When a quadratic equation is not factorable, another method is needed to solve for $x$. The Quadratic Formula can be used to calculate the roots of a quadratic function, that is, the $x$-intercepts of the parabola. The Quadratic Formula can be used with any quadratic equation, factorable or not. There may be two, one, or no solutions, depending on whether the parabola intersects the $x$-axis twice, once, or not at all.

The solution(s) to any quadratic equation $ax^2 + bx + c = 0$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$  

The ± symbol is read as “plus or minus.” It is shorthand notation that tells you to calculate the formula twice, once with + and again with − to get both $x$-values.

To use the formula, the quadratic equation must be written in **standard form**: $ax^2 + bx + c = 0$. This is necessary to correctly identify the values of $a$, $b$, and $c$. Once the equation is in standard form and equal to 0, $a$ is the coefficient of the $x^2$-term, $b$ is the coefficient of the $x$-term and $c$ is the constant term.

For additional information, see the Math Notes boxes in Lessons 9.1.1 through 9.1.4 and 10.2.4.

**Example 1**

Solve $2x^2 - 5x - 3 = 0$.

Identify $a$, $b$, and $c$. Watch your signs carefully. $a = 2, \ b = -5, \ c = -3$

Write the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute $a$, $b$, and $c$ into the formula and do the initial calculations.

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{5 \pm \sqrt{25 - (-24)}}{4}$$

Simplify the $\sqrt{}$.

$$x = \frac{5 \pm \sqrt{49}}{4}$$

Calculate both values of $x$.

$$x = \frac{5 + 7}{4} = \frac{12}{4} = 3 \quad \text{or} \quad x = \frac{5 - 7}{4} = \frac{-2}{4} = -\frac{1}{2}$$

The solutions are $x = 3$ or $x = -\frac{1}{2}$.
Example 2

Solve $3x^2 + 5x + 1 = 0$.

Identify $a$, $b$, and $c$. 

\[ a = 3, \ b = 5, \ c = 1 \]

Write the Quadratic Formula.

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Substitute $a$, $b$, and $c$ into the formula and do the initial calculations.

\[ x = \frac{-5 \pm \sqrt{25 - 12}}{6} \]

Simplify the $\sqrt{}$.

\[ x = \frac{-5 \pm \sqrt{13}}{6} \]

The solutions are $x = \frac{-5 + \sqrt{13}}{6} \approx -0.23$ or $x = \frac{-5 - \sqrt{13}}{6} \approx -1.43$.

Example 3

Solve $25x^2 - 20x + 4 = 0$.

Identify $a$, $b$, and $c$. 

\[ a = 25, \ b = -20, \ c = 4 \]

Write the Quadratic Formula.

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Substitute $a$, $b$, and $c$ into the formula and do the initial calculations.

\[ x = \frac{20 \pm \sqrt{400 - 400}}{50} \]

Simplify the $\sqrt{}$.

\[ x = \frac{20 \pm \sqrt{0}}{50} \]

This quadratic equation has only one solution: $x = \frac{2}{5}$. 

Example 4

Solve \( x^2 + 4x + 10 = 0 \).

Identify \( a, b, \) and \( c \).
\[ a = 1, \quad b = 4, \quad c = 10 \]

Write the Quadratic Formula.
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Substitute \( a, b, \) and \( c \) into the formula and do the initial calculations.
\[ x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(10)}}{2(1)} \]
\[ x = \frac{-4 \pm \sqrt{16 - 40}}{2} \]

Simplify the \( \sqrt{} \).
\[ x = \frac{-4 \pm \sqrt{-24}}{2} \]

It is impossible to take the square root of a negative number; therefore this quadratic equation has no real solutions.

Example 5

Solve \((3x + 1)(x + 2) = 1\).

