LAWS OF EXPONENTS

Here are the basic patterns with examples:

1. \(a^x \cdot a^y = a^{x+y}\)
   Examples: \(x^3 \cdot x^4 = x^{3+4} = x^7\); \(2^7 \cdot 2^4 = 2^{11}\)

2. \(\frac{a^x}{a^y} = a^{x-y}\)
   Examples: \(x^{10} \div x^4 = x^{10-4} = x^6\); \(\frac{2^4}{2^7} = 2^{-3}\)

3. \((a^x)^y = a^{xy}\)
   Examples: \((x^4)^3 = x^{4 \cdot 3} = x^{12}\); \((2x)^4 = 2^4 \cdot x^{12} = 16x^{12}\)

4. \(x^{-a} = \frac{1}{x^a}\) and \(\frac{1}{x^{-b}} = x^b\)
   Examples: \(3x^{-3}y^2 = \frac{3y^2}{x^3}\); \(\frac{2x^5}{y^2} = 2x^5y^2\)

5. \(a^{x/y} = \sqrt[y]{a^x} = (\sqrt[y]{a})^x\)
   Examples: \(x^{2/3} = \sqrt[3]{x^2}\); \(8^{2/3} = \sqrt[3]{64} = 4\)

Example 1
Rewrite in a simpler, equivalent form: \((2xy^3)(5x^2y^4)\)
Multiply the coefficients: \(2 \cdot 5 \cdot xy^3 \cdot x^2y^4 = 10xy^3 \cdot x^2y^4\)
Add the exponents of \(x\), then \(y\): \(10x^{1+2}y^{3+4} = 10x^3y^7\)

Example 2
Rewrite in a simpler, equivalent form: \(\frac{14x^2y^{12}}{7x^3y^7}\)
Divide the coefficients: \(\frac{(14+7)x^2y^{12}}{x^3y^7} = \frac{2x^2y^{12}}{x^3y^7}\)
Subtract the exponents: \(2x^{-5}y^{12-7} = 2x^{-5}y^5\) or \(\frac{2y^5}{x^3}\)

Example 3
Rewrite in a simpler, equivalent form: \((3x^2y^4)^3\)
Cube each factor: \(3^3 \cdot (x^2)^3 \cdot (y^4)^3 = 27(x^2)^3(y^4)^3\)
Multiply the exponents: \(27x^6y^{12}\)

Example 4
Rewrite in a simpler, equivalent form: \((144x^{-12})^{1/2}\)
Convert negative exponents and change to radical: \((144x^{-12})^{1/2} = (\frac{144}{x^{12}})^{1/2} = \sqrt{\frac{144}{x^{12}}} = \frac{12}{x^6}\)
Problems

Rewrite each expression in a simpler, equivalent form.

1. \( y^5 \cdot y^7 \)
2. \( b^4 \cdot b^3 \cdot b^2 \)
3. \( 8^6 \cdot 8^2 \)
4. \( (y^5)^2 \)
5. \( (3a)^4 \)
6. \( \frac{m^8}{m^3} \)
7. \( \frac{12x^9}{4x^4} \)
8. \( (x^3 y^2)^3 \)
9. \( \frac{(y^4)^2}{(y^3)^2} \)
10. \( \frac{15x^2 y^7}{3x^4 y^5} \)
11. \( (4c^4)(ac^3)(3a^5c) \)
12. \( (7x^3 y^5)^2 \)
13. \( (4xy^2)(2y)^3 \)
14. \( \left( \frac{4}{x^2} \right)^3 \)
15. \( \frac{(2a^7)(3a^2)}{6a^3} \)
16. \( \left( \frac{5m^3 n^3}{m^5} \right)^3 \)
17. \( (3a^2x^3)^2(2ax^4)^3 \)
18. \( \left( \frac{x^3 y^4}{y^6} \right)^4 \)
19. \( \left( \frac{6y^2 \cdot 8}{12x^3 y^7} \right)^2 \)
20. \( \frac{(2x^5 y^3)^3(4x^y^4)^2}{8x^7 y^{12}} \)
21. \( (-27)^{1/3} \)
22. \( 16^{-1/2} \)
23. \( (16a^8b^{12})^{3/4} \)
24. \( \frac{144^{1/2} x^{-3}}{(16^{3/4} x^{-1})^0} \)

Answers

1. \( y^{12} \)
2. \( b^9 \)
3. \( 8^8 \)
4. \( y^{10} \)
5. \( 81a^4 \)
6. \( m^5 \)
7. \( 3x^5 \)
8. \( x^9 y^6 \)
9. \( y^2 \)
10. \( \frac{5x^2}{x^2} \)
11. \( 12a^6 c^8 \)
12. \( 49x^6 y^{10} \)
13. \( 32xy^5 \)
14. \( \frac{64}{x^6} \)
15. \( a^6 \)
16. \( \frac{125n^3}{m^6} \)
17. \( 72a^7x^{18} \)
18. \( \frac{x^{12}}{y^{12}} \)
19. \( \frac{x^{10}}{4y^{10}} \)
20. \( 16x^{10}y^5 \)
21. \( -3 \)
22. \( \frac{1}{4} \)
23. \( 8a^6 b^9 \)
24. \( \frac{12}{3} \)
### RADICALS

