exploring these transformations and symmetrical properties of shapes helps to improve students’ visualization skills. These skills are often neglected or taken for granted, but much of mathematics requires the students to visualize a picture, problem, or situation in order to solve it. That is why we ask students to “visualize” or “imagine” what something might look like as well as practice creating some transformations of figures.
Problems

Perform the indicated transformation on each polygon below to create a new figure. You may want to use tracing paper to see how the figure moves.

1. Rotate figure A 90° clockwise about the origin.

2. Reflect figure B across line l.

3. Translate figure C six units to the left.

4. Rotate figure D 270° clockwise about the origin (0, 0).

5. Plot the points A(3, 3), B(6, 1), and C(3, -4). Translate the triangle eight units to the left and one unit up to create ΔA’B’C’. What are the coordinates of this new triangle?

6. How can you translate ΔABC in the last problem to put point A" at (4, -5)?

7. Reflect Z across line l, and then reflect the new figure across line m. What are these two reflections equivalent to?
For each shape below, (i) draw all lines of symmetry, and (ii) describe its rotational symmetry if it exists.

8.  

9.  

10.  

11.  

Answers

1.  Rotate figure A 90° clockwise about the origin.

2.  Reflect figure B across line l.

3.  Translate figure C six units to the left.

4.  Rotate figure D 270° clockwise about the origin (0, 0).
5.  \[ A'(5, -4), \ B'(-2, 2), \ C'(-5, -3) \]

6.  Translate it one unit to the right and eight units down.

7.  The two reflections are the same as rotating \( Z \) about point \( X \).

8.  This has 180° rotational symmetry.

9.  The one line of symmetry. No rotational symmetry.

10. The circle has infinitely many lines of symmetry, everyone of them illustrates reflection symmetry. It also has rotational symmetry for every possible degree measure.

11. This irregular shape has no lines of symmetry and does not have rotational symmetry, nor reflection symmetry.
CHARACTERISTICS AND CLASSIFICATION OF SHAPES  1.3.1 and 1.3.3

Geometric shapes occur in many places. After studying them using transformations, students are starting to see certain characteristics of different shapes. In these sections we look at shapes more closely, noticing similarities and differences. We begin to classify them using Venn Diagrams. Students begin to see the need for accurate names, which expands our geometric vocabulary. The last section introduces probability.

See the Math Notes boxes on pages 51 and 60.

Example 1

Using all the shapes listed on resource page 1.2.5, show which shapes belong in each section on the Venn diagram below.

#1: Has only two pairs of parallel sides
#2: Has all sides the same length

The left circle, circle #1, represents all those shapes on the resource page that have only two pairs of parallel sides. There are four figures on the resource page that have this characteristic: the rectangle, the square, the rhombus, and the parallelogram. These shapes will be contained in circle #1. Circle #2 holds all the shapes that have sides all the same length. From the resource page, we have five figures with this characteristic: the regular hexagon, the equilateral triangle, the square, the rhombus, and the regular pentagon. All five of these shapes will be completely contained in Circle #2. There are two shapes that are on both lists: the square and the rhombus. These two shapes have all sides the same length and they have only two pairs of parallel sides. These two shapes, the square and the rhombus, must be listed in the region that is in both circles, which is shaded above.
Example 2

Based on the markings on each shape below, give the figure the best, most specific name possible.

a)

b)

c)

Students created a Shapes Toolkit, that is, a resource page showing many different shapes. Using terms, definitions, and characteristics we have seen, the students named the shapes on the sheet and added appropriate markings. Certain markings mean certain things in geometry. The figure in (a) appears to be a square, but based on the markings, we cannot conclude that. The markings show that the sides of the quadrilateral are equal in length, but equal sides are not enough to make a square. To be a square it would also need right angles. They look like right angles, but maybe they are not quite 90°. Maybe they are 89° and 91°, so without the appropriate markings or other information, we cannot assume the angles are right angles. This quadrilateral with four sides of equal length is called a **rhombus**. Part (b) shows us two types of markings. The small box in the corner of the triangle tells us it is a right angle (measures 90°), so this is a right triangle. We already know that the markings on the sides mean that the sides are the same length. A triangle with two sides that are the same length is called an **isosceles triangle**. Putting both of these facts together, we can label this figure an **isosceles right triangle**. The arrowheads on the two sides of the quadrilateral of part (c) tells us that those sides are parallel. One pair of parallel sides makes this figure a **trapezoid**.

