Sometimes the information we know about sides and angle of a triangle is not enough to make one unique triangle. Sometimes a triangle may not even exist, as we saw when we studied the Triangle Inequality. When a triangle formed is not unique (that is, more than one triangle can be made with the given conditions) we call this **triangle ambiguity**. This happens when we are given two sides and an angle not between the two sides, known as SSA.

**Example 1**

In \( \triangle ABC \), \( m\angle A = 50^\circ \), \( AB = 12 \), and \( BC = 10 \). Can you make a unique triangle? If so, find all the angle measures and side lengths for \( \triangle ABC \). If not, show more than one triangle that meets these conditions.

As with many problems, we will first make a sketch of what the problem is describing.

Once we label the figure, we see that the information displays the SSA pattern mentioned above. It does seem as if this triangle can exist. First try to find the length of side \( AC \). To do this we will use the Law of Cosines.

Now we have a type of equation we have not seen when solving this sort of problem. This is a quadratic equation. To solve it, we will use the Quadratic Formula (see the Math Notes box in Lesson 4.2.1). Recall that a quadratic equation may have two different solutions. We will use the formula and see what happens.

Both of these answers are positive numbers, and could be lengths of sides of a triangle. So what happened? If we drew the triangle to scale, we would notice that although we drew the triangle with \( \angle C \) acute, it does not have to be. Nothing in the information given says to draw the triangle this way. In fact, since there are no conditions on \( m\angle B \) the side \( BC \) can swing as if it is on a hinge at \( \angle B \). As you move \( BC \) along \( AC \), \( BC \) can intersect \( AC \) at two different places and still be 10 units long. In one arrangement, \( \angle C \) is fairly small, while in the second arrangement, \( \angle C \) is larger. (Note: The triangle formed with the two possible arrangements (the light grey triangle) is isosceles. From that you can conclude that the two possibilities for \( \angle C \) are supplementary.)
Problems

Partial information is given about a triangle in each problem below. Solve for the remaining parts of the triangle, explain why a triangle does not exist, or explain why there is more than one possible triangle.

1. In $\triangle ABC$, $\angle A = 32^\circ$, $AB = 20$, and $BC = 12$.

2. In $\triangle XYZ$, $\angle Z = 84^\circ$, $XZ = 6$, and $YZ = 9$.

3. In $\triangle ABC$, $\angle A = \angle B = 45^\circ$ and $AB = 7$.

4. In $\triangle PQR$, $PQ = 15$, $\angle R = 28^\circ$, and $PR = 23$.

5. In $\triangle XYZ$, $\angle X = 59^\circ$, $XY = 18$, and $YZ = 10$.

6. In $\triangle PQR$, $\angle P = 54^\circ$, $\angle R = 36^\circ$, and $PQ = 6$.

Answers

1. Two triangles: $AC \approx 22.58$, $m\angle B \approx 85.34^\circ$, $m\angle C \approx 62.66^\circ$ or $AC \approx 11.35$, $m\angle B \approx 30.04^\circ$, $m\angle C \approx 117.96^\circ$

2. One triangle: $XY \approx 10.28$, $m\angle X \approx 60.54^\circ$, $m\angle Y \approx 35.46^\circ$

3. One triangle: $m\angle C \approx 90^\circ$, $BC = AC = \frac{7\sqrt{2}}{2} \approx 4.95$

4. Two triangles: $QR \approx 30.725$, $m\angle Q \approx 46.04^\circ$, $m\angle P \approx 105.96^\circ$, or $QR \approx 9.895$, $m\angle Q \approx 133.96^\circ$, $m\angle P \approx 18.04^\circ$

5. No triangle exists. Note: If you use Law of Cosines, you will have a negative number under the square root sign. This means there are no real number solutions.

6. One triangle: $m\angle Q = 90^\circ$, $QR \approx 8.26$, $PR \approx 10.21$