Chapter 8 introduces students to rewriting quadratic expressions and solving quadratic equations. Quadratic functions are functions which can be rewritten in the form \( y = ax^2 + bx + c \) (where \( a \neq 0 \)) and when graphed, create a U-shaped curve called a parabola.

There are multiple methods that can be used to solve quadratic equations. One of them requires factoring the quadratic expression first. In Lessons 8.1.1 through 8.1.4, students factor quadratic expressions.

In previous chapters, students used algebra tiles to build “generic rectangles” of quadratic expressions. In the figure below, the length and width of the rectangle are \((x + 2)\) and \((x + 4)\). Since the area of a rectangle is given by \((\text{base})(\text{height}) = \text{area}\), the area of the rectangle in the figure below can be expressed as a sum, \(4x + 8 + x^2 + 2x\), or \(x^2 + 6x + 8\). Thus students wrote \((x + 2)(x + 4) = x^2 + 6x + 8\).

In the figure at right, the length and width of the rectangle, which are \((x + 2)\) and \((x + 4)\), are factors of the quadratic expression \(x^2 + 6x + 8\), since \((x + 2)\) and \((x + 4)\) multiply together to produce the quadratic expression \(x^2 + 6x + 8\). Notice that the \(4x\) and the \(2x\) are located diagonally from each other. They are like terms and can be combined and written as \(6x\).

The factors of \(x^2 + 6x + 8\) are \((x + 2)\) and \((x + 4)\).

The \(ax^2\) term and the \(c\) term are always diagonal to one another in a generic rectangle. In this example, the \(ax^2\) term is \((1x^2)\) and the \(c\) term is the constant 8; the product of this diagonal is \(1x^2 \cdot 8 = 8x^2\). The two \(x\)-terms make up the other diagonal and can be combined into a sum since they are like terms. The \(b\) of a quadratic expression is the sum of the coefficients of these factors: \(2x + 4x = 6x\), so \(b = 6\). The product of this other diagonal is \((2x)(4x) = 8x^2\). Note that the products of the two diagonals are always equivalent. In the textbook, students may nickname this rule “Casey’s Rule,” after the fictional character Casey in problem 8-4.

To factor a quadratic expression, students need to identify the coefficients of the two \(x\)-terms so that the products of the two diagonals are equivalent, and also the sum of the two \(x\)-terms is \(b\). Students can use a “diamond problem” to help organize their sums and products. For more information on using a diamond problem and generic rectangle to factor quadratic expressions, see the Math Notes box in Lesson 8.1.4.

For additional information, see the Math Notes boxes in Lessons 8.1.1 through 8.1.4. For additional examples and more practice, see the Checkpoint 10B materials at the back of the student textbook.
Example 1

Factor $x^2 + 7x + 12$.

Sketch a generic rectangle.

Place the $x^2$ and the 12 along one diagonal.

Find two terms whose product is $12x^2$ and whose sum is $7x$.
In this case, $3x$ and $4x$. (Students are familiar with this situation as a “diamond problem” from Chapter 1.)

Write these terms along the other diagonal. Either term can go in either diagonal space.

Determine the base and height of the large outer rectangle by using the areas of the small pieces and finding the greatest common factor of each row and column.

Write the sum as a product (factored form).

$x^2 + 7x + 12 = (x + 3)(x + 4)$

Example 2

Factor $x^2 + 7x – 30$.

Sketch a generic rectangle.

Place the $x^2$ and the $-30$ along one diagonal.

Find two terms whose product is $-30x^2$ and whose sum is $7x$. In this case, $-3x$ and $10x$.

Write these terms along the other diagonal. Either term can go in either diagonal space.

Determine the base and height of the large outer rectangle by using the areas of the small pieces and finding the greatest common factor of each row and column.

Write the sum as a product (factored form).

$x^2 + 7x – 30 = (x – 3)(x + 10)$
Example 3

Factor \(x^2 - 15x + 56\).

Sketch a generic rectangle.
Place the \(x^2\) and the 56 along one diagonal.

Find two terms whose product is 56\(x^2\) and whose sum is \(-15x\). Write these terms as the other diagonal.

Determine the base and height of the large outer rectangle by using the areas of the small pieces and finding the greatest common factor of each row and column.

Write the sum as a product (factored form).

\[x^2 - 15x + 56 = (x - 7)(x - 8)\]

Example 4

Factor \(12x^2 - 19x + 5\).

Sketch a generic rectangle.
Place the \(12x^2\) and the 5 along one diagonal.

