In this chapter, students examine three-dimensional shapes, known as solids. Students will work on visualizing these solids by building and then drawing them. Visualization is a useful, often overlooked skill in mathematics. By drawing solids students gain a better understanding of volume and surface area.

See the Math Notes boxes in Lessons 9.1.2, 9.1.3, and 9.1.5.

Example 1

The solid at right is built from individual cubes (blocks) stacked upon each other on a flat surface. (This means that no cubes are “floating.”) Create a mat plan representing this solid. What is the volume of this solid?

This solid consists of stacked blocks. We are looking at the front, right side, and top of this solid. A mat plan shows a different perspective of a solid. It shows the footprint of the solid as well as how many blocks are in each stack. A mat plan is useful because, in the solid above, we cannot see the possible “hidden” blocks. A mat plan tells us exactly how many blocks are in the solid.

In this case, since we are creating the mat plan from the stacked blocks, there is more than one possible answer. If there are no hidden blocks, then the mat plan is the first diagram at right. If there is a hidden block, then the mat plan is the second one at right. It is helpful to visualize solids by building them with cubes. Build solids on a $3 \times 5$ card so that you can rotate the card to see the solid from all of its sides. Do this to make sure that one block is all that can be hidden in this drawing.

The volume of this solid is the number of cubes it would take to build it. In this case, the volume is either 9 cubic units or 10 cubic units.
Example 2

At right is a mat plan of a solid. Build the solid. What is the volume of this solid? Draw the front, right, and top views, as well as the three-dimensional view of this solid.

We find the volume by counting the number of blocks it would take to build this solid or by adding the numbers in the mat plan. The volume of this solid is 12 cubic units. To draw the different views of this solid, it is extremely helpful to build it out of cubes on a $3 \times 5$ card. Label the card with front, right, left, and back so that you can remember which side is which when rotating it. Remember that the standard three-dimensional view shows the top, right, and front views.

The individual views of each side are flat views. It is helpful to look at the solid at your eye level, so that only one side is visible at a time.

Example 3

If the figure at right is made with the fewest amount of cubes possible, what is its surface area?

The surface area is the sum of the areas of all the surfaces (or faces or sides) on the solid. If we build the solid on a card, then we can rotate the card and count the number of squares on each face, except the bottom. Note that because we never have floating cubes in our solids, the bottom has the same surface area as the top. If we draw every face of the solid, we can count the number of squares to find the surface area. Alternately, rather than drawing every face, we can draw only three views — the front, right, and top — and double those areas, because the back, left, and bottom are always their reflections with equivalent respective areas. Either way, we will arrive at the same answer.

From the front and back the solid looks the same and shows 10 squares.

The right and left views are reflections of each other and each shows seven squares.

The top and bottom views are also reflections of each other. They show four squares each.

Therefore, the surface area is $10 + 10 + 7 + 7 + 4 + 4 = 42$ square units.
Example 4

The dimensions of the prism at right are shown. What are the volume and surface area of this prism?

A prism is a special type of polyhedron that has two congruent and parallel bases. In this problem, the bases are right triangles. The volume of a prism is found by multiplying the area of the base by the height of the prism. To understand this process, think of a prism as a stack of cubes. The base area tells you how many cubes are in one layer of the stack. The height tells you how many layers of cubes are in the figure.

In this example, the base is a right triangle, so the area is \( \frac{1}{2}bh \). Looking at the top of the prism might make it easier to find the area of the base represented by \( A_b \).

\[
A_b = \frac{1}{2}bh = \frac{1}{2}(6)(8) = 24 \text{ square units}, \text{ so there are 24 cubes in one layer.}
\]

To find the volume, we multiply this amount by the height, 12.

\[
V = A_bh = (24)(12) = 288 \text{ cubic units}
\]

To find the surface area of this prism, we will find the area of each of its faces, including the bases, and add the areas. One way to illustrate the sub-problems is to make sketches of the surfaces.

\[
\text{Surface Area} = 2 \left( \frac{\sqrt{8^2 + 6^2}}{6} \right) + \left[ \frac{6 \cdot 8}{12} \right] + \left[ \frac{8 \cdot 12}{12} \right] + \left[ \frac{12 \cdot 12}{12} \right]
\]

All of the surfaces are familiar shapes, namely, triangles and rectangles. We need to calculate the length of the rectangle on the back face (the last rectangle in the pictorial equation above). Fortunately, that length is also the hypotenuse of the right triangle of the base, so we can use the Pythagorean Theorem to find that length.

