CIRCLES 7.1.1 – 7.1.2

Circles have special properties. The fact that they can roll smoothly is because the circle has a constant **diameter** (the distance across the circle that passes through the center). A vehicle with square wheels would cause it to bump up and down because, since the diagonals of a square are longer then its width, it does not have a constant diameter. But a circle is not the only shape with a constant diameter. Reuleaux curves, which resemble rounded polygons, also have a constant diameter. It may not appear to be the case, but Reuleaux curves roll smoothly without bumping up and down. See problem 7-3 in the textbook for a picture.

A circle does not include its interior. It is the set of points on a flat surface at a fixed distance (the radius) from a fixed point (the center). This also means that the center, diameters, and radii (plural of radius) are not part of the circle. Remember, a radius is half of a diameter, and connects the center of the circle to a point on the circle. A circle has infinitely many diameters and infinitely many radii.

See the Math Notes box in Lesson 7.1.2.

**Example 1**

Using the circle at right, write an equation and solve for \(x\).

Note: Each part is a different problem.

a. \(AO = 3x - 4, OB = 4x - 12\).

b. \(OB = 2x - 5, AC = x - 7\)

Using the information we have about circles, diameters, and radii, we can write an equation using the expressions in part (a), then solve for \(x\). \(\overline{AO}\) and \(\overline{OB}\) are both radii of circle \(O\), which means that they are equal in length.

In part (b), \(\overline{OB}\) is a radius, but \(\overline{AC}\) is a diameter, so \(\overline{AC}\) is twice as long as \(\overline{OB}\).
Problems

Using the circle below, write an equation and solve for $x$. Note: Each part is a different problem.

1. $OP = 5x - 3, \ OR = 3x + 9$
2. $OQ = 2x + 12, \ OP = 3x - 1$
3. $OR = 12x - 8, \ OQ = 8x - 4$
4. $OP = 5x + 3, \ PR = 3x + 13$
5. $OQ = x - 6, \ PR = x + 7$

Answers

1. $x = 6$
2. $x = 13$
3. $x = 1$
4. $x = 1$
5. $x = 19$
Through the use of rigid transformations (rotations, reflections, and translations), students created new figures and discovered properties of these figures. Students also used transformations to discover the shortest path from one object to another (an optimization problem), and to define and study regular polygons. A **regular polygon** is a polygon in which all sides have equal length and all angles have equal measure.

See the Math Notes boxes in Lesson 7.1.4.

**Example 1**

On a recent scout camping trip, Zevel was walking back to camp when he noticed that the campfire had grown too large. He wants to fill his bucket at the river, then walk to the fire to douse the flames. To ensure the fire does not get out of control, Zevel wants to take the shortest path. What is the shortest path from where Zevel is standing to go to the river and then to the fire?

The shortest distance between two points on a flat surface is a straight line. How can we find the shortest distance if a third point is involved, such as Zevel’s trip to the river? We use reflections to find the point at the river to which Zevel should walk. Reflecting across a line gives a new figure that is the same distance from the line as the original. Reflecting point F across the river line gives F’ with FX = F’X. To find the shortest distance from Z to F’, we connect them. Since ∠FXR is a right angle, ∆FXR ≅ ∆F’XR by the SAS ≅ conjecture. This means that FR = F’R. Since the shortest path from Z to F’ is the straight line drawn, then the shortest path from Zevel to the river and then to the fire is to walk from Z to R and then to F.

**Example 2**

If a hinged mirror is set at 10° and the core region (the region the hinged mirror encloses on the paper) is isosceles, how many sides would there be on the polygon that would be reflected in the mirror?

If the core region is isosceles, the reflected image will be a regular polygon. In a regular polygon, all the interior angles are equal in measure and, if the center is connected to each vertex of the polygon, these central angles are equal. If one central angle measures 10° and there are 360° around the center, there are 360° ÷ 10° = 36 sides on the polygon.
Problems

1. Venetia wants to install two lights in her garden. Each one will be connected to a control timer that will turn the lights on and off automatically. She can mount the timer anywhere on her house, but she wants to minimize the amount of wire she will use. If the wire must run from the light at point $P$ to the timer, and then back out to the light at point $Q$, where should Venetia place the timer?

2. While playing miniature golf last weekend, Myrtle came to the fifth hole and saw that it was a par 1 hole. This meant that she should be able to putt the ball into the hole with one stroke. Explain to Myrtle how knowledge of “shortest distance” problems can help her make the putt.

