THE EQUATION OF A CIRCLE

Students have examined the parts of circles, found measurements of their circumferences areas, chords, secants, tangents, arcs, and angles, and have used circles with probability and expected value. This section places the circle on a coordinate graph so that the students can derive the equation of a circle.

See the Math Notes box in Lesson 12.1.3.

Example 1

What is the equation of the circle centered at the origin with a radius of 5 units?

The key to deriving the equation of this circle is the Pythagorean Theorem. That means we will need to create a right triangle within the circle. First, draw the circle on graph paper, then choose any point on the circle. We do not know the exact coordinates of this point so call it (x, y). Since endpoints of the radius are (0, 0) and (x, y), we can represent the length of the vertical leg as y and the length of the horizontal leg as x. If we call the radius r, then using the Pythagorean Theorem we can write $x^2 + y^2 = r^2$. Since we know the radius is 5, we can write the equation of this circle as $x^2 + y^2 = 5^2$, or $x^2 + y^2 = 25$.

Example 2

Graph the circle $(x - 4)^2 + (y + 2)^2 = 49$.

Based on what we have seen, this is a circle with a radius of 7. This one, however, is not centered at the origin. The general equation of a circle is $(x - h)^2 + (y - k)^2 = r^2$. The center of the circle is represented by $(h, k)$, so in this example the center is $(4, -2)$. The center of this circle has been shifted 4 units right, and 2 units down.
Example 3

What are the center and the radius of the circle \( x^2 - 6x + y^2 + 2y - 5 = 0 \)?

This circle is not in the graphing form, \((x - h)^2 + (y - k)^2 = r^2\), so it is necessary to “complete the square.” To make a perfect square for \(x\) we need to add 9 units and to make a perfect square for \(y\) we need to add 1 unit. Adding 10 to the right side will balance this.

Finally factor and the graphing form is achieved.

\[
(x^2 - 6x + 9) + (y^2 + 2y + 1) - 5 = 10
\]

\[
(x - 3)^2 + (y + 1)^2 = 15
\]

The center is (3, –1) and the radius is \(\sqrt{15}\).

Problems

1. What is the equation of the circle centered at \((0, 0)\) with a radius of 25?
2. What is the equation of the circle centered at the origin with a radius of 7.5?
3. What is the equation of the circle centered at \((5, -3)\) with a radius of 9?

Graph the following circles.

4. \((x + 1)^2 + (y + 5)^2 = 16\)
5. \(x^2 + (y - 6)^2 = 36\)
6. \((x - 3)^2 + y^2 = 64\)

“Complete the square” to convert the equation of each circle to graphing form. Identify the center and the radius.

7. \(x^2 + 6x + y^2 - 4y = -9\)
8. \(x^2 + 10x + y^2 - 8y = -31\)
9. \(x^2 - 2x + y^2 + 4y - 11 = 0\)
10. \(x^2 + 9x + y^2 = 0\)
Answers

1. \( x^2 + y^2 = 625 \)

2. \( x^2 + y^2 = 56.25 \)

3. \( (x - 5)^2 + (y + 3)^2 = 81 \)

4. \( (x + 3)^2 + (y - 2)^2 = 4; (-3, 2), r = 2 \)

5. \( (x + 5)^2 + (y - 4)^2 = 10; (-5, 4), r = \sqrt{10} \)

6. \( (x - 1)^2 + (y + 2)^2 = 16; (1, -2), r = 4 \)

7. \( (x + 9)^2 + y^2 = \frac{81}{4}; (-\frac{9}{2}, 0), r = \frac{9}{2} \)