\[
6^2 + 8^2 = ?^2 \quad 6^2 + 8^2 = ?^2 \quad \text{We need to calculate the length of the rectangle on the back face (the last rectangle in the pictorial equation above). Fortunately, that length is also the hypotenuse of the right triangle of the base, so we can use the Pythagorean Theorem to find that length.}
\]

Therefore the surface area is:

\[
S.A. = 2\left( \frac{1}{2} \cdot 6 \cdot 8 \right) + (6 \cdot 12) + (8 \cdot 12) + (10 \cdot 12)
\]

\[
= 48 + 72 + 96 + 120
\]

\[
= 336 \text{ square units}
\]
**Example 5**

The Styrofoam pieces used in packing boxes, known as “shipping peanuts,” are sold in three box sizes: small, medium, and large. The small box has a volume of 1200 cubic inches. The dimensions on the “medium” box are twice the dimensions of the small box, and the “large” box has triple the dimensions of the small one. All three boxes are similar prisms. What are volumes of the medium and large boxes?

Since the boxes are similar, we can use the ratio of similarity to determine the volume of the medium and large boxes without knowing their actual dimensions. When figures are similar with ratio of similarity $\frac{a}{b}$, the ratio of the areas is $(\frac{a}{b})^2$ and the ratio of the volumes is $(\frac{a}{b})^3$.

Since the medium box has dimensions twice the small box and the large box has dimensions three times the small box, we can write:

\[
\frac{\text{medium box}}{\text{small box}} = \frac{2}{1} \quad \text{and} \quad \frac{\text{large box}}{\text{small box}} = \frac{3}{1}
\]

\[
\frac{\text{volume of medium box}}{\text{volume of small box}} = \left(\frac{2}{1}\right)^3 = 8 \quad \text{and} \quad \frac{\text{volume of large box}}{\text{volume of small box}} = \left(\frac{3}{1}\right)^3 = 27
\]

Solving, $V_m = 8 \cdot 1200$ or $V_m = 9600$ cubic units and $V_l = 27 \cdot 1200$ or $V_l = 32,400$ cubic units.

**Problems**

For each solid, calculate the volume and surface area, then draw a mat plan. Assume there are no hidden or floating cubes.

1.  
2.  
3.  
4.  
5.  
6.  

For each mat plan, draw the solid, then calculate the volume and surface area.
Calculate the volume and surface area of each prism.

7. The base is a rectangle.

8. A cube.

9. 11

10. At Cakes R Us, it is possible to buy round cakes in different sizes. The smallest cake has a diameter of 8 inches and a height of 4 inches, and requires 3 cups of batter. Another similar round cake has a diameter of 13 inches. How much batter would this cake require?

11. Prism A and prism B are similar with a ratio of similarity of 2:3. If the volume of prism A is 36 cubic units, what is the volume of prism B?

12. Two rectangular prisms are similar. The smaller, A, has a height of 4 units while the larger, B, has a height of 6 units.
   a. What is the scale factor from prism A to prism B?
   b. What is the ratio of the lengths of the edges labeled x and y?
   c. What is the ratio of their surface areas? What is the ratio of their volumes?
   d. A third prism C is similar to prisms A and B. Prism C’s height is 10 units. If the volume of prism A is 24 cubic units, what is the volume of prism C?

13. Two rectangular prisms are similar. The smaller, A, has a height of 4 units while the larger, B, has a height of 6 units.
   a. What is the scale factor from prism B to prism A?
   b. What would be the ratio of the lengths of the edges labeled x and y?
   c. What is the ratio of their surface areas? What is the ratio of their volumes?
   d. A third prism, C is similar to prisms A and B. Prism C’s height is 10 units. If the volume of prism A is 20 cubic units, what is the volume of prism C?

14. If rectangle A and rectangle B have a ratio of similarity of 5:4, what is the area of rectangle B if the area of rectangle A is 24 square units?
15. If rectangle \( A \) and rectangle \( B \) have a ratio of similarity of 2:3, what is the area of rectangle \( B \) if the area of rectangle \( A \) is 46 square units?

16. If rectangle \( A \) and rectangle \( B \) have a ratio of similarity of 3:4, what is the area of rectangle \( B \) if the area of rectangle \( A \) is 82 square units?

17. If rectangle \( A \) and rectangle \( B \) have a ratio of similarity of 1:5, what is the area of rectangle \( B \) if the area of rectangle \( A \) is 24 square units?

