ARITHMETIC SEQUENCES

An ordered list of numbers such as: 4, 9, 16, 25, 36… is a sequence. Each number in the sequence is a term. Usually variables with subscripts are used to label terms. For example, in the sequence above, the first term is 4 and the third term is 16. This might be written \( a_1 = 4 \) and \( a_3 = 16 \) where \( a \) is the variable used to label the sequence.

In the sequence 1, 5, 9, 13, …, there is a common difference \( (d = 4) \) between the successive terms and this is called an arithmetic sequence. There are two common methods to define a sequence. An explicit formula tells you exactly how to find any specific term in the sequence. A recursive formula tells first term and how to get from one term to the next. Formally, for arithmetic sequences, this is written:

Explicit: \( a_n = a_1 + (n - 1)d \) where \( n \) = term number and \( d \) = common difference.

Recursive: \( a_1 = \text{some specific value}, a_{n+1} = a_n + d \), and \( d \) = common difference.

For the sequence 1, 5, 9, 13, …, the explicit formula is: \( a_n = 1 + (n - 1)(4) = 4n - 3 \) and the recursive formula is: \( a_1 = 1, a_{n+1} = a_n + 4 \). In each case, successively replacing \( n \) by 1, 2, 3, … will yield the terms of the sequence. See the examples below.

Example 1

List the first five terms of the arithmetic sequence.

\( a_n = 5n + 2 \) (an explicit formula)

\[ a_1 = 5(1) + 2 = 7 \]
\[ a_2 = 5(2) + 2 = 12 \]
\[ a_3 = 5(3) + 2 = 17 \]
\[ a_4 = 5(4) + 2 = 22 \]
\[ a_5 = 5(5) + 2 = 27 \]

The sequence is: 7, 12, 17, 22, 27, …

Example 2

List the first five terms of the arithmetic sequence.

\[ b_1 = 3 \]
\[ b_{n+1} = b_n - 5 \] (A recursive formula)

\[ b_1 = 3 \]
\[ b_2 = b_1 - 5 = 3 - 5 = -2 \]
\[ b_3 = b_2 - 5 = -2 - 5 = -7 \]
\[ b_4 = b_3 - 5 = -7 - 5 = -12 \]
\[ b_5 = b_4 - 5 = -12 - 5 = -17 \]

The sequence is: 3, –2, –7, –12, –17, …

Example 3

Write an explicit and a recursive equation for the sequence: –2, 1, 4, 7, …

Explicit: \( a_1 = -2, d = 3 \) so the equation is \( a_n = a_1 + (n - 1)d = -2 + (n - 1)(3) = 3n - 5 \)

Recursive: \( a_1 = -2, d = 3 \) so the equation is \( a_1 = -2, a_{n+1} = a_n + 3 \).
Problems

List the first five terms of each arithmetic sequence.

1. \( a_n = 5n - 2 \)
2. \( b_n = -3n + 5 \)
3. \( a_n = -15 + \frac{1}{2}n \)
4. \( c_n = 5 + 3(n - 1) \)
5. \( a_1 = 5, a_{n+1} = a_n + 3 \)
6. \( a_1 = 5, a_{n+1} = a_n - 3 \)
7. \( a_1 = -3, a_{n+1} = a_n + 6 \)
8. \( a_1 = \frac{1}{3}, a_{n+1} = a_n + \frac{1}{2} \)

Find the 30\(^{th}\) term of each arithmetic sequence.

9. \( a_n = 5n - 2 \)
10. \( a_n = -15 + \frac{1}{2}n \)
11. \( a_{31} = 53, \ d = 5 \)
12. \( a_1 = 25, a_{n+1} = a_n - 3 \)

For each arithmetic sequence, find an explicit and a recursive formula.

13. \( 4, 8, 12, 16, 20, \ldots \)
14. \( -2, 5, 12, 19, 26, \ldots \)
15. \( 27, 15, 3, -9, -21, \ldots \)
16. \( 3, 3 \frac{1}{3}, 3 \frac{2}{3}, 4, 4 \frac{1}{3}, \ldots \)

Sequences are graphed using points of the form: (term number, term value). For example, the sequence 4, 9, 16, 25, 36, … would be graphed by plotting the points (1, 4), (2, 9), (3, 16), (4, 25), (5, 36), …. Sequences are graphed as points and not connected.

17. Graph the sequences from problems 1 and 2 above and determine the slope of each line.
18. How do the slopes of the lines found in the previous problem relate to the sequences?

Answers

1. 3, 8, 13, 18, 23
2. 2, -1, -4, -7, -10
3. -14 \( \frac{1}{2} \), -14, -13 \( \frac{1}{2} \), -13, -12 \( \frac{1}{2} \)
4. 5, 8, 11, 14, 17
5. 5, 8, 11, 14, 17
6. 5, 2, -1, -4, -7
7. -3, 3, 9, 15, 21
8. \( \frac{1}{3}, \frac{5}{6}, 1 \frac{1}{3}, 1 \frac{5}{6}, 2 \frac{1}{3} \)
9. 148
10. 0
11. 48
12. -62
13. \( a_n = 4n; \ a_1 = 4, a_{n+1} = a_n + 4 \)
14. \( a_n = 7n - 9; \ a_1 = -2, a_{n+1} = a_n + 7 \)
15. \( a_n = -12n + 39; \ a_1 = 27, a_{n+1} = a_n - 12 \)
16. \( a_n = \frac{1}{3}n + 2 \frac{2}{3}; \ a_1 = 3, a_{n+1} = a_n + \frac{1}{3} \)
17. Graph (1): (1, 3), (2, 8), (3, 13), (4, 18), (5, 23); slope = 5
   Graph (2): (1, 2), (2, -1), (3, -4), (4, -7), (5, -10); slope = -3
18. The slope of the line containing the points is the same as the common difference of the sequence.