Sometimes it is convenient to leave square roots in radical form instead of using a calculator to find approximations (decimal values). Look for perfect squares (i.e., 4, 9, 16, 25, 36, 49, …) as factors of the number that is inside the radical sign (radicand). Compute the square root(s) of any perfect square factor(s) and place the root(s) outside the radical to indicate the root multiplies the reduced radical. When there is already an existing value that multiplies the radical, multiply any root(s) by that value.

**Examples:**

\[
\begin{align*}
\sqrt{9} &= 3 \\
\sqrt{18} &= \sqrt{9 \cdot 2} = \sqrt{9} \cdot \sqrt{2} = 3\sqrt{2} \\
\sqrt{80} &= \sqrt{16 \cdot 5} = \sqrt{16} \cdot \sqrt{5} = 4\sqrt{5} \\
\sqrt{45} + 4\sqrt{20} &= \sqrt{9 \cdot 5} + 4\sqrt{4 \cdot 5} = 3\sqrt{5} + 4 \cdot 2\sqrt{5} = 11\sqrt{5}
\end{align*}
\]

When there are no more perfect square factors inside the radical sign, the product of the whole number (or fraction) and the remaining radical is said to be in **simple radical form**.

Simple radical form does not allow radicals in the denominator of a fraction. If there is a radical in the denominator, rationalize the denominator by multiplying the numerator and denominator of the fraction by the radical in the original denominator. Then rewrite the remaining fraction.

**Examples:**

\[
\begin{align*}
\frac{3}{\sqrt{2}} &= \frac{3}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{2} = \sqrt{2} \\
\frac{4\sqrt{5}}{\sqrt{30}} &= \frac{4\sqrt{5}}{\sqrt{30}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{4\sqrt{5} \cdot \sqrt{6}}{\sqrt{30} \cdot \sqrt{6}} = \frac{2\sqrt{30}}{3}
\end{align*}
\]

In the first example, \( \sqrt{2} \cdot \sqrt{2} = \sqrt{4} = 2 \) and \( \frac{3}{2} = 1 \).

In the second example, \( \sqrt{6} \cdot \sqrt{6} = \sqrt{36} = 6 \) and \( \frac{4\sqrt{5}}{\sqrt{30}} = \frac{2\sqrt{30}}{3} \).

### Example 1

Add \( \sqrt{27} + \sqrt{12} - \sqrt{48} \). Factor each radical and rewrite it in a simpler, equivalent form.

\[
\begin{align*}
\sqrt{9 \cdot 3} + \sqrt{4 \cdot 3} - \sqrt{16 \cdot 3} &= 3\sqrt{3} + 2\sqrt{3} - 4\sqrt{3} = 1\sqrt{3} \text{ or } \sqrt{3}
\end{align*}
\]

### Example 2

Rewrite \( \frac{3}{\sqrt{6}} \) in simple radical form.

Multiply by \( \frac{\sqrt{6}}{\sqrt{6}} \) and reduce: \( \frac{3}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{3\sqrt{6}}{6} = \frac{\sqrt{6}}{2} \).
Problems

Rewrite each of the following radicals in simple radical form.

1. \(\sqrt{24}\)  
2. \(\sqrt{48}\)  
3. \(\sqrt{17}\)  
4. \(\sqrt{31}\)  
5. \(\sqrt{75}\)  
6. \(\sqrt{50}\)  
7. \(\sqrt{96}\)  
8. \(\sqrt{243}\)  
9. \(\sqrt{8} + \sqrt{18}\)  
10. \(\sqrt{18} + \sqrt{32}\)  
11. \(\sqrt{27} - \sqrt{12}\)  
12. \(\sqrt{50} - \sqrt{32}\)  
13. \(\sqrt{6} + \sqrt{63}\)  
14. \(\sqrt{44} + \sqrt{99}\)  
15. \(\sqrt{50} + \sqrt{32} - \sqrt{27}\)  
16. \(\sqrt{75} - \sqrt{8} - \sqrt{32}\)  
17. \(\frac{3}{\sqrt{3}}\)  
18. \(\frac{8}{\sqrt{27}}\)  
19. \(\frac{\sqrt{5}}{\sqrt{3}}\)  
20. \(\frac{\sqrt{5}}{\sqrt{7}}\)  
21. \(4\sqrt{5} - \frac{10}{\sqrt{5}}\)

