This chapter will require skills and knowledge from many earlier chapters. In Chapter 11, the students saw cross sections of different solids when they were finding the radius needed to find the distance between two cities as well as when they found the lengths of slant heights in pyramids. In this chapter we look at different cross sections of cones, which produce several different type of curves. These curves are known as conic sections.

See the Math Notes boxes on pages 578, 585, and 589.

Example 1

Use Focus-Directrix paper for each part, (a) through (c) below, to: (1) highlight the focus at the origin, (2) highlight the directrix 6 units away, and (3) graph the resulting curve. What is the name of each conic section you created?

a. The points that are twice the distance from the directrix as from the focus.

b. The points that are the same distance from the directrix as they are from the focus.

c. The points that are six units from the focus.

For part (a), if we find all the points that are twice the distance from the directrix as from the focus, we form an ellipse. To check, look at point A. It is six units from the directrix, but only three units from the focus.

In part (b) we want points that are the same distance from both the directrix and the focus. This forms a parabola. Check that point B is the same distance from the focus and the directrix.

In part (c), the points that are just six units from the focus is a circle with a center at the focus and a radius of six.
Example 2

Using what you have learned about the equations of conic sections, state whether the graph of the equation will be a parabola, circle, ellipse or hyperbola.

a. \((x + 3)^2 + (y - 1)^2 = 100\)  
b. \(y = 3x^2 - 8\)  
c. \(x^2 - 4y^2 = 24\)  
d. \(6x^2 + 3y^2 = 12\)

The key to determining which type of graph the equation represents is remembering general forms of equations. Some general forms are:

- **Parabola**: \(y = a(x - h)^2 + k\)
- **Circle**: \((x - h)^2 + (y - k)^2 = r^2\)
- **Ellipse**: \(a(x - h)^2 + b(y - k)^2 = c\)
- **Hyperbola**: \(a(x - h)^2 - b(y - k)^2 = c\)

In each of these general forms, the variables are \(x\) and \(y\), while \(a\), \(b\), and \(c\) are constants. These general equations can be written in slightly different forms, so you might see variations of them in another math textbook. The differences that distinguish these equations are:

- **Parabola**: exactly one variable is squared.
- **Circle**: Both variables are squared and they do not have coefficients (constants in front of the variable).
- **Ellipse**: Both variables are squared and at least one variable has a coefficient not equal to 1, the coefficients of \(x\) and \(y\) are not equal to each other, and the two variable expressions are summed.
- **Hyperbola**: Same as the ellipse except that one variable expression is subtracted from the other.

With this in mind, we can identify the equations above. The equation in part (a) is a circle. From our previous work, we also know that the center of the circle is \((-3, 1)\), and the radius is 10.

The equation in part (b) is a parabola. Only the \(x\) is squared and its coefficient is not zero.

The equation in part (c) is a hyperbola. Both variables are squared and one is subtracted from the other.

Lastly, the equation in part (d) is an ellipse. Both variables are squared, their coefficients are different, and the variable expressions are added.
Problems

1. Draw a double cone solid similar to the one at right. For each of the conic sections, draw the cross section that produces that curve.

For problems 2 through 4, use a piece of Focus-Directrix paper to highlight a focal point, then highlight a line five units away to be the directrix.

2. Plot three points that are the same distance from the focus as the directrix. What curve do these points lie on?

3. Plot three points that are farther away from the focus than they are from the directrix. What curve do these points lie on?

4. Plot three points that are farther away from the directrix than the focus. What curve do these points lie on?

For each equation below, what is the name of the graph that the equation represents?

5. \( y - x^2 = 16 \)

6. \( x^2 + y = 24 \)

7. \( 2x^2 + y^2 = 18 \)

8. \( \frac{(x+3)^2}{25} + \frac{(y-7)^2}{25} = 1 \)

9. \( (y - 4)^2 - x^2 = 121 \)

10. \( \frac{(x+3)^2}{8} + \frac{(y-7)^2}{6} = 1 \)
Answers

1. The figure at right shows all cross sections.
2. A parabola
3. A hyperbola
4. An ellipse
5. A parabola
6. A parabola
7. An ellipse
8. A circle (Multiply both sides of the equation by 25 to make this look more like the general equation of a circle.)
9. A hyperbola
10. An ellipse (No matter what you multiply both sides of the equation by, there will still be a coefficient for the \(x\) or \(y\).)