Find two terms whose product is 60\(x^2\) and whose sum is \(-19x\). Write these terms as the other diagonal.

Find the base and height of the rectangle. Check the signs of the factors.

Write the sum as a product (factored form).

\[12x^2 - 19x + 5 = (3x - 1)(4x - 5)\]
Example 5

Factor $3x^2 + 21x + 36$.

Note: If a common factor appears in all the terms, it should be factored out first.
For example, $3x^2 + 21x + 36 = 3(x^2 + 7x +12)$.

Then $x^2 + 7x + 12$ can be factored in the usual way, as in Example.
$x^2 + 7x + 12 = (x + 3)(x + 4)$.

Then, since the expression $3x^2 + 21x + 36$ has a factor of 3,
$3x^2 + 21x + 36 = 3(x^2 + 7x + 12) = 3(x + 3)(x + 4)$.

Problems

1. $x^2 + 5x + 6$  
2. $2x^2 + 5x + 3$  
3. $3x^2 + 4x + 1$  
4. $3x^2 + 30x + 75$
5. $x^2 + 15x + 44$  
6. $x^2 + 7x + 6$  
7. $2x^2 + 22x + 48$  
8. $x^2 + 4x – 32$
9. $4x^2 + 12x + 9$  
10. $24x^2 + 22x – 10$  
11. $x^2 + x – 72$  
12. $3x^2 – 20x – 7$
13. $x^3 – 11x^2 + 28x$  
14. $2x^2 + 11x – 6$  
15. $2x^2 + 5x – 3$  
16. $x^2 – 3x – 10$
17. $4x^2 – 12x + 9$  
18. $3x^2 + 2x – 5$  
19. $6x^2 – x – 2$  
20. $9x^2 – 18x + 8$

Answers

1. $(x + 2)(x + 3)$  
2. $(x + 1)(2x + 3)$  
3. $(3x + 1)(x + 1)$  
4. $(3x + 5)(x + 5)$
5. $(x + 11)(x + 4)$  
6. $(x + 6)(x + 1)$  
7. $2(x + 8)(x + 3)$  
8. $(x + 8)(x – 4)$
9. $(2x + 3)(2x + 3)$  
10. $2(3x – 1)(4x + 5)$  
11. $(x – 8)(x + 9)$  
12. $(x – 7)(3x + 1)$
13. $x(x – 4)(x – 7)$  
14. $(x + 6)(2x – 1)$  
15. $(x + 3)(2x – 1)$  
16. $(x – 5)(x + 2)$
17. $(2x – 3)(2x – 3)$  
18. $(3x + 5)(x – 1)$  
19. $(2x + 1)(3x – 2)$  
20. $(3x – 4)(3x – 2)$
FACTORIZING SHORTCUTS

Although most factoring problems can be done with generic rectangles, there are two special factoring patterns that, if recognized, can be done by sight. The two patterns are known as the **Difference of Squares** and **Perfect Square Trinomials**. The general patterns are as follows:

**Difference of Squares**: \( \frac{a^2}{x^2} - \frac{b^2}{y^2} = (ax + by)(ax - by) \)

**Perfect Square Trinomial**: \( \frac{a^2}{x^2} + 2abxy + \frac{b^2}{y^2} = (ax + by)^2 \)

### Example 1

<table>
<thead>
<tr>
<th>Difference of Squares</th>
<th>Perfect Square Trinomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 - 49 = (x + 7)(x - 7) )</td>
<td>( x^2 - 10x + 25 = (x - 5)^2 )</td>
</tr>
<tr>
<td>( 4x^2 - 25 = (2x - 5)(2x + 5) )</td>
<td>( 9x^2 + 12x + 4 = (3x + 2)^2 )</td>
</tr>
<tr>
<td>( x^2 - 36 = (x + 6)(x - 6) )</td>
<td>( x^2 - 6x + 9 = (x - 3)^2 )</td>
</tr>
<tr>
<td>( 9x^2 - 1 = (3x - 1)(3x + 1) )</td>
<td>( 4x^2 + 20x + 25 = (2x + 5)^2 )</td>
</tr>
</tbody>
</table>

### Example 2

Sometimes removing a common factor reveals one of the special patterns:

\[ 8x^2 - 50y^2 \Rightarrow 2(4x^2 - 25y^2) \Rightarrow 2(2x + 5y)(2x - 5y) \]

\[ 12x^2 + 12x + 3 \Rightarrow 3(4x^2 + 4x + 1) \Rightarrow 3(2x + 1)^2 \]
Problems