Answers

1. \(2\sqrt{6}\)  
2. \(4\sqrt{3}\)  
3. \(\sqrt{17}\)  
4. \(\sqrt{31}\)  
5. \(5\sqrt{3}\)  
6. \(5\sqrt{2}\)  
7. \(4\sqrt{6}\)  
8. \(9\sqrt{3}\)  
9. \(5\sqrt{2}\)  
10. \(7\sqrt{2}\)  
11. \(\sqrt{3}\)  
12. \(\sqrt{2}\)  
13. \(3\sqrt{7} + \sqrt{6}\)  
14. \(5\sqrt{11}\)  
15. \(9\sqrt{2} - 3\sqrt{3}\)  
16. \(5\sqrt{3} - 6\sqrt{2}\)  
17. \(\sqrt{3}\)  
18. \(\frac{5\sqrt{3}}{9}\)  
19. \(\frac{\sqrt{15}}{5}\)  
20. \(\frac{\sqrt{35}}{7}\)  
21. \(2\sqrt{5}\)
SOLVING BY REWRITING: FRACTION BUSTERS

Equations with fractions and/or decimals can be rewritten as equivalent equations without fractions and/or decimals and then solved in the usual manner. Equations can also be made simpler by factoring a common numerical factor out of each term. Fractions can be eliminated from an equation by multiplying both sides (i.e., all terms) of an equation by the common denominator. If you cannot easily determine the common denominator, then multiply the entire equation by the product of all of the denominators. We call the term used to eliminate the denominators a fraction buster. Always remember to check your solution(s).

Example 1
Solve: \(0.12x + 7.5 = 0.2x + 3\)

Multiply by 100 to remove the decimals.
\[
100 \cdot (0.12x + 7.5) = 0.2x + 3
\]
\[
12x + 750 = 20x + 300
\]

Solve in the usual manner.
\[
-8x = -450
\]
\[
x = 56.25
\]

Example 2
Solve: \(25x^2 + 125x + 150 = 0\)

Divide by 25 (a common factor).
\[
\frac{25x^2 + 125x + 150}{25} = 0
\]
\[
x^2 + 5x + 6 = 0
\]

Solve in the usual manner.
\[
(x + 2)(x + 3) = 0
\]
\[
x = -2 \text{ or } x = -3
\]

Example 3
Solve: \(\frac{x}{2} + \frac{x}{6} = 7\)

Multiply both sides of the equation by 6, the common denominator, to remove the fractions.
\[
6 \cdot \left(\frac{x}{2} + \frac{x}{6}\right) = 6(7)
\]

(Multiplying both sides by 12 would also have been acceptable.)

Distribute and solve as usual.
\[
6 \cdot \frac{x}{2} + 6 \cdot \frac{x}{6} = 6 \cdot 7
\]
\[
3x + x = 42
\]
\[
4x = 42
\]
\[
x = \frac{42}{4} = \frac{21}{2} = 10.5
\]

Example 4
Solve: \(\frac{5}{2x} + \frac{1}{6} = 8\)

Multiply both sides by \(6x\), the common denominator, to remove the fractions.
\[
6x \cdot \left(\frac{5}{2x} + \frac{1}{6}\right) = 6x(8)
\]

(Multiplying both sides by 12 would also have been acceptable.)

Distribute and solve as usual.
\[
6x \cdot \frac{5}{2x} + 6x \cdot \frac{1}{6} = 6x \cdot 8
\]
\[
15 + x = 48x
\]
\[
x = \frac{15}{47} \approx 0.32
\]
Problems

Rewrite each equation in a simpler form and then solve the new equation.

1. $\frac{5}{2} + \frac{x}{2} = 5$
2. $3000x + 2000 = -1000$
3. $0.02y - 1.5 = 17$

4. $\frac{5}{2} + \frac{x}{2} - \frac{3}{4} = 12$
5. $50x^2 - 200 = 0$
6. $\frac{5}{9} + \frac{2x}{5} = 3$

7. $\frac{3x}{10} + \frac{x}{10} = \frac{15}{10}$
8. $\frac{3}{2x} + \frac{5}{x} = \frac{13}{6}$
9. $x^2 - 2.5x + 1 = 0$

10. $\frac{2}{3x} - \frac{1}{x} = \frac{1}{36}$
11. $0.002x = 5$
12. $10 + \frac{5}{x} + \frac{3}{3x} = 11$

13. $0.3(x + 7) = 0.2(x - 2)$
14. $x + \frac{5}{2} + \frac{3x}{5} = 21$
15. $32 \cdot 3x - 32 \cdot 1 = 32 \cdot 8$

16. $5 + \frac{2}{x} + \frac{5}{4x} = \frac{73}{12}$
17. $\frac{17}{2x+1} = \frac{17}{5}$
18. $2 + \frac{6}{x} + \frac{6}{3x} = 3$

