Students began their study of trigonometric functions in Chapter 7 when they learned about radians and transforming trig functions. Here they will learn how to solve equations with trig functions by operating on the variable. This work will review the inverse trig functions as well as introduce the reciprocal trigonometric functions. See the Math Notes box in Lesson 12.1.4 for more information.

**Example 1**

For what values are the following equations true?

a. \( \cos(\theta) = \frac{\sqrt{3}}{2} \)

b. \( 2 \sin(\theta) = \sqrt{2} \)

c. \( \cos(\theta) = 5 \)

Although the first inclination might be to use the inverse function to solve the equation in part (a), that will not completely answer the question.

\[
\cos(\theta) = \frac{\sqrt{3}}{2}
\]

\[
\cos^{-1}(\cos(\theta)) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)
\]

\( \theta = \frac{\pi}{6} \) radians

You might recall that the graph of \( f(x) = \cos(x) \) is a periodic function, and the graph of \( y = \frac{\sqrt{3}}{2} \) is a horizontal line. By graphing both equations on the same set of axes, we see that they intersect infinitely many times. Solving by using the inverse trig function gives us one solution. How do we find all solutions?

It helps to remember the unit circle. For what values of \( \theta \) does \( \cos(\theta) = \frac{\sqrt{3}}{2} \)? Two points are easily found: \( \frac{\pi}{6} \) and \( -\frac{\pi}{6} \).

How do we find the rest? On the unit circle, we can revisit \( \frac{\pi}{6} \) and \( -\frac{\pi}{6} \) at each rotation of \( 2\pi \). Therefore, not only does \( \frac{\pi}{6} \) make the equation true, but so do \( \frac{\pi}{6} \pm 2\pi, \frac{\pi}{6} \pm 4\pi, \frac{\pi}{6} \pm 6\pi \), etc. Similarly, \( -\frac{\pi}{6} \pm 2\pi, -\frac{\pi}{6} \pm 4\pi, -\frac{\pi}{6} \pm 6\pi, \ldots \) will also make the equation true. We can summarize this information as \( \theta = \pm \frac{\pi}{6} \pm 2\pi n \), for all integers \( n \). Note: There are other ways to write this solution that are equivalent to this expression.
Using the same method for part (b), we find:

In a unit circle, sine is also positive in the second quadrant, so the other solution is \( \theta = \frac{3\pi}{4} \pm 2\pi n \).

Some students might see the solution to part (c) quickly. Since the range of \( f(x) = \cos(x) \) is \(-1 \leq y \leq 1\), this equation has no solution.

**Example 2**

Graph \( f(x) = \cos^{-1}(x) \) and \( g(x) = \tan^{-1}(x) \) on separate axes. How can you restrict the domain so that these are functions?

Because the graphs of \( y = \cos(x) \) and \( y = \tan(x) \) are periodic (repeating), if we used our carbon paper method of graphing the inverses, they too will be periodic. The problem is that they will not be functions, because they will fail the vertical line test.

\[
\begin{align*}
2 \sin \theta &= \sqrt{2} \\
\sin \theta &= \frac{\sqrt{2}}{2} \\
\theta &= \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) \\
\theta &= \frac{\pi}{4} \pm 2\pi n
\end{align*}
\]

By restricting the domain of the original function, we create an inverse that also has a restricted domain and is therefore a function as well. There are infinitely many restrictions we can make; however, the convention is to restrict the domain of \( y = \cos(x) \) to \( 0 \leq x \leq \pi \), and to restrict the domain of \( y = \tan(x) \) to \( -\frac{\pi}{2} < x < \frac{\pi}{2} \). These adjustments produce the necessary restrictions on the inverses so that they are also functions.
Example 3

Let \( f(x) = \sin(x) \), \( g(x) = \cos(x) \), and \( h(x) = \tan(x) \). Graph each of the following equations on separate axes.

\[
\begin{align*}
y &= \frac{1}{f(x)} \\
y &= \frac{1}{g(x)} \\
y &= \frac{1}{h(x)}
\end{align*}
\]

Compare your graphs to the graphs of these equations:

\[
\begin{align*}
y &= \sin^{-1}(x) \\
y &= \cos^{-1}(x) \\
y &= \tan^{-1}(x)
\end{align*}
\]

The first three functions are the reciprocal functions of sine, cosine and tangent. However, rather than writing them as reciprocals \( \frac{1}{f(x)} \), they are given new names:

\[
\begin{align*}
\frac{1}{\sin(x)} &= \text{cosecant } x \\
\frac{1}{\cos(x)} &= \text{secant } x \\
\frac{1}{\tan(x)} &= \text{cotangent } x
\end{align*}
\]

The abbreviation for cosecant is csc, for secant it is sec, and for cotangent it is cot. Their graphs are:

Since these are reciprocal functions, everywhere the first functions were zero, the corresponding reciprocal functions will be undefined. Check that this is the case.

In comparing these functions to the inverse trig functions, it is important to note that \( \frac{1}{\sin(x)} \neq \sin^{-1}(x) \) (and similarly for the other corresponding functions). This is very clear by examining the graphs. The exponent of “−1” tells us the function is the inverse, not the reciprocal.
Problems

For each of the following problems, find all the solutions. You may use your calculator but remember that your calculator only gives one answer.

1. \[2 \cos(x) = \sqrt{2}\]
2. \[5 \tan(x) - 5 = 0\]
3. \[4 \cos^2(x) - 1 = 0\]
4. \[4 \sin^2(x) = 3\]
5. \[\sin(x) + 2 = 3 \sin(x)\]
6. \[\tan^2(x) + \tan(x) = 0\]

Graph each of the following equations on a separate set of axes. Label all the important points.

7. \[y = 3 \csc(x)\]
8. \[y = 4 + \sec(x)\]
9. \[y = \cot(x - \pi)\]

Answers

1. \[x = \pm \frac{\pi}{4} \pm 2\pi n\] for all integers \(n\)
2. \[x = \frac{\pi}{4} \pm \pi n\] for all integers \(n\)
3. \[x = \pm \frac{\pi}{3} n\] for all integers \(n\)
4. \[x = \frac{\pi}{3} \pm \pi n\] or \[x = \frac{2\pi}{3} \pm \pi n\] for all integers \(n\)
5. \[x = \frac{\pi}{2} \pm 2\pi n\] for all integers \(n\)
6. \[x = \pm \pi n\] for all integers \(n\), \[x = \frac{3\pi}{4} \pm \pi n\] for all integers \(n\)

7. [Graph 1]
8. [Graph 2]
9. [Graph 3]