Factor each difference of squares.
1. \( x^2 - 16 \)
2. \( x^2 - 25 \)
3. \( 64m^2 - 25 \)
4. \( 4p^2 - 9q^2 \)
5. \( 9x^2y^2 - 49 \)
6. \( x^4 - 25 \)
7. \( 64 - y^2 \)
8. \( 144 - 25p^2 \)
9. \( 9x^4 - 4y^2 \)

Factor each perfect square trinomial.
10. \( x^2 + 4x + 4 \)
11. \( y^2 + 8y + 16 \)
12. \( m^2 - 10m + 25 \)
13. \( x^2 - 8x + 16 \)
14. \( a^2 + 8ab + 16b^2 \)
15. \( 36x^2 + 12x + 1 \)
16. \( 25x^2 - 30xy + 9y^2 \)
17. \( 9x^2y^2 - 6xy + 1 \)
18. \( 49x^2 + 1 + 14x \)

Factor completely.
19. \( 9x^2 - 16 \)
20. \( 9x^2 + 24x + 16 \)
21. \( 9x^2 - 36 \)
22. \( 2x^2 + 8xy + 8y^2 \)
23. \( x^2y + 10xy + 25y \)
24. \( 8x^2 - 72 \)
25. \( 4x^3 - 9x \)
26. \( 4x^2 - 8x + 4 \)
27. \( 2x^2 + 8 \)

Answers
1. \( (x + 4)(x - 4) \)
2. \( (x + 5)(x - 5) \)
3. \( (8m + 5)(8m - 5) \)
4. \( (2p + 3q)(2p - 3q) \)
5. \( (3xy + 7)(3xy - 7) \)
6. \( (x^2 + 5)(x^2 - 5) \)
7. \( (8 + y)(8 - y) \)
8. \( (12 + 5p)(12 - 5p) \)
9. \( (3x^2 + 2y)(3x^2 - 2y) \)
10. \( (x + 2)^2 \)
11. \( (y + 4)^2 \)
12. \( (m - 5)^2 \)
13. \( (x - 4)^2 \)
14. \( (a + 4b)^2 \)
15. \( (6x + 1)^2 \)
16. \( (5x - 3)^2 \)
17. \( (3xy - 1)^2 \)
18. \( (7x + 1)^2 \)
19. \( (3x + 4)(3x - 4) \)
20. \( (3x + 4)^2 \)
21. \( 9(x + 2)(x - 2) \)
22. \( 2(x + 2y)^2 \)
23. \( y(x + 5)^2 \)
24. \( 8(x + 3)(x - 3) \)
25. \( x(2x + 3)(2x - 3) \)
26. \( 4(x - 1)^2 \)
27. \( 2(x^2 + 4) \)
The graph of a quadratic function, a parabola, is a symmetrical curve. Its highest or lowest point is called the vertex. The graph is created by using the equation \( y = ax^2 + bx + c \). Students have been graphing parabolas by substituting values for \( x \) and solving for \( y \). This can be a tedious process, especially if an appropriate range of \( x \)-values is not known. If only a quick sketch of the parabola is needed, one possible method is to find the \( x \)-intercepts first, then find the vertex and/or the \( y \)-intercept. To find the \( x \)-intercepts, substitute 0 for \( y \) and solve the quadratic equation, \( 0 = ax^2 + bx + c \). Students will learn multiple methods to solve quadratic equations in this chapter and in Chapter 9.

One method to solve quadratic equations uses the Zero Product Property, that is, solving by factoring. This method uses two ideas:

1. When the product of two or more numbers is zero, then one of the numbers must be zero.
2. Some quadratic expressions can be factored into the product of two binomials.

For additional information see the Math Notes box in Lesson 8.2.2.

**Example 1**

Find the \( x \)-intercepts of \( y = x^2 + 6x + 8 \).

The \( x \)-intercepts are located on the graph where \( y = 0 \), so write the quadratic expression equal to zero, then solve for \( x \).

\[
x^2 + 6x + 8 = 0
\]

Factor the quadratic expression.

\[
(x + 4)(x + 2) = 0
\]

Set each factor equal to 0.

\[
(x + 4) = 0 \text{ or } (x + 2) = 0
\]

Solve each equation for \( x \).

\[
x = -4 \text{ or } x = -2
\]

The \( x \)-intercepts are \((-4, 0)\) and \((-2, 0)\).

