So far, students have measured, described, and transformed geometrical shapes. In this chapter we focus on comparing geometrical shapes. We begin by dilating shapes: enlarging them as one might on a copy machine. When the students compare the original and enlarged shapes closely, they discover that the shape of the figure remains exactly the same (this means the angle measures of the enlarged figure are equal to those in the original), but the size changes (the lengths of the sides increase). Although the size changes, the lengths of the sides have a constant ratio, known as the ratio of similarity or zoom factor.

See the Math Notes boxes on pages 138, 142, 145, and 150.

**Example 1**
Enlarge the figure at right from the origin by a factor of three.

The students used rubber bands to create a dilation (enlargement) of several shapes. We can do this using a grid and slope triangles. Create a right triangle so that the segment from the origin to point A is the hypotenuse, one leg lies on the positive x-axis, and the other connects point A to the endpoint of the leg at (0, 2). This triangle is called a slope triangle since it represents the slope of the hypotenuse from (0, 0) to vertex A. The slope triangle to point A has a vertical leg length of 5 and a horizontal leg length of 2. We will add two more slope triangles exactly like this one along the line from (0, 0) to A as shown in the figure below left. This gives us two new points, A'(4, 10) and A''(6, 15). We do the same thing for the other two vertices, forming a new slope triangle for each vertex. This will give us four more new points B'(10, -6), B''(15, -9), C'(-8, -4), and C''(-12, -6). If we connect A', B', and C', we form a triangle which is an enlargement of the original figure by a factor of two. If we connect A'', B'', and C'' we form an enlargement which is three times the original figure.
Example 2

The two quadrilaterals at right are similar. What parts are equal? Can you determine the lengths of any other sides?

Similar figures have the same shape, but not the same size. Since the quadrilaterals are similar, we know that all the corresponding angles have the same measure. This means \( m\angle A = m\angle A' \), \( m\angle B = m\angle B' \), \( m\angle C = m\angle C' \), and \( m\angle D = m\angle D' \). In addition, the corresponding sides are proportional, which means the ratio of corresponding sides is a constant. To find the ratio, we need to know the lengths of one pair of corresponding sides. From the picture we see that \( AD \) corresponds to \( A'D' \). Since these sides correspond, we can write:

\[
\frac{AD}{A'D'} = \frac{4}{6}
\]

Therefore, the ratio of similarity is \( \frac{4}{6} \), or \( \frac{2}{3} \). We can use this value to find the lengths of other sides when we know at least one length of a corresponding pair of sides.

\[
\frac{AB}{A'B'} = \frac{4}{6} \quad \frac{BC}{B'C'} = \frac{4}{6} \quad \frac{CD}{C'D'} = \frac{4}{6}
\]

\[
\frac{AB}{6} = \frac{4}{6} \quad \frac{BC}{8} = \frac{4}{6} \quad \frac{CD}{12} = \frac{4}{6}
\]

\[
AB = 4 \quad 4B'C' = 48 \quad 6CD = 48
\]

\[
B'C' = 12 \quad CD = 8
\]

Example 3

The pair of shapes below is similar. Label the second figure correctly to reflect the similarity statement.

\[ ABCDEF \sim UVWXYZ \]

As we have stated, similar figures have the same shape, just different sizes, and this means that the corresponding angles have equal measure. When we write a similarity statement, we write the letters so that the corresponding equal angles match up. By the similarity statement, we must have \( m\angle A = m\angle U \), \( m\angle B = m\angle V \), \( m\angle C = m\angle W \), \( m\angle D = m\angle X \), \( m\angle E = m\angle Y \), and \( m\angle F = m\angle Z \).

The smaller figure is labeled at right. If it difficult to tell which original angle corresponds to its enlargement or reduction, try rotating the figures so they have the same orientation.
Problems

1. Copy the figure below onto graph paper and then enlarge the shape by a factor of two.

Each pair of figures below is similar. Use what you know about similarity to solve for x.

3.

4.

5.

6.
Solve for the missing lengths in the set of similar figures below.

7. $\triangle ABC \sim \triangle PQR$

8. $\triangle JKLM \sim \triangle WXYZ$

9. $\triangle STUV \sim \triangle MNOP$

10. $\triangle DAV \sim \triangle ISW$

11. $\triangle ABCDE \sim \triangle FGHIJ$

12. $\triangle ABC \sim \triangle DBE$
Answers

1. 