19. $2.5x^2 + 3x + 0.5 = 0$
20. $\frac{x}{x+2} = \frac{7}{x+2}$

Answers

1. $x = 6$
2. $x = -1$
3. $y = 925$

4. $x = \frac{144}{7} \approx 20.57$
5. $x = \pm 2$
6. $x = \frac{135}{23} \approx 5.87$

7. $x = \frac{15}{4} = 3.75$
8. $x = 3$
9. $x = \frac{1}{2}$ or 2

10. $x = -12$
11. $x = 2500$
12. $x = 6$

13. $x = -25$
14. $x = 10$
15. $x = 3$

16. $x = 3$
17. $x = 2$
18. $x = 8$

19. $x = -\frac{1}{5}$ or -1
20. $x = 7$ Note: $x$ cannot be 2.
ARITHMETIC SEQUENCES

An ordered list of numbers such as: 4, 9, 16, 25, 36… is a **sequence**. Each number in the sequence is a **term**. Usually variables with subscripts are used to label terms. For example, in the sequence above, the first term is 4 and the third term is 16. This might be written \( a_1 = 4 \) and \( a_3 = 16 \) where \( a \) is the variable used to label the sequence.

In the sequence 1, 5, 9, 13, …, there is a **common difference** \( (d = 4) \) between the successive terms and this is called an **arithmetic sequence**. There are two common methods to define a sequence. An explicit formula tells you exactly how to find any specific term in the sequence. A recursive formula tells first term and how to get from one term to the next. Formally, for arithmetic sequences, this is written:

**Explicit:**

\[ a_n = a_1 + (n - 1)d \]

where \( n \) = term number and \( d \) = common difference.

**Recursive:**

\[ a_1 = \text{some specific value}, \quad a_{n+1} = a_n + d, \quad \text{and} \quad d = \text{common difference}. \]

For the sequence 1, 5, 9, 13, …, the explicit formula is:

\[ a_n = 1 + (n - 1)(4) = 4n - 3 \]

and the recursive formula is: \( a_1 = 1, a_{n+1} = a_n + 4 \). In each case, successively replacing \( n \) by 1, 2, 3, … will yield the terms of the sequence. See the examples below.

**Example 1**

List the first five terms of the arithmetic sequence.

\[ a_n = 5n + 2 \] (an explicit formula)

\( a_1 = 5(1) + 2 = 7 \)
\( a_2 = 5(2) + 2 = 12 \)
\( a_3 = 5(3) + 2 = 17 \)
\( a_4 = 5(4) + 2 = 22 \)
\( a_5 = 5(5) + 2 = 27 \)

The sequence is: 7, 12, 17, 22, 27, …

**Example 2**

List the first five terms of the arithmetic sequence.

\[ b_1 = 3 \] (A recursive formula)

\[ b_{n+1} = b_n - 5 \]

\( b_1 = 3 \)
\( b_2 = b_1 - 5 = 3 - 5 = -2 \)
\( b_3 = b_2 - 5 = -2 - 5 = -7 \)
\( b_4 = b_3 - 5 = -7 - 5 = -12 \)
\( b_5 = b_4 - 5 = -12 - 5 = -17 \)

The sequence is: 3, –2, –7, –12, –17, …

**Example 3**

Write an explicit and a recursive equation for the sequence: –2, 1, 4, 7, …

Explicit: \( a_1 = -2, \ d = 3 \) so the equation is \( a_n = a_1 + (n - 1)d = -2 + (n - 1)(3) = 3n - 5 \)

Recursive: \( a_1 = -2, \ d = 3 \) so the equation is \( a_1 = -2, a_{n+1} = a_n + 3 \).
Problems

List the first five terms of each arithmetic sequence.

1. \(a_n = 5n - 2\)  
2. \(b_n = -3n + 5\)  
3. \(a_n = -15 + \frac{1}{2}n\)

4. \(c_n = 5 + 3(n - 1)\)  
5. \(a_1 = 5, a_{n+1} = a_n + 3\)  
6. \(a_1 = 5, a_{n+1} = a_n - 3\)

7. \(a_1 = -3, a_{n+1} = a_n + 6\)  
8. \(a_1 = \frac{1}{3}, a_{n+1} = a_n + \frac{1}{2}\)

Find the 30th term of each arithmetic sequence.

9. \(a_n = 5n - 2\)  
10. \(a_n = -15 + \frac{1}{2}n\)

11. \(a_{31} = 53, d = 5\)  
12. \(a_1 = 25, a_{n+1} = a_n - 3\)

For each arithmetic sequence, find an explicit and a recursive formula.

13. \(4, 8, 12, 16, 20, \ldots\)  
14. \(-2, 5, 12, 19, 26, \ldots\)

15. \(27, 15, 3, -9, -21, \ldots\)  
16. \(3, 3 \frac{1}{3}, 3 \frac{2}{3}, 4, 4 \frac{1}{3}, \ldots\)

Sequences are graphed using points of the form: (term number, term value). For example, the sequence \(4, 9, 16, 25, 36, \ldots\) would be graphed by plotting the points (1, 4), (2, 9), (3, 16), (4, 25), (5, 36), …. Sequences are graphed as points and not connected.