You can check your answers by substituting them into the original equation.

\[
(-4)^2 + 6(-4) + 8 \Rightarrow 16 - 24 + 8 \Rightarrow 0
\]

\[
(-2)^2 + 6(-2) + 8 \Rightarrow 4 - 12 + 8 \Rightarrow 0
\]
Example 2

Solve $2x^2 + 7x - 15 = 0$.

Factor the quadratic expression. $(2x - 3)(x + 5) = 0$

Set each factor equal to 0. $(2x - 3) = 0$ or $(x + 5) = 0$

Solve for each $x$. $2x = 3$ or $x = -5$

$x = \frac{3}{2}$ or $x = -5$

Example 3

If the quadratic equation does not equal 0, rewrite it algebraically so that it does, then use the Zero Product Property.

Solve $2 = 6x^2 - x$.

Set the equation equal to 0. $2 = 6x^2 - x$

$0 = 6x^2 - x - 2$

Factor the quadratic expression. $0 = (2x + 1)(3x - 2)$

Solve each equation for $x$. $(2x + 1) = 0$ or $(3x - 2) = 0$

$2x = -1$ or $3x = 2$

$x = -\frac{1}{2}$ or $x = \frac{2}{3}$

Example 4

Solve $9x^2 - 6x + 1 = 0$.

Factor the quadratic expression. $9x^2 - 6x + 1 = 0$

$(3x - 1)(3x - 1) = 0$

Solve each equation for $x$. Notice the factors are the same so there will be only one solution. $(3x - 1) = 0$

$3x = 1$

$x = \frac{1}{3}$
Problems

Solve for $x$.

1. $x^2 - x - 12 = 0$  
   2. $3x^2 - 7x - 6 = 0$  
   3. $x^2 + x - 20 = 0$

4. $3x^2 + 11x + 10 = 0$  
   5. $x^2 + 5x = -4$  
   6. $6x - 9 = x^2$

7. $6x^2 + 5x - 4 = 0$  
   8. $x^2 - 6x + 8 = 0$  
   9. $6x^2 - x - 15 = 0$

10. $4x^2 + 12x + 9 = 0$  
    11. $x^2 - 12x = 28$  
    12. $2x^2 + 8x + 6 = 0$

13. $2 + 9x = 5x^2$  
    14. $2x^2 - 5x = 3$  
    15. $x^2 = 45 - 4x$

Answers

1. $x = 4$ or $-3$  
   2. $x = -\frac{2}{3}$ or $3$  
   3. $x = 5$ or $4$

4. $x = -\frac{5}{3}$ or $-2$  
   5. $x = -4$ or $-1$  
   6. $x = 3$

7. $x = -\frac{4}{3}$ or $\frac{1}{2}$  
   8. $x = 4$ or $2$  
   9. $x = -\frac{3}{2}$ or $\frac{5}{3}$

10. $x = -\frac{3}{2}$  
    11. $x = 14$ or $-2$  
    12. $x = -1$ or $-3$

13. $x = -\frac{1}{3}$ or $2$  
    14. $x = -\frac{1}{2}$ or $3$  
    15. $x = 5$ or $-9$
In Lesson 8.2.3, students found that if the equation of a parabola is written in **graphing form**: \( f(x) = (x - h)^2 + k \) then the vertex can easily be seen as \((h, k)\). For example, for the parabola \( f(x) = (x + 3)^2 - 1 \) the vertex is \((-3, -1)\). Students can then set the function equal to zero to find the \(x\)-intercepts: solve \(0 = (x + 3)^2 - 1\) to find the \(x\)-intercepts. For help in solving this type of equation, see the Lesson 8.2.3 Resource Page, available at cpm.org. Students can set \(x = 0\) to find the \(y\)-intercepts: \(y = (0 + 3)^2 - 1\).

When the equation of the parabola is given in standard form: \( f(x) = x^2 + bx + c \), then using the process of **completing the square** can be used to convert standard form into graphing form. Algebra tiles are used to help visualize the process.

For additional examples and practice with graphing quadratic functions, see the Checkpoint 11 materials at the back of the student textbook.

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**Example 1  (Using algebra tiles)**

Complete the square to change \( f(x) = x^2 + 8x + 10 \) into graphing form, identify the vertex and \(y\)-intercept, and draw a graph.

\[ f(x) = x^2 + 8x + 10 \]

would look like this:

![Algebra tiles](image)

Arrange the tiles as shown in the picture at right to make a square.

16 small unit tiles are needed to fill in the corner, but only ten unit tiles are available. Show the 10 small square tiles and draw the outline of the whole square.

