Students began their study of trigonometric functions in Chapter 7 when they learned about radians and transforming trig functions. Here they will learn how to solve equations with trig functions by operating on the variable. This work will review the inverse trig functions as well as introduce the reciprocal trigonometric functions. See the Math Notes box in Lesson 12.1.4 for more information.

Example 1

For what values are the following equations true?

a. \( \cos(\theta) = \frac{\sqrt{3}}{2} \)  
b. \( 2 \sin(\theta) = \sqrt{2} \)  
c. \( \cos(\theta) = 5 \)

Although the first inclination might be to use the inverse function to solve the equation in part (a), that will not completely answer the question.

\[
\cos(\theta) = \frac{\sqrt{3}}{2} \\
\cos^{-1}(\cos(\theta)) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \\
\theta = \frac{\pi}{6} \text{ radians}
\]

You might recall that the graph of \( f(x) = \cos(x) \) is a periodic function, and the graph of \( y = \frac{\sqrt{3}}{2} \) is a horizontal line. By graphing both equations on the same set of axes, we see that they intersect infinitely many times. Solving by using the inverse trig function gives us one solution. How do we find all solutions?

It helps to remember the unit circle. For what values of \( \theta \) does \( \cos(\theta) = \frac{\sqrt{3}}{2} \)? Two points are easily found: \( \frac{\pi}{6} \) and \( -\frac{\pi}{6} \).

How do we find the rest? On the unit circle, we can revisit \( \frac{\pi}{6} \) and \( -\frac{\pi}{6} \) at each rotation of \( 2\pi \). Therefore, not only does \( \frac{\pi}{6} \) make the equation true, but so do \( \frac{\pi}{6} \pm 2\pi, \frac{\pi}{6} \pm 4\pi, \frac{\pi}{6} \pm 6\pi \), etc. Similarly, \( -\frac{\pi}{6} \pm 2\pi, -\frac{\pi}{6} \pm 4\pi, -\frac{\pi}{6} \pm 6\pi, \ldots \) will also make the equation true. We can summarize this information as \( \theta = \pm \frac{\pi}{6} \pm 2\pi n \), for all integers \( n \). Note: There are other ways to write this solution that are equivalent to this expression.
Using the same method for part (b), we find:

In a unit circle, 
\[
\sin \theta = \frac{\sqrt{2}}{2}.
\]

\[
\sin \theta = \frac{\sqrt{2}}{2}
\]

\[
\theta = \sin^{-1} \left( \frac{\sqrt{2}}{2} \right)
\]

\[
\theta = \frac{\pi}{4} \pm 2\pi n.
\]

Some students might see the solution to part (c) quickly. Since the range of \( f(x) = \cos(x) \) is \(-1 \leq y \leq 1\), this equation has no solution.

**Example 2**

Graph \( f(x) = \cos^{-1}(x) \) and \( g(x) = \tan^{-1}(x) \) on separate axes. How can you restrict the domain so that these are functions?

Because the graphs of \( y = \cos(x) \) and \( y = \tan(x) \) are periodic (repeating), if we used our carbon paper method of graphing the inverses, they too will be periodic. The problem is that they will not be functions, because they will fail the vertical line test.

By restricting the domain of the original function, we create an inverse that also has a restricted domain and is therefore a function as well. There are infinitely many restrictions we can make; however, the convention is to restrict the domain of \( y = \cos(x) \) to \( 0 \leq x \leq \pi \), and to restrict the domain of \( y = \tan(x) \) to \( -\frac{\pi}{2} < x < \frac{\pi}{2} \). These adjustments produce the necessary restrictions on the inverses so that they are also functions.
Example 3

Let \( f(x) = \sin(x) \), \( g(x) = \cos(x) \), and \( h(x) = \tan(x) \). Graph each of the following equations on separate axes.

\[
y = \frac{1}{f(x)} \quad y = \frac{1}{g(x)} \quad y = \frac{1}{h(x)}
\]

