In this chapter, the students examine three-dimensional shapes, known as solids. The students will work on visualizing these solids by building and then drawing them. Visualization is a useful, often overlooked skill in mathematics. By drawing solids students gain a better understanding of volume and surface area.

See the Math Notes boxes on pages 448, 452, and 457.

**Example 1**

The solid at right is built from individual cubes stacked upon each other on a flat surface. (This means that no cubes are “floating.”) Create a mat plan representing this solid. What is the volume of this solid?

This solid consists of stacked blocks. We are looking at the front, right side, and top of this solid. A mat plan shows a different perspective of a solid. It shows the footprint of the solid as well as how many blocks are in each stack. A mat plan is useful because, in the solid above, we do not know if there are any hidden blocks. A mat plan tells us exactly how many blocks are in the solid.

In this case, since we are creating the mat plan from the stacked blocks, there is more than one possible answer. If there are no hidden blocks, then the mat plan is the first diagram at right. If there is a hidden block, then the mat plan is the second one at right. It is helpful to visualize solids by building them with cubes. Build solids on a \(3 \times 5\) card so that you can rotate the card to see the solid from all of its sides. Do this to make sure that one block is all that can be hidden in this drawing.

The volume of this solid is the number of cubes it would take to build it. In this case, the volume is either 9 cubic units or 10 cubic units.
Example 2
At right is a mat plan of a solid. Build the solid. What is the volume of this solid? Draw the front, right and top views, as well as the three-dimensional view of this solid.

We find the volume by counting the number of blocks it would take to build this solid. Adding the numbers, the volume is 12 cubic units. To draw the different views of this solid, it is extremely helpful to build it out of cubes on a 3 x 5 card. Label the card with front, right, left, and back so that you can remember which side is which when rotating it. Remember that the standard three-dimensional view shows the top, right, and front views.

The individual views of each side are flat views. It is helpful to look at the solid at your eye level, so that only one side is visible at a time.

Example 3
If the figure at right is made with the fewest amount of cubes possible, what is its surface area?

The surface area is the sum of the areas of all the surfaces (or faces or sides) on the solid. If we draw every view of the solid, we can count the number of squares to find the surface area. If we build the solid on a card, then we can rotate the card and count the number of squares on each face. Either way, we will arrive at the same answer.

From the front and back the solid looks the same and shows 10 squares.

The right and left views are reflections of each other and each shows seven squares.

The top and bottom views are also reflections of each other. They show four squares each.

Therefore, the surface area is $10 + 10 + 7 + 7 + 4 + 4 = 42$ square units.
Example 4

The dimensions of the prism at right are shown. What are the volume and surface area of this prism?

A prism is a special type of polyhedron that has two congruent and parallel bases. In this problem, the bases are right triangles. The volume of a prism is found by finding the area of the base, multiplied by the height of the prism. To understand this process, think of a prism as a stack of cubes. The base area tells you how many cubes are in one layer of the stack. The height tells you how many layers of cubes are in the figure.

In this example, since the base is a right triangle, the area is \( \frac{1}{2}bh \). Looking at the top of the prism might make it easier to find the area of the base represented by \( A_b \).

\[
A_b = \frac{1}{2}bh = \frac{1}{2}(6)(8) = 24 \text{ square units}, \text{ so there are 24 cubes in one layer.}
\]

To find the volume, we multiply this number by the height, 12.

\[
V = A_bh = (24)(12) = 288 \text{ cubic units}
\]

To find the surface area of this prism, we will find the area of each of its faces, including the bases, and add the areas. One way to illustrate the subproblems is to make sketches of the surfaces.

