Students take on challenging problems using the Fundamental Principle of Counting, permutations, and combinations to compute probabilities. These techniques are essential when the sample space is too large to model or to count.

See the Math Notes boxes in Lessons 10.3.1, 10.3.2, 10.3.3, and 10.3.5.

Example 1

Twenty-three people have entered the pie-eating contest at the county fair. The first place pie-eater (the person eating the most pies in fifteen minutes) wins a pie each week for a year. Second place will receive new baking ware to make his/her own pies, and third place will receive the *Sky High Pies* recipe book. How many different possible top finishers are there?

Since the prizes are different for first, second, and third place, the order of the top finishers matters. We can use a decision chart to determine the number of ways we can have winners. How many different people can come in first? Twenty-three. Once first place is “chosen” (i.e., removed from the list of contenders) how many people are left to take second place? Twenty-two. This leaves twenty-one possible third place finishers. Just as with the branches on the tree diagram, multiply these numbers to determine the number of arrangements: 

\[(23)(22)(21) = 10,626.\]

Example 2

Fifteen students are participating in a photo-shoot for a layout in the new journal *Mathmaticious*. In how many ways can you arrange:

a. Eight of them?  

b. Two of them?  

c. Fifteen of them?

We can use a decision chart for each of these situations, but there is another, more efficient method for answering these questions. An arrangement of items where order matters is called a permutation, and in this case, since changing the order of the students changes the layout, the order matters.

With a permutation, you need to know the total number things to be arranged (in this case \(n = 15\) students) and how many will be taken (\(r\)) at a time. The formula for a permutation is 

\[nPr = \frac{n!}{(n-r)!}.\]

In part (a), we have 15 students taken 8 at a time.

The number of permutations is: 

\[15P_8 = \frac{15!}{(15-8)!} = \frac{15!}{7!} = 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = 259,459,200\]
In part (b) the solution becomes: \[ \begin{align*} 15P_2 &= \frac{15!}{(15-2)!} = \frac{15!}{13!} = 15 \cdot 14 = 210 \end{align*} \]

Part (c) poses a new problem: \[ \begin{align*} 15P_{15} &= \frac{15!}{(15-15)!} = \frac{15!}{0!} \]

What is 0! ? “Factorial” means to calculate the product of the integers from the given value down to one. How can we compute 0! ? If it equals zero, we have a problem because part (c) would not have an answer (dividing by zero is undefined). But this situation must have an answer. In fact, if we used a decision chart to determine how many ways the 15 people can line up, we would find that there are 15! arrangements. Therefore, if \( 15P_{15} = 15! \) and 0! = 1. This is another case of mathematicians defining elements of mathematics to fit their needs. 0! is defined to equal 1 so that other mathematics makes sense.

**Example 3**

In the annual homecoming parade, three students get to ride on the lead float. Seven students are being considered for this coveted position. How many ways can three students be chosen for this honor?

All three students who are selected will ride on the lead float, but whether they are the first, second, or third student selected does not matter. In a case where the order of the selections does not matter, the situation is called a **combination**. This means that if the students were labeled A, B, C, D, E, and F, choosing A, B, and then C would be essentially the same as choosing B, C, and then A. In fact, all the arrangements of A, B, and C could be lumped together. This makes the number of combinations much smaller than the number of permutations. The symbol for a combination is \( \binom{n}{r} \), where \( n \) is the total number of items under consideration, and \( r \) is the number of items we will choose. It is often read as “\( n \) choose \( r \).” In this problem we have \( \binom{7}{3} \), 7 choose 3. The formula is similar to the formula for a permutation, but we must divide out the similar groups.

\[ \binom{n}{r} = \frac{n!}{(n-r)!r!} \]

Here we have: \[ \binom{7}{3} = \frac{7!}{(7-3)!3!} = \frac{7!}{4!3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 35 \]
Problems

Simplify the following expressions.

1. \(10!\)  
2. \(\frac{10!}{3!}\)  
3. \(\frac{35!}{30!}\)  
4. \(\frac{88!}{87!}\)

5. \(\frac{72!}{70!}\)  
6. \(\frac{65!}{62!3!}\)  
7. \(8 P_2\)  
8. \(15 P_0\)

9. \(9 P_9\)  
10. \(12 C_4\)  
11. \(5 C_0\)  
12. \(32 C_{32}\)

Solve the following problems.