17. Graph the sequences from problems 1 and 2 above and determine the slope of each line.

18. How do the slopes of the lines found in the previous problem relate to the sequences?

Answers

1. \(3, 8, 13, 18, 23\)  
2. \(2, -1, -4, -7, -10\)

3. \(-14 \frac{1}{2}, -14, -13 \frac{1}{2}, -13, -12 \frac{1}{2}\)  
4. \(5, 8, 11, 14, 17\)

5. \(5, 8, 11, 14, 17\)  
6. \(5, 2, -1, -4, -7\)  
7. \(-3, 3, 9, 15, 21\)  
8. \(\frac{1}{3}, \frac{5}{6}, 1 \frac{1}{3}, 1 \frac{5}{6}, 2 \frac{1}{3}\)

9. \(148\)  
10. \(0\)  
11. \(48\)  
12. \(-62\)

13. \(a_n = 4n; a_1 = 4, a_{n+1} = a_n + 4\)  
14. \(a_n = 7n - 9; a_1 = -2, a_{n+1} = a_n + 7\)

15. \(a_n = -12n + 39; a_1 = 27, a_{n+1} = a_n - 12\)  
16. \(a_n = \frac{1}{3}n + 2 \frac{2}{3}; a_1 = 3, a_{n+1} = a_n + \frac{1}{3}\)

17. Graph (1): (1, 3), (2, 8), (3, 13), (4, 18), (5, 23); slope = 5  
Graph (2): (1, 2), (2, -1), (3, -4), (4, -7), (5, -10); slope = -3

18. The slope of the line containing the points is the same as the common difference of the sequence.
GEOMETRIC SEQUENCES

In the sequence 2, 6, 18, 54, …, there is a common ratio \( r = 3 \) between the successive terms and this is called an geometric sequence. There are two common methods to define a geometric sequence. The explicit formula tells you exactly how to find any specific term in the sequence. The recursive formula gives first term and how to get from one term to the next. Formally, for geometric sequences, this is written:

Explicit: \( a_n = a_1 \cdot r^{n-1} \) where \( n = \text{term number} \) and \( r = \text{common ratio} \).

Recursive: \( a_1 = \text{some specific value} \) and \( a_{n+1} = a_n \cdot r \) where \( r = \text{common ratio} \).

For the sequence 2, 6, 18, 54, …, the explicit formula is \( a_n = a_1 \cdot r^{n-1} = 2 \cdot 3^{n-1} \), and the recursive formula is \( a_1 = 2, a_{n+1} = a_n \cdot 3 \). In each case, successively replacing \( n \) by 1, 2, 3, … will yield the terms of the sequence. See the examples below.

Example 1

List the first five terms of the geometric sequence.
\[
a_n = 3 \cdot 2^{n-1} \quad \text{(an explicit formula)}
\]

\[
\begin{align*}
a_1 &= 3 \cdot 2^{1-1} = 3 \cdot 2^0 = 3 \\
a_2 &= 3 \cdot 2^{2-1} = 3 \cdot 2^1 = 6 \\
a_3 &= 3 \cdot 2^{3-1} = 3 \cdot 2^2 = 12 \\
a_4 &= 3 \cdot 2^{4-1} = 3 \cdot 2^3 = 24 \\
a_5 &= 3 \cdot 2^{5-1} = 3 \cdot 2^4 = 48
\end{align*}
\]

The sequence is: 3, 6, 12, 24, 48, …

Example 2

List the first five terms of the geometric sequence.
\[
b_1 = 8 \\
b_{n+1} = b_n \cdot \frac{1}{2} \quad \text{(a recursive formula)}
\]

\[
\begin{align*}
b_1 &= 8 \\
b_2 &= b_1 \cdot \frac{1}{2} = 8 \cdot \frac{1}{2} = 4 \\
b_3 &= b_2 \cdot \frac{1}{2} = 4 \cdot \frac{1}{2} = 2 \\
b_4 &= b_3 \cdot \frac{1}{2} = 2 \cdot \frac{1}{2} = 1 \\
b_5 &= b_4 \cdot \frac{1}{2} = 1 \cdot \frac{1}{2} = \frac{1}{2}
\end{align*}
\]

The sequence is: 8, 4, 2, 1, \( \frac{1}{2} \), …

Example 3

Write an explicit and a recursive equation for the sequence: 81, 27, 9, 3, …

Explicit: \( a_1 = 81, r = \frac{1}{3} \) so the equation is \( a_n = a_1 \cdot r^{n-1} = 81 \cdot \left( \frac{1}{3} \right)^{n-1} \).

Recursive: \( a_1 = 81, r = \frac{1}{3} \) so the equation is \( a_1 = 81, a_{n+1} = a_n \cdot \frac{1}{3} \).
Problems

List the first five terms of each geometric sequence.

