Applications of geometry in “real world” settings often involve the measures of angles. In this chapter we begin our study of angle measurement. After describing angles and recognizing their characteristics, students complete an Angle Measurement Toolkit. This sheet lists the names of the angles as well as important information about them. The list includes complementary angles (pairs of angles with measures that sum to 90°), supplementary angles (pairs of angles with measures that sum to 180°), vertical angles (which are always equal in measure), straight angle (which equals 180°), corresponding angles, alternate interior angles, and same-side interior angles. See the Methods and Meanings boxes on pages 76 and 91 for further descriptions and pictures.

See the Math Notes boxes on pages 76, 81, 91, and 100.

Example 1

In each figure below, find the measures of angles \(a\), \(b\), and/or \(c\). Justify your answers.

a.

\[ m\angle a = 72° \]

b.

\[ m\angle a = 22° \]

\[ m\angle c = 97° \]

In part (a), the little box at angle \(b\) tells us that angle \(b\) is a right angle, so \(m\angle b = 90°\). The angle labeled \(c\) is a straight angle (it is opened wide enough to form a straight line) so \(m\angle c = 180°\). To calculate \(m\angle a\) we need to realize that \(\angle a\) and the 72° angle are complementary which means together they sum to 90°. Therefore, \(m\angle a + 72° = 90°\) which tells us that \(m\angle a = 18°\).

In part (b) we will use two pieces of information, one about supplementary angles and one about vertical angles. First, \(m\angle a\) and the 22° angle are supplementary because they form a straight angle (line), so the sum of their measures is 180°. Subtracting from 180° we find that...
\( m\angle a = 158^\circ \). Vertical angles are formed when two lines intersect. They are the two pairs of angles that are opposite (across from) each other where the lines cross. Their angle measures are always equal. Since the 22° angle and \( \angle b \) are a pair of vertical angles, \( m\angle b = 22^\circ \). Similarly, \( \angle a \) and \( \angle c \) are vertical angles, and therefore equal, so \( m\angle c = 158^\circ \).

The figure in part (c) shows two parallel lines that are intersected by a transversal. When this happens we have several pairs of angles with equal measures. \( \angle a \) and the 92° angle are alternate interior angles, and since the lines are parallel (that is what the arrows on the lines mean), these angles are equal. Therefore, \( m\angle a = 92^\circ \). There are several ways to calculate the remaining angles. One way is to realize that \( \angle a \) and \( \angle b \) are supplementary. Another uses the fact that \( \angle b \) and the 92° angle are same-side interior angles, which makes them supplementary because the lines are parallel. Either way gives the same result: \( m\angle b = 180^\circ - 92^\circ = 88^\circ \). There is also more than one way to calculate \( m\angle c \). We know that \( \angle c \) and \( \angle b \) are supplementary. Alternately, \( \angle c \) and the 92° angle are corresponding angles which are equal because the lines are parallel. A third way is to see that \( \angle c \) and \( \angle a \) are vertical angles. With any of these approaches, \( m\angle c = 92^\circ \).

Part (d) is a triangle. In class, the students investigated the measures of the angles of a triangle. They found that the sum of the measures of the three angles always equals 180°. Knowing this, we can calculate \( m\angle a \): \( m\angle a + 50^\circ + 97^\circ = 180^\circ \). Therefore, \( m\angle a = 33^\circ \).

**Problems**

Use what you know about angle measures to find \( x \), \( y \), or \( z \).

1. \[ x + 5 \quad \begin{array}{c} 4x \\ 6x - 4 \end{array} \]

2. \[ \begin{array}{c} x + 13 \\ 2x + 7 \end{array} \]

3. \[ \begin{array}{c} y \\ 4x - 6 \end{array} \]

4. \[ \begin{array}{c} x - 7 \\ y \\ 3x - 3 \end{array} \]

5. \[ \begin{array}{c} 28^\circ \\ 100^\circ \end{array} \]
In Lesson 2.1.5 we used what we have learned about angle measures to create proofs by contradiction. (See the Methods and Meanings box on page 186.) Use this method of proof to justify each of your conclusion to problems 7 and 8 below.

7. Nik scored 40 points lower than Tess on their last math test. The scores could range from 0 to 100 points. Could Tess have scored a 30 on this test? Justify using a proof by contradiction.

8. Can a triangle have two right angles? Justify your answer with a proof by contradiction.

Answers

1. \((x + 5) + 4x = 180\), \(x = 35\)

2. \((x + 13) + (2x + 7) + 5x = 180^\circ\), \(x = 20\)

3. \((6x - 4) + (4x - 6) = 180\), \(x = 19\), \(y = 110^\circ\)

4. \((x - 7) + (3x - 3) = 90\), \(x = 25\), \(y = 90^\circ\)

5. \(x = 28^\circ\), \(y = 52^\circ\), \(z = 80^\circ\)

6. \(x = 150^\circ\), \(y = 160^\circ\), \(z = 130^\circ\)

7. If Tess scored 30 points, then Nik’s score would be -10, which is impossible. So Tess cannot have a score of 30 points.