Compare your graphs to the graphs of these equations:

\[
y = \sin^{-1}(x) \quad y = \cos^{-1}(x) \quad y = \tan^{-1}(x)
\]

The first three functions are the reciprocal functions of sine, cosine and tangent. However, rather than writing them as reciprocals \( \frac{1}{f(x)} \), they are given new names:

\[
\frac{1}{\sin(x)} = \text{cosecant } x \\
\frac{1}{\cos(x)} = \text{secant } x \\
\frac{1}{\tan(x)} = \text{cotangent } x
\]

The abbreviation for cosecant is csc, for secant it is sec, and for cotangent it is cot. Their graphs are:

Since these are reciprocal functions, everywhere the first functions were zero, the corresponding reciprocal functions will be undefined. Check that this is the case.

In comparing these functions to the inverse trig functions, it is important to note that \( \frac{1}{\sin(x)} \neq \sin^{-1}(x) \) (and similarly for the other corresponding functions). This is very clear by examining the graphs. The exponent of “–1” tells us the function is the inverse, not the reciprocal.
Problems

For each of the following problems, find all the solutions. You may use your calculator but remember that your calculator only gives one answer.

1. \(2 \cos(x) = \sqrt{2}\)
2. \(5 \tan(x) - 5 = 0\)
3. \(4 \cos^2(x) - 1 = 0\)
4. \(4 \sin^2(x) = 3\)
5. \(\sin(x) + 2 = 3 \sin(x)\)
6. \(\tan^2(x) + \tan(x) = 0\)

Graph each of the following equations on a separate set of axes. Label all the important points.

7. \(y = 3 \csc(x)\)
8. \(y = 4 + \sec(x)\)
9. \(y = \cot(x - \pi)\)

Answers

1. \(x = \pm \frac{\pi}{4} \pm 2\pi n\) for all integers \(n\)
2. \(x = \frac{\pi}{4} \pm \pi n\) for all integers \(n\)
3. \(x = \pm \frac{\pi}{3} n\) for all integers \(n\)
4. \(x = \frac{\pi}{3} \pm \pi n\) or \(x = \frac{2\pi}{3} \pm \pi n\) for all integers \(n\)
5. \(x = \frac{\pi}{2} \pm 2\pi n\) for all integers \(n\)
6. \(x = \pm \pi n\) for all integers \(n\), \(x = \frac{3\pi}{4} \pm \pi n\) for all integers \(n\)
7. 
8. 
9. 
TRIGONOMETRIC IDENTITIES

By graphing different trigonometric expressions, students realize that some expressions are equivalent to others. These equivalent expressions are known as trig identities. These identities allow students to rewrite and solve many more trigonometric equations. See the Math Notes box in Lesson 12.2.3 for more information.

Example 1

Graph the function \( f(x) = \frac{1}{\cos^2(x)} - \tan^2(x) \). Based on the graph, what can you conclude about this expression? (That is, what trig identity can you write?) What substitution can you make in the identity so that you no longer have a fraction?

Before the widespread availability of calculators, students used tables to look up the trig values of various angle measures. Since tables were cumbersome, students would memorize hundreds of trigonometric identities so that they could quickly rewrite trig expressions. Knowing that \( \sin(2\theta) = 2 \sin(\theta) \cos(\theta) \), for instance, meant that a trig table of values did not need to extend to 120º angles. A student could rewrite \( \sin(120^\circ) \) as \( 2 \sin(60^\circ) \cos(60^\circ) \), and use the values for 60º. This also allowed students to rewrite trig expressions in equations, making the equations simpler and easier to solve.