1. \( a_n = 5 \cdot 2^{n-1} \)
2. \( b_n = -3 \cdot 3^{n-1} \)
3. \( a_n = 40\left(\frac{1}{2}\right)^{n-1} \)
4. \( c_n = 6\left(-\frac{1}{2}\right)^{n-1} \)
5. \( a_1 = 5, a_{n+1} = a_n \cdot 3 \)
6. \( a_1 = 100, a_{n+1} = a_n \cdot \frac{1}{2} \)
7. \( a_1 = -3, a_{n+1} = a_n \cdot (-2) \)
8. \( a_1 = \frac{1}{3}, a_{n+1} = a_n \cdot \frac{1}{2} \)

Find the 15\textsuperscript{th} term of each geometric sequence.

9. \( b_{14} = 232, r = 2 \)
10. \( b_{16} = 32, r = 2 \)
11. \( a_{14} = 9, r = \frac{2}{3} \)
12. \( a_{16} = 9, r = \frac{2}{3} \)

Find an explicit and a recursive formula for each geometric sequence.

13. \( 2, 10, 50, 250, 1250, \ldots \) 14. \( 16, 4, 1, \frac{1}{4}, \frac{1}{16}, \ldots \)
15. \( 5, 15, 45, 135, 405, \ldots \) 16. \( 3, -6, 12, -24, 48, \ldots \)

17. Graph the sequences from problems 1 and 14.

18. How are the graphs of geometric sequences different from arithmetic sequences?

Answers

1. \( 5, 10, 20, 40, 80 \) 2. \( -3, -9, -27, -81, -243 \) 3. \( 40, 20, 10, 5, \frac{5}{2} \)
4. \( 6, -3, \frac{3}{2}, -\frac{3}{4}, \frac{3}{8} \) 5. \( 5, 15, 45, 135, 405 \) 6. \( 100, 50, 25, \frac{25}{2}, \frac{25}{4} \)
7. \( -3, 6, -12, 24, -48 \) 8. \( \frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48} \) 9. \( 464 \)
10. \( 16 \) 11. \( 6 \) 12. \( \frac{27}{2} \)
13. \( a_n = 2 \cdot 5^{n-1}, a_1 = 2, a_{n+1} = a_n \cdot 5 \) 14. \( a_n = 16 \cdot \left(\frac{1}{4}\right)^{n-1}, a_1 = 16, a_{n+1} = a_n \cdot \frac{1}{4} \)
15. \( a_n = 5 \cdot 3^{n-1}, a_1 = 5, a_{n+1} = a_n \cdot 3 \) 16. \( a_n = 3 \cdot (-2)^{n-1}, a_1 = 3, a_{n+1} = a_n \cdot (-2) \)
17. Graph (1): \( (1, 5), (2, 10), (3, 20), (4, 40), (5, 80) \) Graph (14): \( (1, 16), (2, 4), (3, 1), (4, \frac{1}{4}), (5, \frac{1}{16}) \)
18. Arithmetic sequences are linear and geometric sequences are curved (exponential).
An exponential function has an equation of the form $y = ab^x$ (with $b \geq 0$).

In many situations “$a$” represents a starting or initial value, “$b$” represents the multiplier or growth/decay factor, and “$x$” represents the time.

**Example 1**

Graph $y = 3 \cdot 2^x$.

Make a table of values.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>1.5</td>
<td>3</td>
<td>6</td>
<td>12</td>
<td>24</td>
</tr>
</tbody>
</table>

Plot the points and connect them to form a smooth curve.

$y = 3 \cdot 2^x$

This is called an increasing exponential curve.

**Example 2**

Graph $y = 2(0.75)^x$.

Make a table of values using a calculator.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>2.7</td>
<td>2</td>
<td>1.5</td>
<td>1.1</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Plot the points and connect them to form a smooth curve.

$y = 2(0.75)^x$

This is called a decreasing exponential curve.

**Example 3**

Movie tickets now average $9.75 a ticket, but are increasing 15% per year. How much will they cost 5 years from now?

The equation to use is: $y = ab^x$. The initial value $a = 9.75$. The multiplier $b$ is always found by adding the percent increase (as a decimal) to the number 1 (100%), so $b = 1 + 0.15 = 1.15$. The time is $x = 5$.

Substituting into the equation and using a calculator for the calculations:

$y = ab^x = 9.75(1.15)^5 = 19.61$. In five years movie tickets will average about $19.61.

**Example 4**

A powerful computer is $2000, but on the average loses 20% of its value each year. How much will it be worth 4 years from now?

The equation to use is: $y = ab^x$. The initial value $a = 2000$. In this case the value is decreasing so multiplier $b$ is always found by subtracting the percent decrease from the number 1 (100%), so $b = 1 - 0.2 = 0.8$. The time is $x = 4$.

