Students revisit circles in the first part of Chapter 10 to develop “circle tools,” which will help them find lengths and angle measures within circles. In addition to working with the lengths of the radius and diameter of a circle, they will gain information about angles, arcs, and chords. As with the development of many topics we have studied, triangles will again be utilized.

See the Math Notes boxes in Lessons 10.1.1, 10.1.2, 10.1.3, 10.1.4, and 10.1.5.

Example 1

In the circle at right are two chords, $\overline{AB}$ and $\overline{CD}$. Find the center of the circle and label it $P$.

The chords of a circle (segments with endpoints on the circle) are useful segments. In particular, the diameter is a special chord that passes through the center. The perpendicular bisectors of the chords pass through the center of the circle as well. Therefore to find the center, we will find the perpendicular bisectors of each segment. They will meet at the center.

There are several ways to find the perpendicular bisectors of the segments. A quick way is to fold the paper so that the endpoints of the chords come together. The crease will be perpendicular to the chord and bisect it. Another method is to use the construction we learned last chapter. In either case, point $P$ is the center of the circle.
Example 2

In \( \bigcirc O \) at right, use the given information to find the values of \( x \), \( y \), and \( z \).

Pieces of a circle are called arcs, and every arc breaks the circle into two pieces. The large piece is called a major arc, and the smaller piece is called a minor arc. Arcs have lengths, and we found lengths of arcs by finding a fraction of the circumference. But arcs also have measures based on the measure of the corresponding central angle. In the picture at right, \( \angle JOE \) is a central angle since its vertex is at the center, \( O \). An arc’s measure is the same as its central angle. Since \( JE = 100^\circ \), \( x = 100^\circ \).

An angle with its vertex on the circle is called an inscribed angle. Both of the angles \( y \) and \( z \) are inscribed angles. Inscribed angles measure half of their intercepted arc (in this case, \( JE \)). Therefore, \( y = z = \frac{1}{2} (100^\circ) = 50^\circ \).

Example 3

In the figure at right, \( O \) is the center of the circle. \( \overline{TX} \) and \( \overline{TB} \) are tangent to \( \bigcirc O \), and \( m \angle BOX = 120^\circ \). Find the \( m \angle BTX \).

If a line is tangent to a circle, that line intersects the circle in only one point. Also, a radius drawn to the point of tangency is perpendicular to the tangent line. Therefore we know that \( \overline{OB} \perp \overline{BT} \) and \( \overline{OX} \perp \overline{XT} \). At this point there are different ways to solve this problem. One way is to add a segment to the picture. Adding \( \overline{OT} \) will create two triangles, and we know a lot of information about triangles. In fact, these two right triangles are congruent by \( \text{HL} \equiv (\overline{OB} \equiv \overline{OX} \text{ because they are both radii, and } \overline{OT} \equiv \overline{OT}) \). Since the corresponding parts of congruent triangles are also congruent, and \( m \angle BOX = 120^\circ \), we know that \( m \angle BOT = m \angle XOT = 60^\circ \). Using the sum of the angles of a triangle is \( 180^\circ \), we find \( m \angle BTO = m \angle XTO = 30^\circ \). Therefore, \( m \angle BTX = 60^\circ \).

An alternate solution is to note that the two right angles at points \( B \) and \( X \), added to \( \angle BOX \), make \( 300^\circ \). Since we know that the angles in a quadrilateral sum to \( 360^\circ \), \( m \angle BTX = 360^\circ - 300^\circ = 60^\circ \).
**Example 4**

In the circle at right, $DV = 9$ units, $SV = 12$ units, and $AV = 4$ units. Find the length of $IV$.

Although we have been concentrating on angles and their measures, there are some facts about lengths of chords of circles that are useful (and should be part of your “circle tools”). In the figure above, if we drew $SI$ and $DA$ we would form two similar triangles. (See the Math Notes box in Lesson 10.1.4.) The sides of similar triangles are proportional, so we can write the proportion at right, which leads to the simplified equation with the two products.

Substitute the lengths that we know, then solve the equation.

$\frac{SV}{DV} = \frac{IV}{AV}$

$SV \cdot AV = DV \cdot IV$

$
\begin{align*}
SV &= 12 \\
AV &= 4 \\
DV &= 9 \\
IV &= 48 \\
IV &= 5.33 \text{ units}
\end{align*}
$

**Problems**

Find each measure in $\odot P$ if $m\angle WPX = 28^\circ$, $m\angle ZPY = 38^\circ$, and $\overline{WZ}$ and $\overline{XV}$ are diameters.

1. $m\overline{YZ}$
2. $m\overline{WX}$
3. $m\angle VPZ$
4. $m\overline{WVX}$
5. $m\angle XPY$
6. $m\overline{XY}$
7. $m\overline{XWY}$
8. $m\overline{WZX}$

In each of the following figures, $O$ is the center of the circle. Calculate the value of $x$ and justify your answer.

9.
10.
11.
12.

13.
14.
15.
16.

17.
18.
19.
20.
In \( \odot O \), \( m\widehat{WT} = 86^\circ \) and \( m\widehat{EA} = 62^\circ \).

21. Find \( m\angle EWA \).
22. Find \( m\angle WET \).
23. Find \( m\angle WES \).
24. Find \( m\angle WST \).

In \( \odot O \), \( m\angle EWA = 36^\circ \) and \( m\angle WST = 42^\circ \).

25. Find \( m\angle WES \).
26. Find \( m\angle TW \).
27. Find \( m\angle EA \).
28. Find \( m\angle TKE \).

29. In the figure at right, \( m\widehat{SD} = 92^\circ \), \( m\widehat{DA} = 103^\circ \), \( m\widehat{AI} = 41^\circ \) and \( \overline{SW} \) is tangent to \( \odot O \). Find \( m\angle AKD \) and \( m\angle VAS \).

30. In the figure at right, \( m\angle KE = 43^\circ \), \( \overline{EW} \equiv \overline{KW} \), and \( \overline{ST} \) is tangent to \( \odot O \). Find \( m\angle WEO \) and \( m\angle SEW \).
Answers

1. 38°  2. 28°  3. 28°  4. 180°  5. 114°
6. 114°  7. 246°  8. 332°  9. 68°  10. 73°
11. 98°  12. 124°  13. 50°  14. 55°  15. 18°
16. 27°  17. 55°  18. 77°  19. 35°  20. 50°
21. \(\frac{1}{2}(62°) = 31°\)  22. \(\frac{1}{2}(86°) = 43°\)
25. 180° – 36° – 42° = 102°  26. \(m\angle TEW = 180° – 102° = 78°, 2(78°) = 156°\)
27. 2(36°) = 72°  28. 180° – 36° – 78° = 66°
29. \(m\angle SAD = \frac{1}{2}(92°), m\angle IDA = \frac{1}{2}(41°), 180° – 46° – 20.5° = 113.5°,\)
    \(m\angle VAS = 180° – 46° = 134°\)
30. \(m\angle EWK = \frac{1}{2}(43°) = 21.5°, m\angle EOK = 43°, \) so 317° remain for the other angle at \(O\).
    \(m\angle WEO = m\angle WKO\) and for \(WEO\), 360° – 21.5° – 317° = 21.5° = \(m\angle WEO + m\angle WKO\),
    so \(m\angle WEO = \frac{1}{2}(21.5°) = 10.75°. m\angle SEO = 90°, m\angle WEO = 10.75°, \) so \(m\angle SEW = 79.25°.\)