2. 

3. \( x = 12 \)

4. \( x = 9 \)

5. \( x = 0.8 \)

6. \( x = \frac{40}{7} \approx 13.33 \)

7. \( x = 7.5 \)

8. \( x = 1.25 \)

9. \( x = 16 \)

10. \( x \approx 3.69 \)

11. \( x = 13.5 \)

12. \( x = 12 \)
Rather than always measuring all the angles and sides of two triangles to check for similarity, in this section the students develop conjectures to shorten the process. These are the SSS Triangle Similarity Conjecture (SSS~), AA Triangle Similarity Conjecture (AA~), and the SAS Triangle Similarity Conjecture (SAS~). The first conjecture states that if all three corresponding side lengths share a common ratio, then the triangles are similar. The second conjecture says if two pairs of corresponding angles have equal measures, then the triangles are similar. The third conjecture says if two pairs of corresponding sides lengths share a common ratio, and the included angles have the same measure, then the triangles are similar. By included angle we mean that the equal angles must be between the two pairs of corresponding sides. Additionally, the students found that if similar figures have a ratio of similarity of one, then the shapes are congruent, that is, they have the same size and shape. The students used flow charts in this section to help organize their information and make logical conclusions about similar triangles. Now students are able to use similar triangles to find side lengths, perimeters, heights, and other measurements.

See the Math Notes Boxes on pages 155, 159, 167, and 171.

**Example 1**

Based on the information given, which pair of triangles is similar? If they are similar, write the similarity statement. Justify your answer completely.

a. 

b. 

c. 

d.
We will use the three conjectures to test whether or not the triangles are similar. In part (a), we have the lengths of the three sides, so it makes sense to check whether the SSS~ holds true. Write the ratios of the corresponding side lengths and compare them to see if each one is the same, as shown at right. Each ratio reduces to 3, so they are equal. Therefore, \( \triangle TES \sim \triangle AWK \) by SSS~.

The measurements given in part (b) suggest we look at SAS~. \( \angle A \) and \( \angle R \) are the included angles. Since they are both right angles, they have equal measure. Now we need to check that the corresponding sides lengths have the same ratio, as shown at right. Although the triangles display the SAS~ pattern and the included angles have equal measures, the triangles are not similar because the corresponding side lengths do not have the same ratio.

In part (c), we are given the measures of two angles of each triangle, but not corresponding angles. \( \angle K = 55^\circ = \angle N \) which is one pair of corresponding angles, but for AA~, we need two pairs of equal angles. If we use the fact that the measures of the three angles of a triangle add up to 180°, we can find the measures of \( \angle O \) and \( \angle E \). Now we see that all pairs of corresponding angles have equal measures, so \( \triangle POK \sim \triangle EMN \) by AA~.

Part (d) shows the SAS~ pattern and we can see that the included angles have equal measures, \( m\angle G = m\angle H \). We also need to have the ratio of the corresponding side lengths to be equal. Since the two fractions are equal (the second reduces to the first), the corresponding side lengths have the same ratio. Therefore, \( \triangle YUG \sim \triangle IOH \) by SAS~.
In part (e), we see that the included angles have equal measures, \( m\angle B = m\angle N \). Since \( \frac{45}{15} = \frac{9}{3} = \frac{3}{1} \), the corresponding sides are proportional. Therefore, \( \triangle BOX \sim \triangle NTE \) by SAS \( \sim \).

In part (f), we only have one pair of angles that are equal (the right angles), but those angles are not between the sides with known lengths. However, we can find the length of the third sides by using the Pythagorean Theorem.

\[
8^2 + (IL)^2 = 10^2
\]
\[
64 + (IL)^2 = 100
\]
\[
(IL)^2 = 36
\]
\[
IL = 6
\]

\[
12^2 + (AB)^2 = 20^2
\]
\[
144 + (AB)^2 = 400
\]
\[
(AB)^2 = 256
\]
\[
AB = 16
\]

Now that we know all three sides we can check to see if the triangles are similar by SSS \( \sim \). Since the ratio of corresponding sides is a constant, \( \triangle ELI \sim \triangle BZA \) by SSS\( \sim \).