![Diagram](image)

The **complete square** would have length and width both equal to \((x + 4)\), so the complete square can be represented by the quadratic expression \((x + 4)^2\). But the tiles from \(x^2 + 8x + 10\) do not form a complete square—the expression \(x^2 + 8x + 10\) has six fewer tiles than a complete square. So \(x^2 + 8x + 10\) is a complete square minus 6, or \((x + 4)^2\) minus 6. That is,

\[ x^2 + 8x + 10 = (x + 4)^2 - 6. \]

From graphing form, the vertex is at \((h, k) = (-4, -6)\). The \(y\)-intercept is where \(x = 0\), so \(y = (0 + 4)^2 - 6 = 10\). The \(y\)-intercept is \((0, 10)\), and the graph is shown at right.
Example 2 (Using the general process)

Complete the square to change \( f(x) = x^2 + 5x + 2 \) into graphing form. Identify the vertex and \( y \)-intercept, and draw a graph.

Rewrite the expression as: \( f(x) = x^2 + 5x + 2 \)

\[
f(x) = (x^2 + 5x + ?) + 2
\]

Make \((x^2 + 5x + ?)\) into a perfect square by taking half of the \(x\)-term coefficient and squaring it: \((\frac{5}{2})^2 = \frac{25}{4}\). Then \((x^2 + 5x + \frac{25}{4})\) is a perfect square trinomial.

We need to write an equivalent function with \(+ \frac{25}{4}\), but if we add \(\frac{25}{4}\), we also need to subtract it:

\[
f(x) = x^2 + 5x + 2 \quad \text{which can be rewritten as: } \quad f(x) = (x + \frac{5}{2})^2 - \frac{17}{4}.
\]

The function is now in graphing form.

The vertex is \((-\frac{5}{2}, -\frac{17}{4})\) or \((-2.5, -4.25)\).

The \(y\)-intercept is where \(x = 0\).

Thus, \(y = (0 + 2.5)^2 - 4.25 = 2\) and the \(y\)-intercept is \((0, 2)\).

Alternatively, if the process above is unclear, draw a generic rectangle of \(x^2 + 5x + 2\) and imagine algebra tiles.

\[
\begin{array}{c|c}
2.5x & \\
\hline
x^2 & 2.5x
\end{array}
\]

There should be \((-2.5)^2 = 6.25\) tiles in the upper right corner to complete the square. But the expression \(x^2 + 5x + 2\) only provides 2 unit tiles. So there are 4.25 unit tiles missing. Thus, 4.25 is missing from the rectangle below:

\[
\begin{array}{c|c|c|c|c|c}
+2.5 & 2.5x & \\
\hline
x & x^2 & 2.5x & \\
\hline
& x & +2.5 &
\end{array}
\]

That is, \((x + 2.5)^2\) minus 4.25 unit tiles is the equivalent of \(x^2 + 5x + 2\), or,

\[x^2 + 5x + 2 = (x + 2.5)^2 - 4.25.\]
Problems

Complete the square to write each equation in graphing form. Then state the vertex.

1. \( f(x) = x^2 + 6x + 7 \)
2. \( f(x) = x^2 + 4x + 11 \)
3. \( f(x) = x^2 + 10x \)
4. \( f(x) = x^2 + 7x + 2 \)
5. \( f(x) = x^2 - 6x + 9 \)
6. \( f(x) = x^2 + 3 \)
7. \( f(x) = x^2 - 4x \)
8. \( f(x) = x^2 + 2x - 3 \)
9. \( f(x) = x^2 + 5x + 1 \)
10. \( f(x) = x^2 - \frac{1}{3}x \)

Answers

1. \( f(x) = (x + 3)^2 - 2; \ (-3, -2) \)
2. \( f(x) = (x + 2)^2 + 7; \ (-2, 7) \)
3. \( f(x) = (x + 5)^2 - 25; \ (-5, -25) \)
4. \( f(x) = (x + 3.5)^2 - 10.25; \ (-3.5, -10.25) \)
5. \( f(x) = (x - 3)^2; \ (3, 0) \)
6. \( f(x) = x^2 + 3; \ (0, 3) \)
7. \( f(x) = (x - 2)^2 - 4; \ (2, -4) \)
8. \( f(x) = (x + 1)^2 - 4; \ (-1, -4) \)
9. \( f(x) = (x + \frac{5}{2})^2 - \frac{21}{4}; \ (-\frac{5}{2}, -\frac{21}{4}) \)
10. \( f(x) = (x - \frac{1}{6})^2 - \frac{1}{36}; \ (\frac{1}{6}, -\frac{1}{36}) \)