3. If the center of a regular dodecagon (12 sides) is connected to each vertex of the figure, what is the measure of each angle at the center?

4. If a central angle of a regular polygon measures 18°, how many sides does the polygon have?

5. The center point of a regular pentagon is connected to each vertex forming five congruent isosceles triangles. Find the measure of each base angle in the isosceles triangles and use that result to find the measure of one interior angle of the pentagon.

Answers

1. Venetia should place the timer about 11.54 feet from the point $X$ or 18.45 feet from the point $Y$ on the diagram above.

2. If Myrtle can aim correctly and hit a straight shot, she can make a hole in one. She can imagine the hole reflected across the top boundary to find the direction to aim. If she hits the wall at the point $X$, the ball will travel to the hole.

3. $30^\circ$

4. 20 sides

5. Each base angle measures $54^\circ$. Two together makes one interior angle of the pentagon, so an interior angle measures $108^\circ$. 
By tracing and reflecting triangles to form quadrilaterals, students discover properties about quadrilaterals. More importantly, they develop a method to prove that what they have observed is true. Students are already familiar with using flowcharts to organize information, so they will use flowcharts to present proofs. Since they developed their conjectures by reflecting triangles, their proofs will rely heavily on the triangle congruence conjectures developed in Chapter 6. Once students prove that their observations are true, they can use the information in later problems.

See the Math Notes boxes in Lessons 7.2.1, 7.2.3, 7.2.4, and 7.2.6.

Example 1

$ABCD$ at right is a parallelogram. Use this fact and other properties and conjectures to prove that:

a. the opposite sides are congruent.

b. the opposite angles are congruent.

c. the diagonals bisect each other.

Because $ABCD$ is a parallelogram, the opposite sides are parallel. Whenever we have parallel lines, we should be looking for some pairs of congruent angles. In this case, since $AB \parallel CD$, $\angle BAC \equiv \angle DCA$ because alternate interior angles are congruent. Similarly, since $AD \parallel CB$, $\angle DAC \equiv \angle BCA$. Also, $\overline{AC} \equiv \overline{CA}$ by the Reflexive Property. Putting all three of these pieces of information together tells us that $\triangle BAC \equiv \triangle DCA$ by the ASA $\equiv$ conjecture. Now that we know that the triangles are congruent, all the other corresponding parts are also congruent. In particular, $\overline{AB} \equiv \overline{CD}$ and $\overline{AD} \equiv \overline{CB}$, which proves that the opposite sides are congruent. As a flowchart proof, this argument would be presented as shown below.
For part (b), we can continue the previous proof, again using congruent parts of congruent triangles, to justify that \( \angle ADC \cong \angle CBA \). That gives one pair of opposite angles congruent. To get the other pair, we need to draw in the other diagonal. As before, the alternate interior angles are congruent, \( \angle ADB \cong \angle CBD \) and \( \angle ABD \cong \angle CDB \), because the opposite sides are parallel. Using the Reflexive Property, \( \overline{BD} \cong \overline{BD} \). Therefore, \( \triangle ABD \cong \triangle CDB \) by the ASA \( \cong \) conjecture. Now that we know that the triangles are congruent, we can conclude that the corresponding parts are also congruent. Therefore, \( \angle DAB \cong \angle BCD \). We have just proven that the opposite angles in the parallelogram are congruent.

Lastly, we will prove that the diagonals bisect each other. To begin, we need a picture with both diagonals included. There are many triangles in the figure now, so our first task will be deciding which ones we should prove congruent to help us with the diagonals. To show that the diagonals bisect each other we will show that \( \overline{AE} \cong \overline{CE} \) and \( \overline{BE} \cong \overline{DE} \) since “bisect” means to cut into two equal parts.

We have already proven facts about the parallelogram that we can use here. For instance, we know that the opposite sides are congruent, so \( \overline{AD} \cong \overline{CB} \). We already know that the alternate interior angles are congruent, so \( \angle ADE \cong \angle CBE \) and \( \angle DAE \cong \angle BCE \). Once again we have congruent triangles: \( \triangle ADE \cong \triangle CBE \) by ASA \( \cong \). Since congruent triangles give us congruent corresponding parts, \( \overline{AE} \cong \overline{CE} \) and \( \overline{BE} \cong \overline{DE} \), which means the diagonals bisect each other.

**Example 2**

\( PQRS \) at right is a rhombus. Do the diagonals bisect each other? Justify your answer. Are the diagonals perpendicular? Justify your answer.