**Example 2**

In the figure at right, \( \overline{AY} \parallel \overline{HP} \). Decide whether or not there are any similar triangles in the figure. Justify your answer with a flowchart. Can you find the length of \( \overline{AY} \)? If so, find it. Justify your answer.

Recalling information we studied in earlier chapters, the parallel lines give us angles with equal measures. In this figure, we have two pairs of corresponding angles with equal measures: \( m\angle PHR = m\angle YAR \) and \( m\angle HPR = m\angle AYR \). With two pairs of corresponding angles with equal measures, we can say the triangles are similar: \( \triangle PHR \sim \triangle YAR \) by AA\( \sim \). Since the triangles are similar, the lengths of corresponding sides are proportional (i.e., have the same ratio).

This means we can write

\[
\frac{RA}{RH} = \frac{AY}{HP}
\]
\[
\frac{12}{16} = \frac{AY}{9}
\]
\[
AY = \frac{9 \cdot 12}{16} = 6.75
\]

We can justify this result with a flowchart as well. The flowchart at right organizes and states what is written above.
Problems
Each pair of figures below is similar. Write the correct similarity statement and solve for $x$.

1. \[ \begin{align*}
A & \quad 4 \\
F & \quad 8 \\
E & \quad 2 \\
B & \quad 10 \\
Z & \quad 10 \\
C & \quad 9 \\
D & \quad 7
\end{align*} \]

2. \[ \begin{align*}
R & \quad 6 \\
E & \quad 3 \\
T & \quad 3 \\
C & \quad x
\end{align*} \]

3. \[ \begin{align*}
I & \quad x \\
M & \quad 24
\end{align*} \]

4. \[ \begin{align*}
L & \quad 15 \\
Y & \quad 18 \\
E & \quad x
\end{align*} \]

Decide if each pair of triangles is similar. Write a correct similarity statement, and justify your answer.

5. \[ \begin{align*}
O & \quad 16^\circ \\
B & \quad 82^\circ \\
X & \quad \text{X}
\end{align*} \]

6. \[ \begin{align*}
O & \quad 12 \\
H & \quad 18 \\
W & \quad 22.5
\end{align*} \]

7. \[ \begin{align*}
A & \quad 12 \\
M & \quad 12 \\
I & \quad 20
\end{align*} \]

8. \[ \begin{align*}
A & \quad 13 \\
M & \quad 8 \\
S & \quad 10
\end{align*} \]

9. In the figure at right $\overline{AB} \parallel \overline{DE}$. Is $\triangle ABC$ similar to $\triangle EDC$? Use a flowchart to organize and justify your answer.

10. Standing four feet from a mirror resting on the flat ground, Palmer, whose eye height is 5 feet, 9 inches, can see the reflection of the top of a tree. He measures the mirror to be 24 feet from the tree. How tall is the tree? Draw a picture to help solve the problem.
Answers

1. $ABCDEF \sim UZYXWV, x = 3.75$

2. $RECT \sim NGLA, x = 8$

3. $\triangle IMS \sim \triangle RCH, x = 72$

4. $LACEY \sim ITHOM, x = 16.5$

5. $\triangle BOX \sim \triangle NCA$ by $AA\sim$

6. The triangles are not similar because the sides are not proportional.
   \[
   \frac{12}{15} = \frac{18}{22.5} = 0.8, \quad \frac{10}{13} \approx 0.76
   \]

7. $\triangle ALI \sim \triangle MES$ by $SAS\sim$.

8. The triangles are not similar. On $\triangle SAM$, the $60^\circ$ is included between the two given sides, but on $\triangle UEL$ the angle is not included.

9. \[
\begin{align*}
&AB \parallel DE \\
&\text{Given} \\
&m\angle ACB = m\angle ECD \\
&m\angle ABD = m\angle EDC \\
&\parallel \text{lines} \rightarrow \text{equal corresp. angles} \\
&\triangle ABC \sim \triangle ECD \\
&\text{AA }\sim
\end{align*}
\]

Note: there is more than one way to solve this problem. Corresponding angles could have been used twice rather than mentioning vertical angles.

10. The figures at right show a sketch of the situation and how it translates into a diagram with triangles. $\triangle PFM \sim \triangle TRM$ by $AA\sim$. The proportion is

\[
\frac{x}{5.75} = \frac{24}{4} \\
4x = 138 \\
x = 34.5
\]

Therefore, the tree is 34.5 feet tall.