13. How many ways can you arrange the letters from the word “KAREN”?

14. How many ways can you arrange the letters from the word “KAREN” if you want the arrangement to begin with a vowel?

15. All standard license plates in Alaska start with three letters followed by three digits. If repetition is allowed, how many different license plates are there?

16. For $3.99, The Creamery Ice Cream Parlor will put any three different flavored scoops, out of their 25 flavors of ice cream, into a bowl. How many different “bowls” are there? (Note: A bowl of chocolate, strawberry, and vanilla is the same bowl as a bowl of chocolate, vanilla, and strawberry.)

17. Suppose those same three scoops of ice cream are on a cone. Now how many arrangements are there? (Note: Ice cream on a cone must be eaten “top down” because you cannot eat the bottom or middle scoop out, keeping the cone intact.)

18. A normal deck of playing cards contains 52 cards. How many five-card poker hands can be made?

19. How many ways are there to make a full house (three of one kind, two of another)?

20. What is the probability of getting a full house (three of one kind of card and two of another)? Assume a standard deck and no wild cards.

For problems 21–25, a bag contains 36 marbles. There are twelve blue marbles, eight red marbles, seven green marbles, five yellow marbles, and four white marbles. Without looking, you reach into the bag and pull out eight marbles. What is the probability you pull out:

21. All blue marbles?  
22. Four blue and four white marbles?

23. Seven green and one yellow marble?  
24. At least one red and at least two yellow?

25. No blue marbles?
Answers

1. 3,628,800  
2. 604,800  
3. 38,955,840  
4. 88

5. 5,112  
6. 43,680  
7. 56  
8. 1

9. 362,880  
10. 495  
11. 1  
12. 1

13. $5! = 120$  
14. $2(4!) = 48$  
15. $(26)(26)(26)(10)(10)(10) = 17,576,000$

16. $25 \binom{3}{} = 2300$

17. $25 \binom{3}{} = 13,800$ (On a cone, order matters!)

18. $52 \binom{5}{} = 2,598,960$

19. This is tricky and tough! There are 13 different “types” of cards: twos, threes, fours, ..., Jacks, Queens, Kings, and Aces. We need to choose which of the 13 we want three of ($\binom{13}{3}$). Once we choose what type (for example, we pick Jacks) then we need to choose which three out of the four to take ($\binom{4}{3}$). Then from the remaining 12 types, we choose which type to have two of ($\binom{12}{2}$). Then again we need to choose which two out of the four ($\binom{4}{2}$). This gives us $\binom{13}{3}\cdot\binom{4}{3}\cdot\binom{12}{2}\cdot\binom{4}{2} = 3,744$.

20. We already calculated the numbers we need in problems 18 and 19 so: $\frac{3,744}{2,598,960} \approx 0.0014$.

21. Each time we reach in and pull out 8 marbles, order does not matter. The number of ways to do this is $\binom{36}{8}$. This is the number in the sample space, i.e., the denominator. How many ways can we pull out all blue? $\binom{12}{8}$. Therefore the probability is $\frac{\binom{12}{8}}{\binom{36}{8}} = 0.0000164$.

22. Same denominator. Now we want to choose 4 from the 12 blue, $\binom{12}{4}$, and 4 from the 4 whites, $\binom{4}{4}$. $\frac{\binom{12}{4}\cdot\binom{4}{4}}{\binom{36}{8}} = 0.0000164$, the same answer!

23. Seven green: $\binom{7}{1}$, one yellow: $\binom{5}{1}$. $\frac{\binom{7}{1}\cdot\binom{5}{1}}{\binom{36}{8}} = 0.0000001652$

24. Here we have to get at least one red: $\binom{8}{1}$, and at least two yellow: $\binom{5}{2}$, but the other five marbles can come from the rest of the pot: $\binom{33}{5}$. Therefore, $\frac{\binom{8}{1}\cdot\binom{5}{2}\cdot\binom{33}{5}}{\binom{36}{8}} \approx 0.627$.

25. To get no blue marbles means we want all eight from the other 24 non-blue marbles. $\frac{\binom{24}{8}}{\binom{36}{8}} = 0.0243$. 

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