18. Rectangle \( A \) is similar to rectangle \( B \). The area of rectangle \( A \) is 81 square units while the area of rectangle \( B \) is 49 square units. What is the ratio of similarity between the two rectangles?

19. Rectangle \( A \) is similar to rectangle \( B \). The area of rectangle \( B \) is 18 square units while the area of rectangle \( A \) is 12.5 square units. What is the ratio of similarity between the two rectangles?

20. Rectangle \( A \) is similar to rectangle \( B \). The area of rectangle \( A \) is 16 square units while the area of rectangle \( B \) is 100 square units. If the perimeter of rectangle \( A \) is 12 units, what is the perimeter of rectangle \( B \)?

21. If prism \( A \) and prism \( B \) have a ratio of similarity of 2:3, what is the volume of prism \( B \) if the volume of prism \( A \) is 36 cubic units?

22. If prism \( A \) and prism \( B \) have a ratio of similarity of 1:4, what is the volume of prism \( B \) if the volume of prism \( A \) is 83 cubic units?

23. If prism \( A \) and prism \( B \) have a ratio of similarity of 6:11, what is the volume of prism \( B \) if the volume of prism \( A \) is 96 cubic units?

24. Prism \( A \) and prism \( B \) are similar. The volume of prism \( A \) is 72 cubic units while the volume of prism \( B \) is 1125 cubic units. What is the ratio of similarity between these two prisms?

25. Prism \( A \) and prism \( B \) are similar. The volume of prism \( A \) is 27 cubic units while the volume of prism \( B \) is approximately 512 cubic units. If the surface area of prism \( B \) is 128 square units, what is the surface area of prism \( A \)?

26. The corresponding diagonals of two similar trapezoids are in the ratio of 1:7. What is the ratio of their areas?

27. The ratio of the perimeters of two similar parallelograms is 3:7. What is the ratio of their areas?

28. The ratio of the areas of two similar trapezoids is 1:9. What is the ratio of their altitudes?

29. The areas of two circles are in the ratio of 25:16. What is the ratio of their radii?
30. The ratio of the volumes of two similar circular cylinders is 27:64. What is the ratio of the diameters of their similar bases?

31. The surface areas of two cubes are in the ratio of 49:81. What is the ratio of their volumes?

32. The ratio of the weights of two spherical steel balls is 8:27. What is the ratio of the diameters of the two steel balls?

Answers

1. \[ \frac{2}{2} = \frac{2}{2} \]
2. \[ \frac{3}{2} : \frac{2}{1} \]
3. \[ \frac{2}{2} : \frac{1}{1} \]

4. \[ V = 20 \text{ cu. units} \]
5. \[ V = 11 \text{ cu. units} \]
6. \[ V = 15 \text{ cu. units} \]

7. \[ V = 630 \text{ cu. units} \]
8. \[ V = 45 \text{ cu. units} \]
9. \[ V = 1331 \text{ cu. units} \]

10. \[ \approx 12.87 \text{ cups} \]
11. \[ 121.5 \text{ cu. units} \]

12. a. \[ \frac{4}{6} = \frac{2}{3} \]
   b. \[ \frac{x}{y} = \frac{4}{6} = \frac{2}{3} \]
   c. \[ \frac{16}{36} = \frac{4}{9}, \frac{64}{216} = \frac{8}{27} \]
   d. \[ 375 \text{ cu. units} \]

13. a. \[ \frac{6}{2} = \frac{3}{1} \]
   b. \[ \frac{x}{y} = \frac{2}{6} = \frac{1}{3} \]
   c. \[ \frac{4}{36} = \frac{1}{9}, \frac{8}{216} = \frac{1}{27} \]
   d. \[ 2500 \text{ cu. units} \]

14. \[ 15.36 \text{ units}^2 \]
15. \[ 103.5 \text{ units}^2 \]
16. \[ \approx 145.8 \text{ units}^2 \]
17. \[ 600 \text{ units}^2 \]

18. \[ \frac{9}{7} \]
19. \[ \frac{6}{5} \]
20. \[ 30 \text{ units} \]
21. \[ 121.5 \]

22. \[ 5312 \]
23. \[ \approx 591.6 \]
24. \[ \frac{2}{5} \]
25. \[ \approx 18 \text{ units}^2 \]

26. \[ \frac{1}{49} \]
27. \[ \frac{9}{49} \]
28. \[ \frac{1}{3} \]
29. \[ \frac{5}{4} \]

30. \[ \frac{3}{4} \]
31. \[ \frac{343}{729} \]
32. \[ \frac{2}{3} \]