Students will generalize what they have learned about transforming the graph of $f(x) = x^2$ to change the shape and position of the graphs of several other functions. The students start with the simplest form of each function’s graph, which is called the “parent graph.” Students use $y = x^3$, $y = \frac{1}{x}$, $y = \sqrt{x}$, and $y = b^x$ as the equations for parent graphs, and what they learn while studying these graphs will apply to all functions. They also learn to apply their knowledge to non-functions. For further information see the Math Notes boxes in Lessons 2.2.2 and 2.2.3.

Example 1

For each of the following equations, state the parent equation, and use it to graph each equation as a transformation of its parent equation.

\[
\begin{align*}
y &= (x + 4)^3 - 1 \\
y &= 3\sqrt{x} - 2 \\
y &= -\frac{1}{x} \\
y &= 3^x - 6
\end{align*}
\]

For each of these equations, we will graph both it and its parent on the same set of axes to help display the change and movement.

The first equation is a cubic (the term given to a polynomial with 3 as the highest power of $x$), thus its parent is $y = x^3$. The given equation will have the same shape as $y = x^3$, but it will be shifted to the left 4 units (from the “+ 4” within the parentheses), and down one unit (from the “– 1”). The new graph is the darker curve shown on the graph at right. Notice that the point (0, 0) on the original graph has shifted left 4, and down 1, and now is at (–4, –1). This point is known as a locator point. It is a key point of the graph, and graphing its position helps us to graph the rest of the curve.

The second curve, $y = -\frac{1}{x}$, has had only one change from the parent graph $y = \frac{1}{x}$: the negative sign. Just as a negative at the front of $f(x) = x^2$ would flip this graph upside down, the negative sign here “flips” the parts of the parent graph across the x-axis. The lighter graph shown at right is the parent $y = \frac{1}{x}$, and the darker graph is $y = -\frac{1}{x}$. 

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In the equation \( y = 3\sqrt{x - 2} \) the radical is multiplied by 3, hence the transformed graph will grow vertically more quickly than the parent graph \( y = \sqrt{x} \). It is also shifted to the right 2 units because of the “–2” under the radical sign. The new graph is the darker curve on the graph shown at right. Notice that the point (0, 0) on the original graph (the locator point) has shifted right 2 units.

This last graph is an exponential function. The parent graph, \( y = b^x \), changes in steepness as \( b \) changes. The larger \( b \) is, the quicker the graph rises, making a steeper graph. With \( y = 3^x - 6 \), the graph is a bit steeper than \( y = 2^x \), often thought of as the simplest exponential function, but also shifted down 6 units. The lighter graph is \( y = 2^x \), while the darker graph is \( y = 3^x - 6 \).

**Example 2**

Suppose \( f(x) \) is shown at right. From all you have learned about changing the graphs of functions:

a. Graph \( f(x) + 3 \).

b. Graph \( f(x) - 2 \).

c. Graph \( f(x - 1) \).

d. Graph \( f(x + 3) \).

e. Graph \( 3f(x) \).

f. Graph \( \frac{1}{2}f(x) \).

Each time we alter the equation slightly, the graph is changed. Even though we have no idea what the equation of this function is, we can still shift it on the coordinate grid. Remember that \( f(x) \) represents the range or \( y \)-values. Therefore, in part (a), \( f(x) + 3 \) says “the \( y \)-values, plus 3.” Adding three to all the \( y \)-values will shift the graph up three units. This is shown at left. Notice that the shape of the graph is identical to the original, just shifted up three units. Check this by comparing the \( y \)-intercepts.

If \( f(x) + 3 \) shifts the graph up three units, then \( f(x) - 2 \) will shift the graph down two units. This graph is shown at right. Again, compare the \( y \)-intercept on the graph at left to the original. (Note: Using the \( y \)-intercept or the \( x \)-intercepts to help you graph is an effective way to create a graph.)
What happens when the change is made within the parentheses as in parts (c) and (d)? Here the shift is with the \(x\)-coordinates, thus the graph will move left or right.

In part (c), the graph of \(f(x - 1)\) is shifted to the **right** 1 unit.  

\[
\begin{align*}
  &\quad \quad \text{In part (c), the graph of } f(x - 1) \\
  &\quad \quad \text{is shifted to the **right** 1 unit.}
\end{align*}
\]

The graph in part (d), \(f(x + 3)\), is shifted to the **left** 3 units.

\[
\begin{align*}
  &\quad \quad \text{The graph in part (d), } f(x + 3), \\
  &\quad \quad \text{is shifted to the **left** 3 units.}
\end{align*}
\]

When multiplying \(f(x)\) by a number as in parts (e) and (f), look at some key points. In particular, consider the \(x\)-intercepts. Since the \(y\)-value is zero at these points, multiplying by any number will not change the \(y\)-value. Therefore, the \(x\)-intercepts do not change at all, but the \(y\)-intercept will. In the original graph, the \(y\)-intercept is 1, so \(f(0) = 1\). The larger the constant by which you multiply, the more stretched out the graph becomes. A smaller number flattens the graph.