8. If a triangle has two right angles, then the measure of the third angle must be zero. However, this is impossible, so a triangle cannot have two right angles. OR: If a triangle has two 90° angles, the two sides that intersect with the side between them would be parallel and never meet to complete the triangle, as shown in the figure.
After measuring various angles, students look at measurement in more familiar situations, that of length and area on a flat surface. The students develop methods and formulas for calculating the areas of triangles, parallelograms and trapezoids. They also find the areas of more complicated shapes by partitioning them into shapes for which they can use the basic area formulas. Students also learn how to determine the height of a figure with respect to a particular base.

See the Math Notes box on page 112.

Example 1

In each figure, one side is labeled “base.” For this “base,” draw in a corresponding height.

a. 

![Image of a triangle with a height drawn in] 

b. 

![Image of a parallelogram with a height drawn in] 

c. 

![Image of a triangle with a height drawn in] 

d. 

![Image of a trapezoid with a height drawn in] 

To find how tall a person is, we have them stand erect and measure the distance from the highest point on their head straight down to the floor. We measure the height of figures in a similar way. One way to visualize height is to imagine that the shape, with its base horizontal, needs to slide into a tunnel. How high must the tunnel be so that the shape will slide into it? The length of the height will answer this. The height, then, is perpendicular to the base (or a line that contains the base) from any of the shape’s “highest” point(s). The students also used a 3 x 5 card to help them draw in the height.

a. It is often easier to draw in the height of a figure when the base is horizontal, or the “bottom” of the figure. The height of the triangle below is drawn from the highest point down to the base and forms a right angle with the base.

![Image of a triangle with a height drawn in] 

Even though the shape below is not a triangle, it still has a height. In fact, the height can be drawn in any number of places from the side opposite the base. Three heights, all of equal length, are shown.

![Image of a trapezoid with three heights drawn in] 

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c. The base of the first triangle below is different from the one in part (a) in that it is not horizontal nor at the bottom. Rotate the shape, then draw the height as we did in part (a).

Shapes like the trapezoid below or the parallelogram in part (b) have at least one pair of parallel sides. Because the base is one of the parallel sides, we can draw several heights. The height at far right below shows a situation where the height is drawn to a line that contains the base segment.

Example 2

Find the area of each shape or its shaded region below. Be sure to include the appropriate units of measurement.

a. b. c. d. e. f.

The students have the formulas for the areas of different shapes on their Area Toolkit. For part (a), the area of a triangle is \( A = \frac{1}{2}bh \), where \( b \) and \( h \) are perpendicular to each other. In this case, the base is 13 feet and the height is 4 feet. The side which is 5 feet is not a height because it does not meet the base at a right angle. Therefore, \( A = \frac{1}{2}(13)(4) = 26 \) square feet. Area is measured in square units, while length (such as a perimeter) is measured in linear units, such as feet.
The figure in part (b) is a parallelogram and the area of a parallelogram is \( A = bh \) where \( b \) and \( h \) are perpendicular. Therefore \( A = (13)(8) = 104 \) square cm.

The figure in part (c) is a rectangle so the area is also \( A = bh \), but in this case, we have variable expressions representing the lengths. We still calculate the area in the same way. \( A = (4x+1)(x) = 4x^2 + x \) square units. Since we do not know in what units the lengths are measured, we say the area is just “square units.”

Part (d) shows a trapezoid; the students found several different ways to calculate its area. The most common way is: \( A = \frac{1}{2}(b_1 + b_2)h \) where \( b_1 \) is the upper base and \( b_2 \) is the lower base. As always, \( b \) and \( h \) must be perpendicular. The area is \( A = \frac{1}{2}(6+13)5 = 47.5 \) square inches.

The figures shown in parts (e) and (f) are more complicated and one formula alone will not give us the area. In part (e), there are several ways to divide the area into basic familiar shapes. One way is to divide the figure into three rectangles. The areas of the rectangles on either end are easy to find since the dimensions are labeled on the figure. The area of rectangle (1) is \( A = (2)(8) = 16 \) square units.

The area of rectangle (3) is \( A = (3)(6) = 18 \) square units. To find the area of rectangle (2), we have to determine its height. We know the length is 5; the height is 2 shorter than 6, so the height is 4. Therefore, the area of rectangle (2) is \( A = (5)(4) = 20 \) square units. Now that we know the area of each rectangle, we can add them together to find the area of the whole shape: \( A(\text{whole figure}) = 16 + 18 + 20 = 54 \) square units.

In part (f), we are finding the area of the shaded region, and again, there are several ways to do this. One way is to see it as the sum of a rectangle and a triangle. Another way is to see the shaded figure as a tall rectangle with a triangle cut out of it. Either way will give the same answer.