Rewrite the equation in standard form.
\( (3x + 1)(x + 2) = 1 \)
That is, rewrite the product as a sum and then set the equation equal to zero.
\[ 3x^2 + 7x + 2 = 1 \]
\[ 3x^2 + 7x + 1 = 0 \]

Identify \( a, b, \) and \( c \).
\[ a = 3, \quad b = 7, \quad c = 1 \]

Write the Quadratic Formula.
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Substitute \( a, b, \) and \( c \) into the formula and do the initial calculations.
\[ x = \frac{-7 \pm \sqrt{(7)^2 - 4(3)(1)}}{2(3)} \]
\[ x = \frac{-7 \pm \sqrt{49 - 12}}{6} \]
\[ x = \frac{-7 \pm \sqrt{37}}{6} \]

Simplify.
\[ x = \frac{-7 \pm \sqrt{37}}{6} \]

The solutions are \( x = \frac{-7 \pm \sqrt{37}}{6} \), or \( x \approx -0.15 \) or \( x \approx -2.18 \).
Example 6

Solve \(3x^2 + 6x + 1 = 0\).

Identify \(a, b,\) and \(c\).  
\[a = 3, \ b = 6, \ c = 1\]

Write the Quadratic Formula.  
\[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\]

Substitute \(a, b,\) and \(c\) into the formula and do the initial calculations.

\[
x = \frac{-6 \pm \sqrt{(6)^2 - 4(3)(1)}}{2(3)}
\]

\[
x = \frac{-6 \pm \sqrt{36 - 12}}{6}
\]

Simplify.

\[
x = \frac{-6 \pm \sqrt{24}}{6}
\]

The solutions are \(x = \frac{-6 \pm \sqrt{24}}{6}\), or, \(x \approx -1.82\) or \(x \approx -0.18\).

The Math Notes box in Lesson 9.1.4 describes another form of the expression \(\frac{-6 \pm \sqrt{24}}{6}\) that can be written by simplifying the square root. The result is equivalent to the exact values above.

Factor the \(\sqrt{24}\), then simplify by taking the square root of 4.  
\[\sqrt{24} = \sqrt{4}\sqrt{6} = 2\sqrt{6}\]

Simplify the fraction by dividing every term by 2.  
\[
x = \frac{-6 \pm 2\sqrt{6}}{6}
\]

\[
x = \frac{-3 \pm \sqrt{6}}{3}
\]
Problems

Solve each equation by using the Quadratic Formula.

1. \( x^2 - x - 2 = 0 \)  
2. \( x^2 - x - 3 = 0 \)  
3. \( -3x^2 + 2x + 1 = 0 \)  
4. \( -2 - 2x^2 = 4x \)  
5. \( 7x = 10 - 2x^2 \)  
6. \( -6x^2 - x + 6 = 0 \)  
7. \( 6 - 4x + 3x^2 = 8 \)  
8. \( 4x^2 + x - 1 = 0 \)  
9. \( x^2 - 5x + 3 = 0 \)  
10. \( 0 = 10x^2 - 2x + 3 \)  
11. \( x(-3x + 5) = 7x - 10 \)  
12. \( (5x + 5)(x - 5) = 7x \)

Answers

1. \( x = 2 \) or \(-1\)  
2. \( x = \frac{1 \pm \sqrt{13}}{2} \)  
    \( \approx 2.30 \) or \(-1.30\)  
3. \( x = -\frac{1}{3} \) or \(1\)  
4. \( x = -1\)  
5. \( x = \frac{-7 \pm \sqrt{129}}{4} \)  
    \( \approx 1.09 \) or \(-4.59\)  
6. \( x = \frac{1 \pm \sqrt{145}}{-12} \)  
    \( \approx -1.09 \) or \(0.92\)  
7. \( x = \frac{4 \pm \sqrt{40}}{6} = \frac{2 \pm \sqrt{10}}{3} \)  
    \( \approx 1.72 \) or \(-0.39\)  
8. \( x = \frac{-1 \pm \sqrt{17}}{8} \)  
    \( \approx 0.39 \) or \(-0.64\)  
9. \( x = \frac{5 \pm \sqrt{13}}{2} \)  
    \( \approx 4.30 \) or \(0.70\)  
10. no solution  
11. \( x = \frac{2 \pm \sqrt{124}}{-6} = \frac{1 \pm \sqrt{31}}{-3} \)  
    \( \approx -2.19 \) or \(1.52\)  
12. \( x = \frac{27 \pm \sqrt{1229}}{10} \)  
    \( \approx 6.21 \) or \(-0.81\)
SOLVING ONE-VARIABLE INEQUALITIES

