Students explore inverses, that is, equations that “undo” the actions of functions. Although they may not be aware of it, students have already seen inverses without calling them that. When solving equations, students reverse operations to undo the equation leaving just $x$, and that is what inverses do. Students further explore composition of functions by considering what happens when inverses are combined. For further information see the Math Notes boxes in Lessons 5.1.2 and 5.1.3.

**Example 1**

Find the inverse (undo) rule for the functions below. Use function notation and give the inverse rule a name different from the original function.

a. $f(x) = \frac{x-6}{3}$

The function in part (a) subtracts 6 from the input then divides by 3. The undo rule, or inverse, reverses this process. Therefore, the inverse first multiplies by 3 then adds 6. If we call this inverse $h(x)$, we can write $h(x) = 3x + 6$.

The function $g(x)$ adds 4 to the input, squares that value, then adds 1. The inverse will first subtract 1, take the square root then subtract 4. Calling this rule $j(x)$ we can write $j(x) = \pm \sqrt{x-1} - 4$.

Rather than give the inverse a new name, we can use the notation for inverses. The inverse of $f(x)$ is written as $f^{-1}(x)$.

Note: The inverses $h(x)$ and $j(x)$, are fundamentally different. $h(x) = 3x + 6$ is the equation of a non-vertical line, therefore $h(x)$ is a function. $j(x) = \pm \sqrt{x-1} - 4$, however, is not a function. By taking the square root, we created a positive value and a negative value. This gives two outputs for each input, and so by definition it is not a function.

Although we computed the inverses through a verbal description of what each function does, the students learn an algorithm for finding an inverse. They switch the $x$ and $y$, and then solve for $y$. Using this algorithm on the equations above:

$$f(x) = \frac{x-6}{3} \quad g(x) = (x + 4)^2 + 1$$

$$y = \frac{x-6}{3} \quad y = (x + 4)^2 + 1$$

$$x = \frac{y-6}{3} \quad x = (y + 4)^2 + 1$$

$$3x = y - 6 \quad x - 1 = (y + 4)^2$$

$$3x + 6 = y \quad \pm \sqrt{x-1} = y + 4$$

$$\quad \quad -4 \pm \sqrt{x-1} = y$$
Example 2

The graph of \( f(x) = 0.2x^3 - 2.4x^2 + 6.4x \) is shown at right. Graph the inverse of this function.

Following the algorithm for determining the equation for an inverse, as we did above, would be difficult here. The students do not have a method for solving cubic equations. Nevertheless, students can graph the inverse because they know a special property about the graphs of functions and their inverses: they are symmetrical about the line \( y = x \).

If we add the line \( y = x \) to the graph, the inverse is the reflection across this line. Here we can fold the paper along the line \( y = x \), and trace the result to create the reflection.

Example 3

Consider the function \( f(x) = \frac{2x-1}{7} \). Determine the inverse of \( f(x) \) and label it \( g(x) \). Verify that these two functions are inverses by calculating \( f(g(x)) \) and \( g(f(x)) \).

Using the algorithm, we can determine the inverse.

\[
\begin{align*}
y &= \frac{2x-1}{7} \\
x &= \frac{2y-1}{7} \\
7x &= 2y-1 \\
7x+1 &= 2y \\
y &= \frac{7x+1}{2} \\
g(x) &= \frac{7x+1}{2} \\
f(x) &= \frac{2x-1}{7} \\
f(g(x)) &= f\left(\frac{7x+1}{2}\right) = \frac{2\left(\frac{7x+1}{2}\right)-1}{7} = \frac{7x+1-1}{7} = \frac{7x}{7} = x \\
g(f(x)) &= g\left(\frac{2x-1}{7}\right) = \frac{7\left(\frac{2x-1}{7}\right)+1}{2} = \frac{2x-1+1}{2} = \frac{2x}{2} = x
\end{align*}
\]

Since \( f(g(x)) = g(f(x)) = x \), the functions are inverses.
Problems

Find the inverse of each of the following functions.

1. \( f(x) = 8(x - 13) \)  
2. \( y = -\frac{3}{4} x + 6 \)
3. \( y = \frac{5(x+2)}{3} \)  
4. \( f(x) = x^2 + 6 \)
5. \( f(x) = \frac{3}{x} + 6 \)  
6. \( g(x) = \frac{5}{x} \)
7. \( g(x) = (x + 1)^2 - 3 \)  
8. \( y = (x + 2)^3 \)
9. \( y = 3 + \sqrt{x - 4} \)  
10. \( g(x) = 6x + 2 \)

Sketch the graph of the inverse of each of the following functions.

11. \( y = \frac{x}{6} + 2 \)
12. \( f(x) = 2x^2 - 1 \)
13. \( g(x) = x \)
14. \( y = \frac{1}{5x} \)
15. 

For each of the following pairs of functions, determine \( f(g(x)) \) and \( g(f(x)) \), then use the result to decide whether or not \( f(x) \) and \( g(x) \) are inverses of each other.

16. \( f(x) = 5x + 7 \)  
\( g(x) = \frac{x-7}{5} \)
17. \( f(x) = 8x \)  
\( g(x) = \frac{1}{8} x \)
18. \( f(x) = x + 5 \)  
\( g(x) = \frac{1}{x+5} \)
19. \( f(x) = \frac{2}{3x} \)  
\( g(x) = \frac{3x}{2} \)
20. \( f(x) = \frac{2}{3} x + 6 \)  
\( g(x) = \frac{3(x-6)}{2} \)
21. \( f(x) = x\sqrt{3} + 9 \)  
\( g(x) = \left(\frac{x-9}{\sqrt{3}}\right)^2 \)
Answers

1. \( y = \frac{x}{8} + 13 \)
2. \( y = -\frac{4}{3}x + 8 \)
3. \( y = \frac{3}{5}x - 2 \)
4. \( y = \pm \sqrt{x - 6} \)
5. \( y = \frac{3}{x-6} \)
6. \( y = \frac{5}{x} \)
7. \( y = -1 \pm \sqrt{x + 3} \)
8. \( y = -2 + \frac{3}{x} \)
9. \( y = (x - 3)^2 + 4, \text{ for } x \geq 3 \)
10. \( y = \frac{x^2}{6} \)

11. \( y = 6x - 12 \)
12. \( y = \sqrt{\frac{x+1}{2}} \)
13. \( y = x \)

14. 
15. 

16. \( f(g(x)) = g(f(x)) = x \). They are inverses.
17. \( f(g(x)) = g(f(x)) = x \). They are inverses.
18. \( f(g(x)) = \frac{1}{x+5} + 5, \ g(f(x)) = \frac{1}{x+10} \). No, they are not inverses.
19. \( f(g(x)) = \frac{4}{x^2}, \ g(f(x)) = \frac{1}{x} \). No, they are not inverses.
20. \( f(g(x)) = g(f(x)) = x \). They are inverses.
21. \( f(g(x)) = \frac{\sqrt{3(x-9)^2}}{3} + 9, \ g(f(x)) = x^2 \). No, they are not inverses.