Substituting into the equation and using a calculator for the calculations:

$y = ab^x = 2000(0.8)^4 = 819.2$. In four years the computer will only be worth $819.20.
**Example 5**

Dinner at your grandfather’s favorite restaurant now costs $25.25 and has been increasing steadily at 4% per year. How much did it cost 35 years ago when he was courting your grandmother?

The equation is the same as above and $a = 25.56$, $b = 1.04$, but since we want to go back in time, $x = -35$. A common mistake is to think that $b = 0.96$. The equation is $y = ab^x = 25.25(1.04)^{-35} = 6.40$.

**Example 6**

If a gallon of milk costs $3 now and the price is increasing 10% per year, how long before milk costs $10 a gallon?

In this case we know the starting value $a = 3$, the multiplier $b = 1.1$, the final value $y = 10$, but not the time $x$. Substituting into the equation we get $3(1.1)^x = 10$. To solve this, you will probably need to guess and check with your calculator. Doing so yields $x ≈ 12.6$ years. In Algebra 2 you will learn to solve these equations without guess and check.

**Problems**

Make a table of values and draw a graph of each exponential function.

1. $y = 4(0.5)^x$
2. $y = 2(3)^x$
3. $y = 5(1.2)^x$
4. $y = 10(\frac{2}{3})^x$

5. The number of bacteria present in a colony at 12 noon is 180 and the bacteria grows at a rate of 22% per hour. How many will be present at 8 p.m.?

6. A house purchased for $226,000 has lost 4% of its value each year for the past five years. What is it worth now?

7. A 1970 comic book has appreciated 10% per year and originally sold for $0.35. What will it be worth in 2010?

8. A certain car depreciates at 15% per year. Six years ago it was purchased for $21,000. What is it worth now?

9. Inflation is at a rate of 7% per year. Today Janelle’s favorite bread costs $3.79. What would it have cost ten years ago?

10. Ryan’s motorcycle is now worth $2500. It has decreased in value 12% each year since it was purchased. If he bought it four years ago, what did it cost new?

11. The cost of a high definition television now averages $1200, but the cost is decreasing about 15% per year. In how many years will the cost be under $500?

12. A two-bedroom house in Nashville is worth $110,000. If it appreciates at 2.5% per year, when will it be worth $200,000?
13. Last year the principal’s car was worth $28,000. Next year it will be worth $25,270. What is the annual rate of depreciation? What is the car worth now?

14. A concert has been sold out for weeks, and as the date of the concert draws closer, the price of the ticket increases. The cost of a pair of tickets was $150 yesterday and is $162 today. Assuming that the cost continues to increase at this rate:
   a. What is the daily rate of increase? What is the multiplier?
   b. What will be the cost one week from now, the day before the concert?
   c. What was the cost two weeks ago?

Answers

1. 

<table>
<thead>
<tr>
<th>x</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
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<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>16</td>
<td>8</td>
<td>4</td>
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<td>1</td>
<td>1/2</td>
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2. 

<table>
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<td>2/3</td>
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<td>6</td>
<td>18</td>
<td>54</td>
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</table>

3. 

<table>
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<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
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<td>5</td>
<td>6</td>
<td>7.2</td>
<td>8.64</td>
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</tbody>
</table>

4. 

<table>
<thead>
<tr>
<th>x</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>15</td>
<td>10</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

5. $883
6. $184,274
7. $15.84
8. $7920
9. $1.92
10. $4169
11. $5, $26,600
12. a. 8%, 1.08
13. b. $277.64
14. c. $55.15
SOLVING MIXED EQUATIONS AND INEQUALITIES

Problems

Solve these various types of equations.

1. \(2(x - 3) + 2 = -4\)  
2. \(6 - 12x = 108\)  
3. \(3x - 11 = 0\)

4. \(0 = 2x - 5\)  
5. \(y = 2x - 3\)  
6. \(ax - b = 0\)  
\(x + y = 15\)  
(solve for \(x\))

7. \(0 = (2x - 5)(x + 3)\)  
8. \(2(2x - 1) = -x + 5\)  
9. \(x^2 + 5^2 = 13^2\)

10. \(2x + 1 = 7x - 15\)  
11. \(\frac{5-2x}{3} = \frac{x}{5}\)  
12. \(2x - 3y + 9 = 0\)  
(solve for \(y\))

13. \(x^2 + 5x + 6 = 0\)  
14. \(x^2 = y\)  
15. \(x - y = 7\)  
\(100 = y\)  
\(y = 2x - 1\)

16. \(x^2 - 4x = 0\)  
17. \(x^2 - 6 = -2\)  
18. \(\frac{x}{2} + \frac{x}{3} = 2\)

19. \(x^2 + 7x + 9 = 3\)  
20. \(y = x + 3\)  
\(x + 2y = 3\)  
21. \(3x^2 + 7x + 2 = 0\)