To solve an inequality in one variable, first change it to an equation and solve. Place the solution, called a “boundary point,” on a number line. This point separates the number line into two regions. The boundary point is included in the solution for situations that involve ≥ or ≤, and excluded from situations that involve strictly > or <. On the number line graph, boundary points that are included in the solutions are shown with a solid filled-in circle, and excluded solutions are shown with an open circle. Next, choose a number from within each region separated by the boundary point, and check if the number is true or false in the original inequality. If it is true, then every number in that region is a solution to the inequality. If it is false, then no number in that region is a solution to the inequality.

For additional information, see the Math Notes boxes in Lessons 9.2.1 and 9.3.2.

Example 1

Solve: $3x - (x + 2) \geq 0$

change to an equation and solve.

place the solution (boundary point) on the number line. because $x = 1$ is also a solution to the inequality (≥), we use a solid dot.

test a number from each side of the boundary point in the original inequality.

the solution is $x \geq 1$.

Example 2

Solve: $-x + 6 > x + 2$

change to an equation and solve.

place the solution (boundary point) on the number line. because the original problem is a strict inequality (>), $x = 2$ is not a solution, so we use an open dot.

test a number from each side of the boundary point in the original inequality.

the solution is $x < 2$.
Problems

Solve each inequality.

1. \(4x - 1 \geq 7\)  
2. \(2(x - 5) \leq 8\)  
3. \(3 - 2x < x + 6\)  
4. \(\frac{1}{2}x > 5\)  
5. \(3(x + 4) > 12\)  
6. \(2x - 7 \leq 5 - 4x\)  
7. \(3x + 2 < 11\)  
8. \(4(x - 6) \geq 20\)  
9. \(\frac{1}{4}x < 2\)  
10. \(12 - 3x > 2x + 1\)  
11. \(\frac{x - 5}{7} \leq -3\)  
12. \(3(5 - x) \geq 7x - 1\)  
13. \(3y - (2y + 2) \leq 7\)  
14. \(\frac{m + 2}{5} < \frac{2m}{3}\)  
15. \(\frac{m - 2}{3} \geq \frac{2m + 1}{7}\)

Answers

1. \(x \geq 2\)  
2. \(x \leq 9\)  
3. \(x > -1\)  
4. \(x > 10\)  
5. \(x > 0\)  
6. \(x \leq 2\)  
7. \(x < 3\)  
8. \(x \geq 11\)  
9. \(x < 8\)  
10. \(x < \frac{11}{5}\)  
11. \(x \leq -16\)  
12. \(x \leq 1.6\)  
13. \(y \leq 9\)  
14. \(m > \frac{6}{7}\)  
15. \(m \geq 17\)
To graph the solutions to an inequality in two variables, first graph the corresponding equation. This graph is the boundary line (or curve), since all points that make the inequality true lie on one side or the other of the line. Before you graph the equation, decide whether the line or curve is part of the solution or not, that is, whether it is solid or dashed. If the inequality symbol is either $\leq$ or $\geq$, then the boundary line is part of the inequality and it must be solid. If the inequality symbol is either $<$ or $>$, then the boundary line is dashed.

Next, decide which side of the boundary line must be shaded to show the part of the graph that represents all values that make the inequality true. Choose a point not on the boundary line. Substitute this point into the original inequality. If the inequality is true for the test point, then shade the graph on this side of the boundary line. If the inequality is false for the test point, then shade the opposite side of the line.

The shaded portion represents all the solutions to the original inequality.

Caution: If you need to rearrange the inequality in order to graph it, such as putting it in slope-intercept form, always use the original inequality to test a point, not the rearranged form.

For additional information, see the Math Notes box in Lesson 9.4.1.

Example 1

Graph the solutions to the inequality $y > 3x - 2$.

First, graph the line $y = 3x - 2$, but draw it dashed since $>$ means the boundary line is not part of the solution. For example, the point $(0, -2)$ is on the boundary line, but it is not a solution to the inequality because $-2 \not> 3(0) - 2$ or $-2 \not> -2$.

Next, test a point that is not on the boundary line. For this example, use the point $(-2, 4)$.