The definition of a rhombus is a quadrilateral with four sides of equal length. Therefore, \( \overline{PQ} \cong \overline{QR} \cong \overline{RS} \cong \overline{SP} \). By the Reflexive Property, \( \overline{PR} \cong \overline{RP} \). With sides congruent, we can use the SSS \( \cong \) conjecture to write \( \triangle SPR \cong \triangle QRP \). Since the triangles are congruent, all corresponding parts are also congruent. Therefore, \( \angle SPR \cong \angle QRP \) and \( \angle PRS \cong \angle RPQ \). The first pair of congruent angles means that \( \overline{SP} \parallel \overline{QR} \). (If the alternate interior angles are congruent, the lines are parallel.) Similarly, the second pair of congruent angles means that \( \overline{PQ} \parallel \overline{RS} \). With both pairs of opposite sides congruent, this rhombus is a parallelogram. Since it is a parallelogram, we can use what we have already proven about parallelograms, namely, that the diagonals bisect each other. Therefore, the answer is yes, the diagonals bisect each other.
To determine if the diagonals are perpendicular, use what we did to answer the first question. Then gather more information to prove that other triangles are congruent. In particular, since $PQ \cong RQ$, $QT \cong QT$, and $PT \cong RT$ (since the diagonal is bisected), $\triangle QPT \cong \triangle RQT$ by SSS $\cong$. Because the triangles are congruent, all corresponding parts are also congruent, so $\angle QTP \cong \angle QTR$. These two angles also form a straight angle. If two angles are congruent and their measures sum to $180^\circ$, each angle measures $90^\circ$. If the angles measure $90^\circ$, the lines must be perpendicular. Therefore, $QS \perp PR$.

**Example 3**

In the figure at right, if $\overline{AI}$ is the perpendicular bisector of $\overline{DV}$, is $\triangle DAV$ isosceles? Prove your conclusion using the two-column proof format.

Before starting a two-column proof, it is helpful to think about what we are trying to prove. If we want to prove that a triangle is isosceles, then we must show that $DA \cong VA$ because an isosceles triangle has two sides congruent. By showing that $\triangle AID \cong \triangle AIV$, we can then conclude that this pair of sides is congruent. Now that we have a plan, we can begin the two-column proof.

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overline{AI}$ is the perpendicular bisector of $\overline{DV}$</td>
<td>Given</td>
</tr>
<tr>
<td>$DI \cong VI$</td>
<td>Definition of bisector</td>
</tr>
<tr>
<td>$\angle DIA$ and $\angle VIA$ are right angles</td>
<td>Definition of perpendicular</td>
</tr>
<tr>
<td>$\angle DIA \cong \angle VIA$</td>
<td>All right angles are congruent</td>
</tr>
<tr>
<td>$\overline{AI} \cong \overline{AI}$</td>
<td>Reflexive Property of Equality</td>
</tr>
<tr>
<td>$\triangle DAI \cong \triangle VAI$</td>
<td>SAS $\cong$</td>
</tr>
<tr>
<td>$DA \cong VA$</td>
<td>$\equiv \triangle s \rightarrow \equiv$ parts</td>
</tr>
<tr>
<td>$\triangle DAV$ is isosceles</td>
<td>Definition of isosceles</td>
</tr>
</tbody>
</table>
Problems

Find the required information and justify your answers.

For problems 1-4 use the parallelogram at right.

1. Find the perimeter.
2. If \( CT = 9 \), find \( AT \).
3. If \( m\angle CDA = 60^\circ \), find \( m\angle CBA \) and \( m\angle BAD \).
4. If \( AT = 4x - 7 \) and \( CT = -x + 13 \), solve for \( x \).

For problems 5-8 use the rhombus at right.

5. If \( PS = \sqrt{6} \), what is the perimeter of \( PQRS \)?
6. If \( PQ = 3x + 7 \) and \( QR = -x + 17 \), solve for \( x \).
7. If \( m\angle PSM = 22^\circ \), find \( m\angle RSM \) and \( m\angle SPQ \).
8. If \( m\angle PMQ = 4x - 5 \), solve for \( x \).

For problems 9-12 use the quadrilateral at right.

9. If \( WX = YZ \) and \( WZ = XY \), must \( WXYZ \) be rectangle?
10. If \( m\angle WZY = 90^\circ \), must \( WXYZ \) be a rectangle?
11. If the information in problems 9-10 are both true, must \( WXYZ \) be a rectangle?
12. If the information in problems 9-10 are both true, \( WY = 15 \), and \( WZ = 9 \), what are \( YZ \) and \( XZ \)?

