MORE ON COMPLETING THE SQUARE

Although students can find the vertex of a parabola by averaging the $x$-intercepts, they also can use the algebraic method known as completing the square. This allows students to go directly from standard (or non-graphing) form to graphing form without the intermediate step of finding the $x$-intercepts. Completing the square is also used when the equation of a circle is written in an expanded form. When the students first looked at how to complete the square, they used tiles so that they could see how the method works. When they tried to create a square (complete it) by arranging the tiles, there were either too many or missing parts. This visual representation helps students see how to rewrite the equation algebraically.

Example 1

The function $f(x) = x^2 + 6x + 3$ is written in standard form. Complete the square to write it in graphing form. Then state the vertex of the parabola and sketch the graph.

The general equation of a parabola in graphing form is $f(x) = a(x - h)^2 + k$, where $(h, k)$ is the vertex. The original equation needs to be changed into a set of parentheses squared, with a constant either added to or subtracted from it. To do this, we must know that $(x - h)^2 = x^2 - 2xh + h^2$. We will use this form of a perfect square to complete the square of the given equation or function.

$$f(x) = x^2 + 6x + 3$$
$$= x^2 + 6x + 9 - 9 + 3 - 9$$
$$= (x + 3)^2 - 6$$

The first box holds a space for the number we have to add to complete the square. The second box is to subtract that same number so as not to change the balance of the equation. To determine the missing number, take half the coefficient of $x$ (half of 6), and then square it and place the result in both boxes. With the equation in graphing form, we know the vertex is $(-3, -6)$. The graph is shown below.
Example 2

The equation \( x^2 - 8x + y^2 + 16y = 41 \) is the equation of a circle. Complete the square to determine the coordinates of its center and the length of the radius.

As with the last example, we will fill in the blanks to create perfect squares. We need to do this twice: once for \( x \), and again for \( y \).

\[
\begin{align*}
x^2 - 8x + \underline{16} - 16 + y^2 + 16y + \underline{64} - 64 &= 41 \\
(x - 4)^2 - 16 + (y + 8)^2 - 64 &= 41 \\
(x - 4)^2 + (y + 8)^2 &= 41 + 16 + 64 \\
(x - 4)^2 + (y + 8)^2 &= 121
\end{align*}
\]

This is a circle with center \((4, -8)\) and a radius of \(\sqrt{121} = 11\).

Problems

Write each of the following equations in graphing form. Then state the vertex and the direction the parabola opens.

1. \( y = x^2 - 8x + 18 \)  
2. \( y = -x^2 - 2x + 7 \)  
3. \( y = 3x^2 - 24x + 42 \)
4. \( y = 2x^2 - 6 \)  
5. \( y = \frac{1}{2}x^2 - 3x + \frac{1}{2} \)  
6. \( y = x^2 + 18x + 97 \)

Find the center and radius of each circle.

7. \( (x + 2)^2 + (y + 7)^2 = 25 \)  
8. \( 3(x - 9)^2 + 3(y + 1)^2 = 12 \)  
9. \( x^2 + 6x + y^2 = 91 \)
10. \( x^2 - 10x + y^2 + 14y = -58 \)  
11. \( x^2 + 50x + y^2 - 2y = -602 \)  
12. \( x^2 + y^2 - 8x - 16y = 496 \)

Answers

1. \( y = (x - 4)^2 + 2 \), vertex \((4, 2)\), up  
2. \( y = -(x + 1)^2 - 6 \), vertex \((-1, -6)\), down

3. \( y = 3(x - 4)^2 - 6 \), vertex \((4, -6)\), up  
4. \( y = 2(x - 0)^2 - 6 \), vertex \((0, -6)\), up

5. \( y = \frac{1}{2}(x - 3)^2 - 4 \), vertex \((3, -4)\), up  
6. \( y = (x + 9)^2 + 16 \), vertex \((-9, -16)\), up

7. Center: \((-2, -7)\), radius: 5  
8. Center: \((9, -1)\), radius: 2

9. Center: \((-3, 0)\), radius: 10  
10. Center: \((5, -7)\), radius: 4

11. Center: \((-25, 1)\), radius: \(\sqrt{24} = 2\sqrt{6}\)  
12. Center: \((4, 8)\), radius: 24
Answers

1. D
2. A
3. D
4. C
5. B
6. C
7. 250
8. 5
9. 20
10. 1.5