$4 > 3(-2) - 2$, so $4 > -8$ which is a true statement.

Since the inequality is true for this test point, shade the region containing the point $(-2, 4)$. All of the coordinate pairs that are solutions lie in the shaded region.
Example 2

Graph the solutions to the inequality $y \leq x^2 - 2x - 8$.

First, graph the parabola $y = x^2 - 2x - 8$ and draw it solid, since $\le$ means the boundary curve is part of the solution.

Next, test the point $(2, 2)$ above the boundary curve.

$$2 \leq 2^2 - 2 \cdot 2 - 8$$

Since the inequality is false for this test point above the curve, shade below the boundary curve.

The solutions are the shaded region.

Problems

Graph the solutions to each of the following inequalities on a separate set of axes.

1. $y \leq 3x + 1$
2. $y \geq -2x + 3$
3. $y > 4x - 2$
4. $y < -3x - 5$
5. $y \leq 3$
6. $x > 1$
7. $y > \frac{2}{3}x + 8$
8. $y < -\frac{3}{5}x - 7$
9. $3x + 2y \geq 7$
10. $-4x + 2y < 3$
11. $y \geq x^2 - 3$
12. $y \leq x^2 + 2x$
13. $y < 4 - x^2$
14. $y \leq |x + 2|$
15. $y \geq -|x| + 3$

Answers

1.
2.
3.
To **graph the solutions to a system of inequalities**, follow the same procedure outlined in the previous section but do it twice—once for each inequality. The solution to the system of inequalities is the **overlap** of the shading from the individual inequalities.

**Example 1**

Graph the solutions to the system \( y \leq \frac{1}{2}x + 2 \) and \( y > -\frac{2}{3}x + 1 \).

Graph the lines \( y = \frac{1}{2}x + 2 \) and \( y = -\frac{2}{3}x + 1 \). The first is solid and the second is dashed. Test the point \((-4, 5)\) in the first inequality.

\[
5 \leq \frac{1}{2}(-4) + 2, \text{ so } 5 \leq 0
\]

This inequality is false, so shade on the opposite side of the boundary line from \((-4, 5)\), that is, below the line.

\[
5 > -\frac{2}{3}(-4) + 1, \text{ so } 5 > \frac{11}{3}
\]

Test the same point in the second inequality. This inequality is true, so shade on the same side of the boundary line as \((-4, 5)\), that is, above the line.

The solutions are represented by the overlap of the two shaded regions shown by the darkest shading in the second graph above right.
Example 2
Graph the solutions to the system \( y \leq -x + 5 \) and \( y \geq x^2 - 1 \).

Graph the line \( y = -x + 5 \) and the parabola \( y = x^2 - 1 \) with a solid line and curve.

Test the point \((-1, 2)\) in the first inequality. This inequality is true, so shade on the same side of the boundary line as \((-1, 2)\), that is, below the line.

Test the same point in the second inequality. This inequality is also true, so shade on the same side of the boundary curve as \((-1, 2)\), that is, inside the curve.

The solutions are in the overlap of the two shaded regions shown by the darkest shading in the second graph above right.

Problems
Graph the solutions to each of the following pairs of inequalities on the same set of axes.

1. \( y > 3x - 4 \) and \( y \leq -2x + 5 \)
2. \( y \geq -3x - 6 \) and \( y > 4x - 4 \)
3. \( y < -\frac{3}{2}x + 4 \) and \( y < \frac{1}{3}x + 3 \)
4. \( y < -\frac{3}{4}x - 1 \) and \( y > \frac{4}{3}x + 1 \)
5. \( y < 3 \) and \( y > \frac{1}{2}x + 2 \)
6. \( x \leq 3 \) and \( y < \frac{3}{4}x - 4 \)
7. \( y \leq 2x + 1 \) and \( y \geq x^2 - 4 \)
8. \( y < -x + 5 \) and \( y \geq x^2 + 1 \)
9. \( y < -x + 6 \) and \( y \geq |x - 2| \)
10. \( y < -x^2 + 5 \) and \( y \geq |x| - 1 \)
Answers

1.

2.

3.

4.

5.

6.

7.

8.
9.

10.