For problems 17-20 use the kite at right.

17. If \( m\angle XWZ = 95^\circ \), find \( m\angle XYZ \).
18. If \( m\angle WZY = 110^\circ \) and \( m\angle WXY = 40^\circ \), find \( m\angle ZWX \).
19. If \( WZ = 5 \) and \( WT = 4 \), find \( ZT \).
20. If \( WT = 4 \), \( TZ = 3 \), and \( TX = 10 \), find the perimeter of \( WXYZ \).

21. If \( \overline{PQ} \cong \overline{RS} \) and \( \overline{QR} \cong \overline{SP} \), is \( PQRS \) a parallelogram? Prove your answer.

22. \( WXYZ \) is a rhombus. Does \( \overline{WY} \) bisect \( \angle ZWX \)? Prove your answer.
For problems 23-25, use the figure at right. Base your decision on the markings, not appearances.

23. Is $\triangle ABC \cong \triangle EDC$? Prove your answer.

24. Is $AB \cong ED$? Prove your answer.

25. Is $AB \cong DC$? Prove your answer.

26. If $NIFH$ is a parallelogram, is $ES \cong ET$? Prove your answer.

27. If $DSIA$ is a parallelogram and $IA \cong IV$, is $\angle D \cong \angle V$? Prove your answer.

28. If $A$, $W$, and $K$ are midpoints of $TS$, $SE$, and $ET$ respectively, is $TAWK$ a parallelogram? Prove your answer.
## Answers

1. 44 units  
2. 9 units  
3. 60°, 120°  
4. 4  
5. $4\sqrt{6}$  
6. 2.5  
7. 22°, 136°  
8. 23.75  
9. no  
10. no  
11. yes  
12. 12, 15  
13. 60°  
14. 18.75  
15. 3  
16. 10 + 4\sqrt{29}  
17. 95°  
18. 105°  
19. 3  
20. 

21. Yes. $QS \cong SQ$ (reflexive). This fact, along with the given information means that $\triangle PQS \cong \triangle RSQ$ (SSS $\cong$). That tells us the corresponding parts are also congruent, so $\angle PQS \cong \angle RSQ$ and $\angle PSQ \cong \angle RQS$. These angles are alternate interior angles, so both pairs of opposite sides are parallel. Therefore, $PQRS$ is a parallelogram.

22. Yes. Since the figure is a rhombus, all the sides are congruent. In particular, $WZ \cong WX$ and $ZY \cong XY$. Also, $WY \cong WY$ (reflexive), so $\triangle WZY \cong \triangle WXY$ (SSS $\cong$). Congruent triangles give us congruent parts so $\angle ZYW \cong \angle XYW$. Therefore, $WY$ bisects $\angle ZWX$.

23. Yes. Since the lines are parallel, alternate interior angles are congruent so $\angle BDC \cong \angle ECD$. Also, $DC \cong CD$ (reflexive) so the triangles are congruent by SAS $\cong$.

24. Not necessarily, since we have no information about $\overline{AC}$.

25. Not necessarily.

26. Yes. Because $MIFH$ is a parallelogram, we know several things. First, $\angle TME \cong \angle SFE$ (alternate interior angles) and $ME \cong EF$ (diagonals of a parallelogram bisect each other). Also, $\angle TEM \cong \angle SEF$ because vertical angles are congruent. This gives us $\triangle MTE \cong \triangle FSE$ by ASA $\cong$. Therefore, the corresponding parts of the triangle are congruent, so $ES \cong ET$.

27. Since $DSIA$ is a parallelogram, $\overline{DS} \parallel \overline{AI}$ which gives us $\angle D \cong \angle IAV$ (corresponding angles). Also, since $IA \cong IV$, $\triangle IAV$ is isosceles, so $\angle IAV \cong \angle V$. The two angle congruence statements allow us to conclude that $\angle D \cong \angle V$.