Example 3

Suppose we cut out the three shapes shown in the last example and place them into a bag. If we reach into the bag and randomly pull out a figure without looking, what is the probability that the shape is a triangle? What is the probability the shape has at least two sides of equal length? What is the probability that the shape has more than four sides?

To calculate probability, we count the number of ways a desired outcome can happen (successes) and divide that by the total number of possible outcomes. This explains why the probability of flipping tails with a fair coin is $\frac{1}{2}$. The number of ways we can get tails is one since there is only one tail, and the total number of outcomes is two (either heads or tails). In our example, to calculate the probability that we pull out a triangle, we need to count the number of triangles in the bag (which is one) and divide that by the total number of shapes in the bag (three). This means the probability that we randomly pull out a triangle is $\frac{1}{3}$. To calculate the probability that we pull out a shape with at least two sides of equal length, we first count the number of shapes that would be a success (i.e., would fit this condition). Both figures in (a) and (b) have at least two sides of equal length, so there are two ways to be successful. When we reach into the bag, there are three possible shapes we could pull out, so the total number of outcomes is three. Therefore, the probability of pulling out a shape with at least two sides of equal length is $\frac{2}{3}$. The
probability that we reach into the bag and pull out a shape with more than four sides is done the same way. We know that there are still three outcomes (shapes), so three is still the denominator. But how many ways can we be successful? Are there any shapes with more than four sides? No, so there are zero ways to be successful. Therefore the probability that we pull out a shape with more than four sides is \( \frac{0}{3} = 0 \).

**Problems**

Place the shapes from your Shapes Toolkit into the appropriate regions on the Venn Diagram below. The conditions that the shapes must meet to be placed in each circle, #1 and #2, are listed in each problem. Note: each problem, 1 through 3, is separate and will create a new Venn Diagram.

1. Circle #1: Has more than three sides; Circle #2: Has at least one pair of parallel sides.

2. Circle #1: Has fewer than four sides; Circle #2: Has at least two sides equal in length.

3. Circle #1: Has at least one curved side; Circle #2: Has at least one obtuse angle.

Each shape below is missing markings. Add the correct markings so that the shape represents the term listed. Note: the pictures may not be drawn to scale.

4. A rectangle.  

5. A scalene trapezoid.

6. An isosceles right triangle.

7. An equilateral quadrilateral.

Based on the markings, name the figure below with the most specific name. Note: the pictures are not drawn to scale.

8. 

9. 

10.
11. On a roulette wheel, there are the numbers 1 through 36 along with 0 and 00. What is the probability that the ball will stop on the number 17?

12. When Davis was finished with his checkers board, he decided to turn it into a dartboard. If he is guaranteed to hit the board, but his dart will hit it randomly, what is the probability he will hit a shaded square?

Answers

1. Circle #1 contains: square, rectangle, parallelogram, trapezoid, rhombus, quadrilateral, kite, regular pentagon, and regular hexagon. Circle #2 contains: square, rectangle, parallelogram, trapezoid, rhombus, and regular hexagon. Common to both circles and placed in the overlapping region are: square, rectangle, parallelogram, trapezoid, rhombus, and regular hexagon.

2. Circle #1 contains: equilateral triangle, isosceles triangle, scalene triangle, scalene right triangle, and isosceles right triangle. Circle #2 contains: equilateral triangle, isosceles triangle, isosceles right triangle, square, rectangle, parallelogram, rhombus, kite, regular pentagon, and regular hexagon. Common to both circles and placed in the overlapping region are: equilateral triangle, isosceles triangle, and isosceles right triangle.

3. Circle #1 contains the semicircle and the circle. Circle #2 contains: isosceles right triangle (as pictured), scalene triangle (as pictured), parallelogram, trapezoid, quadrilateral (as pictured), kite, regular pentagon, and regular hexagon. There are no shapes with both characteristics, so there is nothing listed in the overlapping region.

4. A rectangle.

5. A scalene trapezoid.

6. An isosceles right triangle.

7. An equilateral quadrilateral.

8. A parallelogram

9. An isosceles triangle

10. An isosceles trapezoid

11. \( \frac{1}{38} \)

12. \( \frac{1}{2} \)