Surface Area:  \[
2 \left( \frac{1}{2} b \cdot h \right) + 6 \cdot 12 + 8 \cdot 12 + ? \cdot 12
\]

All of the surfaces are familiar shapes, namely, triangles and rectangles. We need to do a subproblem to find the length of the rectangle on the back face (the last rectangle in the pictorial equation above). Fortunately, that length is also the hypotenuse of the right triangle in the base, so we can use the Pythagorean Theorem to find that length.

\[
6^2 + 8^2 = c^2 \quad \Rightarrow \quad 36 + 64 = c^2 \quad \Rightarrow \quad c^2 = 100 \quad \Rightarrow \quad c = \sqrt{100} = 10
\]

Therefore the surface area is:

\[
S.A. = 2\left( \frac{1}{2} \cdot 6 \cdot 8 \right) + (6 \cdot 12) + (8 \cdot 12) + (10 \cdot 12) = 48 + 72 + 96 + 120 = 336 \text{ square units}
\]
Example 5

The Styrofoam pieces used in packing boxes, known as “shipping peanuts,” are sold in three box sizes: small, medium and large. The small box has a volume of 1200 cubic inches. The dimensions on the “medium” box are twice the dimensions of the small box, and the “large” box is triple the dimensions of the small one. All three boxes are similar prisms. What are volumes of the medium and large boxes?

Since the boxes are similar, we can use the ratio of similarity to determine the volume of the medium and large boxes without knowing their actual dimensions. When figures are similar with ratio of similarity \( \frac{a}{b} \), the ratio of the areas is \( \left(\frac{a}{b}\right)^2 \) and the ratio of the volumes is \( \left(\frac{a}{b}\right)^3 \). Since the medium box has dimensions twice the small box and the large box has dimensions three times the small box, we can write:

\[
\frac{\text{medium box}}{\text{small box}} = \frac{2}{1}
\]

\[
\frac{\text{volume of medium box}}{\text{volume of small box}} = \left(\frac{2}{1}\right)^3
\]

\[
\frac{x}{1200} = \frac{8}{1}
\]

Solving, \( x = 8 \cdot 1200, x = 9600 \) cubic units and \( x = 27 \cdot 1200, x = 32,400 \) cubic units.

Problems

For each solid, calculate the volume and surface area, then draw a mat plan. Assume there are no hidden or floating cubes.

1.  
2.  
3.  

For each mat plan, draw the solid, then calculate the volume and surface area.

4.  
5.  
6.
Calculate the volume and surface area of each prism.

7. The base is a rectangle.

8. A cube.


10. At Cakes R Us, it is possible to buy round cakes in different sizes. The smallest cake has a diameter of 8 inches and a height of 4 inches, and requires 3 cups of batter. Another similar round cake has a diameter of 13 inches. How much batter would this cake require?

11. Prism A and prism B are similar with a ratio of similarity of 2:3. If the volume of prism A is 36 cubic units, what is the volume of prism B?

12. Two rectangular prisms are similar. The smaller, A, has a height of four units while the larger, B, has a height of six units.
   a. What is the magnification factor from prism A to prism B?
   b. What would the ratio of the lengths of the edges labeled $x$ and $y$?
   c. What is the ratio of their surface areas? What is the ratio of their volumes?
   d. A third prism C is similar to prisms A and B. Prism C’s height is ten units. If the volume of prism A is 24 cubic units, what is the volume of prism C?
Answers

   SA = 28 sq. un.

   SA = 36 sq. un.

   SA = 38 sq. un.

   SA = 60 sq. un.

   SA = 36 sq. un.

   SA = 46 sq. un.

   SA = 482 sq. un.

   SA = 91.8 sq. un.

   SA = 726 sq. un.

10. \( \approx 12.87 \) cups

11. 121.5 cu. un.

12. a. \( \frac{4}{6} = \frac{2}{3} \)
    b. \( \frac{4}{5} = \frac{8}{10} \)
    c. \( \frac{16}{36} = \frac{4}{9}, \frac{64}{216} = \frac{8}{27} \)
    d. 375 cu. un.
Example 1

Using only a straightedge and compass, construct the perpendicular bisector of $\overline{AB}$ at right. Then bisect one of the right angles.