By graphing the given function above, we can readily see that the function is a constant, that is, a horizontal line. This graph is equivalent to the graph of \( y = 1 \). Because their graphs are equivalent for all values of \( x \), the expressions are also equivalent. Therefore we can write:

\[
\frac{1}{\cos^2(x)} - \tan^2(x) = 1
\]

This is now a trig identity. How can we rewrite this so it no longer has a fraction? Since \( \frac{1}{\cos(x)} = \sec(x) \), we can write:

\[
\sec^2(x) - \tan^2(x) = 1.
\]

This trig identity is more commonly written as:

\[
1 + \tan^2(x) = \sec^2(x).
\]
Example 2

Prove the following trig identity:

\[
\frac{\sin(x)}{1-\cos(x)} + \frac{1-\cos(x)}{\sin(x)} = 2 \csc(x)
\]

Another practice common before the advent of calculators was the proving of trig identities. These proofs usually employ algebraic steps and previously proven identities to show that one side of the equation equals the other. For the identity above, we will start with the left side of the equation, get common denominators so we can add the fraction, and see where it takes us. Often with these proofs, you have to try different things to see where they take you. It is also important to be aware of the right hand side of the equation, which is our goal. Remember that 

\[
2 \csc(x) = \frac{2}{\sin(x)}.
\]

\[
\frac{\sin(x)}{1-\cos(x)} + \frac{1-\cos(x)}{\sin(x)} = \frac{\sin(x)}{1-\cos(x)} \cdot \left( \frac{\sin(x)}{\sin(x)} \right) + \frac{1-\cos(x)}{\sin(x)} \cdot \left( \frac{1-\cos(x)}{\sin(x)} \right)
\]

\[
= \frac{\sin^2(x)}{(\sin(x))(1-\cos(x))} + \frac{(1-\cos(x))^2}{(\sin(x))(1-\cos(x))}
\]

\[
= \frac{\sin^2(x)+(1-\cos(x))^2}{(\sin(x))(1-\cos(x))}
\]

\[
= \frac{\sin^2(x)+(1-2 \cos(x)+\cos^2(x))}{(\sin(x))(1-\cos(x))}
\]

\[
= \frac{\sin^2(x)+\cos^2(x)+1-2 \cos(x)}{(\sin(x))(1-\cos(x))}
\]

\[
= \frac{1+1-2 \cos(x)}{(\sin(x))(1-\cos(x))}
\]

\[
= \frac{2-2 \cos(x)}{(\sin(x))(1-\cos(x))}
\]

\[
= \frac{2}{(\sin(x))(1-\cos(x))}
\]

\[
= 2 \csc(x)
\]

This proves that this identity is true.
Problems

1. Show graphically that $\sin(x + y)$ does not equal $\sin(x) + \sin(y)$.
2. Graphically, determine what $\cos(x + 90^\circ)$ equals.
3. Graphically, determine what $\sin(180^\circ - x)$ equals.

Prove the following identities.

4. \[
\frac{\sin(2x)}{2\sin^2(x)} = \cot(x)
\]
5. \[
\sin^2(x) - \cos^2(x) = \frac{\tan(x) - \cot(x)}{\tan(x) + \cot(x)}
\]
6. \[
\frac{\sin^2(x)}{1 + \cos(x)} = 1 - \frac{1}{\sec(x)}
\]
7. \[
\cos^4(x) - \sin^4(x) = 2 \cos^2(x) - 1
\]
8. \[
\frac{1}{1 - \sin(x)} + \frac{1}{1 + \sin(x)} = 2 \sec^2(x)
\]

Answers

1. The graphs are not the same.
2. $\cos(x + 90^\circ) = -\sin(x)$
3. $\sin(180^\circ - x) = \sin(x)$
4. \[
\frac{\sin(2x)}{2\sin^2(x)} = \cot(x)
\]
   \[
\frac{\sin(2x)}{2\sin^2(x)} = \frac{2 \sin(x) \cos(x)}{2 \sin(x) \sin(x)}
\]
   \[
= \frac{2\sin(x) \cos(x)}{2\sin^2(x) \sin(x)}
\]
   \[
= \frac{\cos(x)}{\sin(x)}
\]
   \[
= \cot(x)
\]
5. \( \sin^2(x) - \cos^2(x) \ngtr \frac{\tan(x) - \cot(x)}{\tan(x) + \cot(x)} 
= \frac{\sin(x)}{\cos(x)} \cdot \frac{\cos(x)}{\sin(x)} 
= \frac{\sin^2(x) - \cos^2(x)}{\sin(x) \cos(x)} 
= \frac{\sin^2(x) - \cos^2(x)}{\sin(x) \cos(x)} 
= \sin^2(x) - \cos^2(x) 
= \sin^2(x) - \cos^2(x) 