22. \(\frac{x}{x+1} = \frac{5}{7}\)  
23. \(x^2 + 2x - 4 = 0\)  
24. \(\frac{1}{x} + \frac{1}{3x} = 2\)

25. \(3x + y = 5\)  
\(x - y = 11\)  
26. \(y = -\frac{3}{4} x + 4\)  
\(\frac{1}{4} x - y = 8\)  
27. \(3x^2 = 8x\)

28. \(|x| = 4\)  
29. \(\frac{2}{7}x + 1 = \frac{1}{2} x - 3\)  
30. \(x^2 - 4x = 5\)

31. \(3x + 5y = 15\)  
\(\text{(solve for } y\text{)}\)  
32. \((3x)^2 + x^2 = 15^2\)  
33. \(y = 11\)  
\(y = 2x^2 + 3x - 9\)

34. \((x + 2)(x + 3)(x - 4) = 0\)  
35. \(|x + 6| = 8\)  
36. \(2(x + 3) = y + 2\)  
\(y + 2 = 8x\)

37. \(2x + 3y = 13\)  
\(x - 2y = -11\)  
38. \(2x^2 = -x + 7\)  
39. \(1 - \frac{5}{6x} = \frac{5}{6}\)

40. \(\frac{x-1}{5} = \frac{3}{x+1}\)  
41. \(\sqrt{2x+1} = 5\)  
42. \(2|x-2| + 3 = 7\)

43. \(\sqrt{3x-1} + 1 = 7\)  
44. \((x + 3)^2 = 49\)  
45. \(\frac{4x-1}{x-1} = x + 1\)
Solve these various types of inequalities.

46. \(4x - 2 \leq 6\) 
47. \(4 - 3(x + 2) \geq 19\) 
48. \(\frac{x}{2} > \frac{3}{7}\)

49. \(3(x + 2) \geq -9\)
50. \(-\frac{2}{3}x < 6\)
51. \(y < 2x - 3\)

52. \(|x| > 4\)
53. \(x^2 - 6x + 8 \leq 0\)
54. \(|x + 3| > 5\)

55. \(2x^2 - 4x \geq 0\)
56. \(y \leq -\frac{2}{3}x + 2\)
57. \(y < -x + 2\)

58. \(|2x - 1| \leq 9\)
59. \(5 - 3(x - 1) \geq -x + 2\)
60. \(y \leq 4x + 16\)

Answers

1. \(x = 0\)
2. \(x = -8.5\)
3. \(x = \frac{11}{3}\)
4. \(x = \frac{5}{2}\)

5. \((6, 9)\)
6. \(x = \frac{b}{a}\)
7. \(x = \frac{5}{2}, -3\)
8. \(x = \frac{7}{3}\)

9. \(x = \pm 12\)
10. \(x = \frac{16}{5}\)
11. \(x = \frac{25}{13}\)
12. \(y = \frac{2}{3}x + 3\)

13. \(x = -2, -3\)
14. \((\pm 10, 100)\)
15. \((-6, -13)\)
16. \(x = 0, 4\)

17. \(x = \pm 2\)
18. \(x = \frac{12}{5}\)
19. \(x = -1, -6\)
20. \((-1, 2)\)

21. \(x = -\frac{1}{3}, -2\)
22. \(x = \frac{5}{2}\)
23. \(x = \frac{-2 + \sqrt{20}}{2}\)
24. \(x = \frac{2}{3}\)

25. \((4, -7)\)
26. \((12, -5)\)
27. \(x = 0, \frac{8}{3}\)
28. \(x = \pm 4\)

29. \(x = -24\)
30. \(x = 5, -1\)
31. \(y = -\frac{3}{5}x + 3\)
32. \(x \approx \pm 4.74\)

33. \((-4, 11)\) and \((\frac{5}{2}, 11)\)
34. \(x = -2, -3, 4\)
35. \(x = 2, -14\)

36. \((1, 6)\)
37. \((-1, 5)\)
38. \(x = \frac{1 + \sqrt{57}}{4}\)
39. \(x = 1, 5\)

40. \(x = \pm 4\)
41. \(x = 12\)
42. \(x = \frac{3}{2}, -\frac{1}{2}\)
43. \(x = \frac{32}{5}\)

44. \(x = 4, -10\)
45. \(x = 0, 4\)
46. \(x \leq 2\)
47. \(x \leq -7\)

48. \(x > \frac{6}{7}\)
49. \(x \geq -5\)
50. \(x > -9\)
51. \(\text{See next page.}\)

52. \(x > 4, x < -4\)
53. \(2 \leq x \leq 4\)
54. \(x > 2 \text{ or } x < -8\)
55. \(x \leq 0 \text{ or } x \geq 2\)

56. \(\text{See next page.}\)
57. \(\text{See next page.}\)
58. \(-4 \leq x \leq 5\)
59. \(x \leq 3\)

60. \(\text{See next page.}\)