28. Yes. By the Triangle Midsegment Theorem (see the Math Notes box in Lesson 7.2.6), since $A$, $W$, and $K$ are midpoints of $\overline{TS}$, $\overline{SE}$, and $\overline{ET}$ respectively, $\overline{AW} \parallel \overline{TE}$ and $\overline{KW} \parallel \overline{TS}$. Therefore $TAWK$ is a parallelogram.
Now that students are familiar with many of the properties of various triangles, quadrilaterals, and special quadrilaterals, they can apply their algebra skills and knowledge of the coordinate grid to study **coordinate geometry**. In this section, the shapes are plotted on a graph. Using familiar ideas, such as the Pythagorean Theorem and slope, students can prove whether or not quadrilaterals have special properties.

See the Math Notes boxes in Lessons 7.3.2 and 7.3.3.

**Example 1**

On a set of coordinate axes, plot the points \(A(-3, -1)\), \(B(1, -4)\), \(C(5, -1)\), and \(D(1, 2)\) and connect them in the order given. Is this quadrilateral a rhombus? Justify your answer.

To show that this quadrilateral is a rhombus, we must show that all four sides are the same length because that is the definition of a rhombus. When we want to find the length of a segment on the coordinate graph, we use the Pythagorean Theorem. To begin, we plot the points on a graph.

Although the shape appears to be a parallelogram, and possibly a rhombus, we cannot base our decision on appearances. To use the Pythagorean Theorem, we outline a **slope triangle**, creating a right triangle with \(\overline{AB}\) as the hypotenuse. The lengths of the legs of this right triangle are 3 and 4 units. Using the Pythagorean Theorem,

\[
3^2 + 4^2 = (AB)^2 \\
9 + 16 = (AB)^2 \\
25 = (AB)^2 \\
AB = 5
\]

Similarly, we set up slope triangles for the other three sides of the quadrilateral and use the Pythagorean Theorem again. In each case, we find the lengths are all 5 units. Therefore, since all four sides have the same length, the figure is a rhombus.
Example 2

On a set of coordinate axes, plot the points \(A(-4, 1), B(1, 3), C(8, -1),\) and \(D(4, -3),\) and connect them in the order given. Is this quadrilateral a parallelogram? Prove your answer.

When we plot the points, the quadrilateral appears to be a parallelogram, but we cannot base our decision on appearances. To prove it is a parallelogram, we must show that the opposite sides are parallel. On the coordinate graph, we show that lines are parallel by showing that they have the same slope. We can use slope triangles to find the slope of each side.

\[
\begin{align*}
\text{Slope of } \overline{BC} &= \frac{-4}{7} = -\frac{4}{7} \\
\text{Slope of } \overline{BA} &= \frac{2}{5} \\
\text{Slope of } \overline{AD} &= \frac{-4}{8} = -\frac{1}{2} \\
\text{Slope of } \overline{DC} &= \frac{2}{4} = \frac{1}{2}
\end{align*}
\]

Although the values for the slopes of the opposite sides are close, they are not equal. Therefore this quadrilateral is not a parallelogram.

Problems

1. If \(ABCD\) is a rectangle, and \(A(1, 2), B(5, 2),\) and \(C(5, 5),\) what are the coordinates of point \(D?\)

2. If \(P(2, 1)\) and \(Q(6, 1)\) are the endpoints of the base of an isosceles right triangle, what is the \(x\)-coordinate of the third vertex?

3. The three points \(S(-1, -1), A(1, 4),\) and \(M(2, -1)\) are vertices of a parallelogram. What are the coordinates of three possible points for the other vertex?

4. Graph the following lines on the same set of axes. These lines enclose a shape. What is the name of that shape? Justify your answer.

\[
\begin{align*}
y &= \frac{3}{5} x + 7 & y &= 0.6x \\
y &= -\frac{10}{6} x - 1 & y &= -\frac{5}{3} x + 9
\end{align*}
\]

5. If \(W(-4, -5), X(1, 0), Y(-1, 2),\) and \(Z(-6, -3),\) what shape is \(WXYZ?\) Prove your answer.

6. If \(\overline{DT}\) has endpoints \(D(2, 2)\) and \(T(6, 4),\) what is the equation of the perpendicular bisector of \(\overline{DT}\)?
Answers

1. (1, 5)

2. (4, 4)

3. (4, 4), (0, –6), or (–2, 4)

4. Since the slopes of opposites side are equal, this is a parallelogram. Additionally, since the slopes of intersecting lines are negative reciprocals of each other, they are perpendicular. This means the angles are all right angles, so the figure is a rectangle.

5. The slopes are: \( WX = 1, \ XY = -1, \ YZ = 1, \) and \( ZW = -1. \) This shows that \( WXYZ \) is a rectangle.

6. \( y = -2x + 11 \)