Multiplying by 3 will raise that point three times as high, to the point \((0, 3)\).

Multiplying by \(\frac{1}{2}\) changes the \(y\)-intercept to \(\left(0, \frac{1}{2}\right)\).

\[
\begin{align*}
  &\quad \quad \text{Multiplying by } 3 \text{ will raise that point} \\
  &\quad \quad \text{three times as high, to the point } (0, 3). \\
  &\quad \quad \text{Multiplying by } \frac{1}{2} \text{ changes} \\
  &\quad \quad \text{the } y\text{-intercept to } \left(0, \frac{1}{2}\right). \quad
\end{align*}
\]

**Example 3**

Apply your knowledge of parent graphs and transformations to graph the following two non-functions.

\[
\begin{align*}
  &\quad \quad x = y^2 + 3 \quad &\quad \quad b. \quad (x - 2)^2 + (y + 3)^2 = 36
\end{align*}
\]

**Solution follows on next page →**
Not every equation is a function, and the two non-functions students consider are \( x = y^2 \) and \( x^2 + y^2 = r^2 \). The first is the equation of a “sleeping parabola,” or a parabola lying on its side. The second equation is the general form of a circle with center \((0, 0)\), and radius of length \( r \). As written, neither of these equations can be entered into a graphing calculator. Students need to solve each of these as “\( y = \)” to use the calculator. But rather than doing that, the students can use what they have already learned to make accurate graphs of each equation. The parent of the equation in part (a) is \( x = y^2 \). The “+ 3” tells us the graph will shift 3 units, but is it up, down, left, or right? Rewriting the equation as \( \pm \sqrt{x-3} = y \) helps us see that this graph is shifted to the right 3 units. At right, the grey curve is the graph of \( x = y^2 \), and the darker curve is the graph of \( x = y^2 + 3 \).

Graphing the equation of a circle is straightforward: a circle with center \((h, k)\) and radius \( r \) has the equation \((x - h)^2 + (y + k)^2 = r^2 \). Therefore the graph of the equation in part (b) is a circle with a center at \((2, -3)\), and a radius of 6. The graph of the circle is shown at right.

**Problems**

For each of the following equations, state the parent equation and then sketch its graph. Be sure to include any key and/or locator points.

1. \( y = (x - 5)^2 \)
2. \( y = -\frac{1}{2} (x + 4)^2 + 7 \)
3. \( (x - 2)^2 + (y + 1)^2 = 9 \)
4. \( y = |x + 5| - 2 \)
5. \( y = \frac{1}{x+1} + 10 \)
6. \( y = 2^x - 8 \)
7. \( y = -(x - 2)^3 + 1 \)
8. \( y = \sqrt{x + 7} \)
9. \( y = 3|x - 5| \)
10. \( y = \pm \sqrt{x - 9} \)

For each of the following problems, state whether or not it is a function. If it is not a function, explain why not.

11.\[ y = 7 \pm \sqrt{9 - x^2} \]
12.\[ y = 3(x - 4)^2 \]

Parent Guide and Extra Practice

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Answers

1. Parent graph \( f(x) = x^2 \), vertex (5, 0)

2. Parent graph \( f(x) = x^2 \), vertex (-4, 7)

3. Parent graph \((x - h)^2 + (y - k)^2 = r^2\), center (2, -1), radius 3

4. Parent graph \( f(x) = |x| \), vertex (-5, -2)

5. Parent graph \( f(x) = \frac{1}{x} \), asymptotes \( x = -1, y = 10 \)

6. Parent graph \( f(x) = 2^x \), asymptote \( x = -8 \)

7. Parent graph \( f(x) = x^3 \), locator point (2, 1)

8. Parent graph \( f(x) = \sqrt{x} \), vertex (-7, 0)

9. Parent graph \( f(x) = |x| \), vertex (5, 0)

10. Parent graph \( y^2 = x \), vertex (9, 0)

11. Yes.

12. No, on the left part of the graph, for each \( x \)-value there are two possible \( y \)-values. You can see this by drawing a vertical line through the graph. If a vertical line passes through the graph more than once, it is not a function.

13. No, because the equation has “±,” for each value substituted for \( x \), there will be two \( y \)-values produced. A function can have only one output for each input.

14. Yes.