The perpendicular bisector of $\overline{AB}$ is a line that is perpendicular to $\overline{AB}$ and also goes through the midpoint of $\overline{AB}$. Although we could find this line by folding point $A$ onto point $B$, we want a way to find it with just the straightedge and compass. Also, with no markings on the straightedge, we cannot measure to find the midpoint.

Because of the nature of a compass, circles are the basis for constructions. For this construction, we draw two congruent circles, one with its center at point $A$, the other with its center at point $B$. The radii of these circles must be large enough so that the circles intersect at two points. Drawing a line through the two intersection points of intersection gives $l$, the perpendicular bisector of $\overline{AB}$.
To bisect the right angle, we begin by drawing a circle with a center at point $M$. There are no restrictions on the length of the radius, but we need to see the points of intersection, $P$ and $Q$. Also, we are only concerned with the arc of the circle that is within the interior of the angle we are bisecting, $\overline{PQ}$. Next we use points $P$ and $Q$ as the centers of two congruent circles that intersect in the interior of $\angle PMQ$. This gives us point $X$ which, when connected to point $M$, bisects the right angle.

Note: with this construction, we have created two $45^\circ$ angles. From this we also get a $135^\circ$ angle, $\angle XMA$. Another bisection (of $\angleXMQ$) would give us a $22.5^\circ$ angle.

**Example 2**

Construct $\triangle MUD$ so that $\triangle MUD$ is congruent to $\triangle ABC$ by SAS $\cong$.

To construct a congruent triangle, we will need to use two constructions: copying a segment and copying an angle. In this example we want to construct the triangle with the SAS $\cong$, so we will copy a side, then an angle, and then the adjacent side. It does not matter which side we start with as long as we do the remainder of the parts in a SAS order. Here we will start by copying $\overline{BC}$ with a compass. First draw a ray like the one at right. Next, put the compass point on point $B$ and open the compass so that it reaches to point $C$. Keeping that measurement, mark off a congruent segment on the ray ($\overline{UD}$). Next copy $\angle BCA$ so that its vertex $C$ is at point $D$ on the ray and one of its sides is $\overline{UD}$. Then copy $\overline{CA}$ to create $\overline{DM}$. Finally, connect point $U$ to point $M$ and $\triangle MUD \cong \triangle ABC$. 
Problems

1. Construct a triangle congruent to $\triangle XYZ$ using SSS $\equiv$.

2. Construct a rhombus with sides congruent to $AB$.

3. Construct a regular hexagon with sides congruent to $PQ$.

4. Use constructions to find the centroid of $\triangle WKD$.

5. Construct the perpendicular bisectors of each side of $\triangle TES$. Do they all meet in one point?
Answers

1. Draw a ray and copy one side of $\triangle XYZ$ on it—for example, $\overline{XZ}$. Copy a second side ($\overline{XY}$), put one endpoint at $X$, and swing an arc above $\overline{XZ}$. Finally, copy the third side, place the compass point at $Z$, and swing an arc above $\overline{XZ}$ so that it intersects the arc from $X$. Connect points $X$ and $Z$ to the point of intersection of the arcs and label it $Y$.

2. Copy $\overline{AB}$ on a ray. Draw another ray from $A$ above the ray and mark a point $C$ on it at the length of $\overline{AB}$. Swing arcs of length $\overline{AB}$ from $B$ and $C$ and label their intersection $D$.

3. Construct a circle of radius $\overline{PQ}$. Mark a point on the circle, then make consecutive arcs around the circle using length $\overline{PQ}$. Connect the six points to form the hexagon. Alternately, construct an equilateral triangle using $\overline{PQ}$, then make five copies of the triangle to complete the hexagon.

4. Bisect each side of $\triangle WKD$, then draw a segment from each vertex to the midpoint of the opposite side. The point where the medians intersect is the centroid.

5. Yes.