6. \( \frac{\sin^2(x)}{1 + \cos(x)} \ngtr 1 - \frac{1}{\sec(x)} 
= (1 - \cos(x)) \cdot \left( \frac{1 + \cos(x)}{1 + \cos(x)} \right) 
= \frac{(1 - \cos(x))(1 + \cos(x))}{1 + \cos(x)} 
= \frac{1 - \cos^2(x)}{1 + \cos(x)} 
= \frac{\sin^2(x)}{1 + \cos(x)} 

7. \( \cos^4(x) - \sin^4(x) \ngtr 2 \cos^2(x) - 1 
\cos^4(x) - \sin^4(x) = (\cos^2(x) + \sin^2(x))(\cos^2(x) - \sin^2(x)) 
= 1 \cdot (\cos^2(x) - \sin^2(x)) 
= \cos^2(x) - (1 - \cos^2(x)) 
= \cos^2(x) - 1 + \cos^2(x) 
= 2 \cos^2(x) - 1 

8. \( \frac{1}{1 - \sin(x)} + \frac{1}{1 - \sin(x)} \ngtr 2 \sec^2(x) 
= \left( \frac{1}{1 + \sin(x)} \right) \left( \frac{1}{1 - \sin(x)} + \frac{1}{1 - \sin(x)} \right) 
= \frac{1 + \sin(x) + 1 - \sin(x)}{(1 + \sin(x))(1 - \sin(x))} 
= \frac{2}{1 - \sin^2(x)} 
= \frac{2}{\cos^2(x)} = 2 \sec^2(x) 

SAT PREP

1. If $7x < 2y$ and $2y < 9z$, which of the following statements is true?
   a. $7x < 9z$  b. $9z < 7x$  c. $z < x$  d. $7x = 9z$  e. $7x + 1 = 9z$

2. If $f(t) = 5t - 15$, then at what value of $t$ does the graph of $f(t)$ cross the x-axis?
   a. $-15$  b. $-5$  c. $0$  d. $2$  e. $3$

3. If $p^5 + 3 = p^5 + w$, then $w = ?$
   a. $-3$  b. $-\sqrt[5]{3}$  c. $\sqrt[5]{3}$  d. $3$  e. $3^5$

4. For all positive numbers $j$ and $k$, let $j\n k$ be defined as $\frac{j + 4k}{j - 4k}$. What is the value of $1018\n 4.5$?
   a. 1.036  b. 10.36  c. 103.6  d. 1036  e. 10360

5. If a number is rounded to 26.7, which of the following values could have been the original number?
   a. 26  b. 26.605  c. 26.655  d. 26.776  e. 27

6. On a coordinate plane, the center of a circle is at $(9, -2)$. If the circle touches the y-axis in only one point, what is the radius of the circle?

7. The figure at right shows three squares with sides of length 6, 8, and $k$, respectively. If points $A$, $B$, and $C$ lie on line $l$, what is the value of $k$?

8. Exactly 875 out of 7000 seniors at college are majoring in mathematics. What percent of seniors are NOT majoring in mathematics?

9. Five SnookerBars cost as much as 2 Sodiepop Swirls. If the cost of one Sodiepop Swirl and one SnookerBar is $1.75, what is the cost in dollars of 1 Sodiepop Swirl?

10. The highest score possible on Professor Snape’s test is 100 and the lowest is 0. Harry, Ron, Hermione, and Neville’s tests had an average of 86. If Neville got the lowest score, what is the lowest possible score he could have gotten?

Answers

6. 9  7. $\frac{32}{3}$  8. 87.5%  9. $1.25$  10. 44