Using the first method,
\[
A = 4(7) + \frac{1}{2}(4)(7) = 42 \text{ square units.}
\]

The bottom method gives the same answer:
\[
A = 4(14) - \frac{1}{2}(4)(7) = 42 \text{ square units.}
\]
Problems

For each figure below, draw in a corresponding height for the labeled base.

1. 

2. 

3. 

4. 

Find the area of each shape and/or shaded region. Be sure to include the appropriate units.

5. 

6. 

7. 

8. 

9. 

10. 

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11. $2x(3x + 5) = 6x^2 + 10x$ square units

12. $(12)(7) - \frac{1}{2}(9)(9) = 84 - 40.5 = 43.5$ square units

Answers

1. 

3. Any of these:

5. $2x(3x + 5) = 6x^2 + 10x$ square units

7. $\frac{1}{2}(12)5 = 30$ square units

9. $2(12) + 7(6.5) + 2(2.5) = 74.5$ square cm

11. $\frac{1}{2}(7)(24) - (3)(5) = 84 - 15 = 69$ square unit

12. $(12)(7) - \frac{1}{2}(9)(9) = 84 - 40.5 = 43.5$ square units

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After a reminder of what a “square root” is, the students look at different ways to determine lengths of segments through calculation rather than measurement. They use a method that reinforces the understanding of “square root” (the length of a side of a square with a given area) then use it to apply the Pythagorean Theorem in right triangles. We also study the Triangle Inequality, which determines the restrictions on the possible lengths of the third side of a triangle once we know the lengths of two of its sides.

See the Math Notes Boxes on pages 115, 119, and 123.

**Example 1**

Name the shape at right. Then calculate the value for x. What do we call this number? Explain.

The quadrilateral has four sides of equal length and four right angles. That means the shape is a square. Since the area is 81 sq. cm, we can find the value of x by finding a positive number that, when we multiply it by itself, we get 81. That number is \(x = 9\). We say that 9 is the square root of 81 and write \(\sqrt{81} = 9\).

**Example 2**

The triangle at right does not have the lengths of its sides labeled. Can the sides have lengths of:

a) 3, 4, 5?  
b) 8, 2, 12?

At first, students might think that the lengths of the sides of a triangle can be any three lengths, but that is not so. Students used a dynamic tool to explore the restrictions on the lengths of the sides of a triangle. The Triangle Inequality says that the length of any side must be less than the sum of the lengths of the other two sides. For the triangle in part (a) to exist, all of these statements must be true: \(3 + 4 > 5\), \(4 + 5 > 3\), and \(5 + 3 > 4\). Since each of them is true, we could draw a triangle with sides of lengths 3, 4, and 5. In part (b) we need to check whether \(8 + 2 > 12\), \(2 + 12 > 8\), and \(12 + 8 > 2\). In this case, only two of the three conditions are true, namely, the last two. The first inequality is not true so we cannot draw a triangle with side lengths of 8, 2, and 12. One way to make a convincing argument about this is to cut linguine or coffee stirrers to these lengths and see if you can put the pieces together at their endpoints to form a triangle.
Example 3

Use the Pythagorean Theorem to find the value of $x$.

a. 

\[ 7^2 + 24^2 = x^2 \]
\[ 49 + 576 = x^2 \]
\[ 625 = x^2 \]

To find the value of $x$, use a calculator to find the square root of 625:

\[ x = \sqrt{625}, \text{ so } x = 25 \]

Part (b) is a bit different in that the variable is not the hypotenuse so the equation is:

\[ 8^2 + x^2 = 15^2 \]
\[ 64 + x^2 = 225 \]
\[ x^2 = 225 - 64 \]
\[ x^2 = 161 \]
\[ x = \sqrt{161} \]
\[ x = 12.69 \]
Problems

1. A square has an area of 144 square feet. What is the length of one of its sides?

2. A square has an area of 484 square inches. What is the length of one of its sides?

3. A square has an area of 200 square cm. What is the length of one of its sides?

4. A square has an area of 169 square units. What is the perimeter of the square?

The triangle at right does not have any of the lengths of the sides labeled. Can the triangle have side lengths of:

5. 1, 2, 3?
6. 7, 8, 9?
7. 4.5, 2.5, 6?
8. 9.5, 1.25, 11.75?

Use the Pythagorean Theorem to find the value of x. When necessary, round your answer to the nearest hundredth.

9.

10.

11.

12.

Answers

1. 12 feet
2. 22 inches
3. \approx 14.14 \text{ cm}
4. 52 units
5. No. 1 + 2 is not greater than 3.
6. Yes
7. Yes
8. No. 9.5 + 1.25 is not greater than 11.75
9. \approx 23.85 \text{ units}
10. 9 units
11. \approx 5.66 \text{ units}
12. \approx 9.64 \text{ units}