By graphing different trigonometric expressions, students realize that some expressions are equivalent to others. These equivalent expressions are known as trig identities. These identities allow students to rewrite and solve many more trigonometric equations. See the Math Notes box in Lesson 12.2.3 for more information.

Example 1

Graph the function \( f(x) = \frac{1}{\cos^2(x)} - \tan^2(x) \). Based on the graph, what can you conclude about this expression? (That is, what trig identity can you write?) What substitution can you make in the identity so that you no longer have a fraction?

Before the widespread availability of calculators, students used tables to look up the trig values of various angle measures. Since tables were cumbersome, students would memorize hundreds of trigonometric identities so that they could quickly rewrite trig expressions. Knowing that \( \sin(2\theta) = 2\sin(\theta)\cos(\theta) \), for instance, meant that a trig table of values did not need to extend to 120º angles. A student could rewrite \( \sin(120^\circ) \) as \( 2\sin(60^\circ)\cos(60^\circ) \), and use the values for 60º. This also allowed students to rewrite trig expressions in equations, making the equations simpler and easier to solve.

By graphing the given function above, we can readily see that the function is a constant, that is, a horizontal line. This graph is equivalent to the graph of \( y = 1 \). Because their graphs are equivalent for all values of \( x \), the expressions are also equivalent. Therefore we can write:

\[
\frac{1}{\cos^2(x)} - \tan^2(x) = 1
\]

This is now a trig identity. How can we rewrite this so it no longer has a fraction? Since \( \frac{1}{\cos(x)} = \sec(x) \), we can write:

\[
\sec^2(x) - \tan^2(x) = 1.
\]

This trig identity is more commonly written as:

\[
1 + \tan^2(x) = \sec^2(x).
\]
Example 2

Prove the following trig identity:

\[
\frac{\sin(x)}{1-\cos(x)} + \frac{1-\cos(x)}{\sin(x)} = 2 \csc(x)
\]

Another practice common before the advent of calculators was the proving of trig identities. These proofs usually employ algebraic steps and previously proven identities to show that one side of the equation equals the other. For the identity above, we will start with the left side of the equation, get common denominators so we can add the fraction, and see where it takes us. Often with these proofs, you have to try different things to see where they take you. It is also important to be aware of the right hand side of the equation, which is our goal. Remember that

\[2 \csc(x) = \frac{2}{\sin(x)}\].

\[
\frac{\sin(x)}{1-\cos(x)} + \frac{1-\cos(x)}{\sin(x)} = 2 \csc(x)
\]

\[
= \frac{\sin^2(x)}{(\sin(x))(1-\cos(x))} + \frac{(1-\cos(x))^2}{(\sin(x))(1-\cos(x))}
\]

\[
= \frac{\sin^2(x) + (1-\cos(x))^2}{(\sin(x))(1-\cos(x))}
\]

\[
= \frac{\sin^2(x) + 1 - 2\cos(x) + \cos^2(x)}{(\sin(x))(1-\cos(x))}
\]

\[
= \frac{\sin^2(x) + \cos^2(x) + 1 - 2\cos(x)}{(\sin(x))(1-\cos(x))}
\]

\[
= \frac{1 + 1 - 2\cos(x)}{(\sin(x))(1-\cos(x))}
\]

\[
= \frac{2 - 2\cos(x)}{(\sin(x))(1-\cos(x))}
\]

\[
= \frac{2(1-\cos(x))}{(\sin(x))(1-\cos(x))}
\]

\[
= \frac{2}{\sin(x)}
\]

\[
= 2 \csc(x)
\]

This proves that this identity is true.
Problems

1. Show graphically that \( \sin(x + y) \) does not equal \( \sin(x) + \sin(y) \).

2. Graphically, determine what \( \cos(x + 90^\circ) \) equals.

3. Graphically, determine what \( \sin(180^\circ - x) \) equals.

Prove the following identities.

4. \[ \frac{\sin(2x)}{2 \sin^2(x)} = \cot(x) \]

5. \[ \sin^2(x) - \cos^2(x) = \frac{\tan(x) - \cot(x)}{\tan(x) + \cot(x)} \]

6. \[ \frac{\sin^2(x)}{1 + \cos(x)} = 1 - \frac{1}{\sec(x)} \]

7. \[ \cos^4(x) - \sin^4(x) = 2 \cos^2(x) - 1 \]

8. \[ \frac{1}{1 - \sin(x)} + \frac{1}{1 + \sin(x)} = 2 \sec^2(x) \]

Answers

1. The graphs are not the same.

2. \( \cos(x + 90^\circ) = -\sin(x) \)

3. \( \sin(180^\circ - x) = \sin(x) \)

4. \[ \frac{\sin(2x)}{2 \sin^2(x)} = \cot(x) \]
\[ = \frac{\sin(2x)}{2 \sin^2(x)} = \frac{2 \sin(x) \cos(x)}{2 \sin(x) \sin(x)} \]
\[ = \frac{\cos(x)}{\sin(x)} = \cot(x) \]
5. \( \sin^2(x) - \cos^2(x) = \frac{\tan(x) - \cos(x)}{\tan(x) + \cos(x)} \)

\[ = \frac{\sin(x)}{\cos(x)} - \frac{\cos(x)}{\sin(x)} \]

\[ = \frac{\sin^2(x) - \cos^2(x)}{\sin(x) \cos(x)} \]

\[ = \frac{\sin^2(x) - \cos^2(x)}{\sin^2(x) + \cos^2(x)} \]

\[ = \sin^2(x) - \cos^2(x) \]

\[ = \sin^2(x) - \cos^2(x) \]

6. \( \frac{\sin^2(x)}{1 + \cos(x)} = 1 - \frac{1}{\sec(x)} \)

\[ = (1 - \cos(x)) \left( \frac{1 + \cos(x)}{1 + \cos(x)} \right) \]

\[ = (1 - \cos(x)) \frac{1 + \cos(x)}{1 + \cos(x)} \]

\[ = 1 - \cos(x) \]

\[ = \frac{\sin^2(x)}{1 + \cos(x)} \]

7. \( \cos^4(x) - \sin^4(x) = 2 \cos^2(x) - 1 \)

\( \cos^4(x) - \sin^4(x) = (\cos^2(x) + \sin^2(x))(\cos^2(x) - \sin^2(x)) \)

\[ = 1 \cdot (\cos^2(x) - \sin^2(x)) \]

\[ = \cos^2(x) - (1 - \cos^2(x)) \]

\[ = \cos^2(x) - 1 + \cos^2(x) \]

\[ = 2 \cos^2(x) - 1 \]

8. \( \frac{1}{1 - \sin(x)} + \frac{1}{1 + \sin(x)} = 2 \sec^2(x) \)

\( \left( \frac{1 + \sin(x)}{1 + \sin(x)} \right) \left( \frac{1}{1 - \sin(x)} \right) + \left( \frac{1 - \sin(x)}{1 - \sin(x)} \right) \left( \frac{1}{1 + \sin(x)} \right) = \)

\[ \frac{1 + \sin(x)}{1 + \sin(x)} \cdot \frac{1}{1 - \sin(x)} + \frac{1 - \sin(x)}{1 - \sin(x)} \cdot \frac{1}{1 + \sin(x)} = \]

\[ \frac{1 + \sin(x) + 1 - \sin(x)}{(1 + \sin(x))(1 - \sin(x))} = \]

\[ \frac{2}{1 - \sin^2(x)} = \]

\[ \frac{2}{\cos^2(x)} = 2 \sec^2(x) \]