CORE CONNECTIONS INTEGRATED I
Parent Guide with Extra Practice

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DESCRIBING FUNCTIONS 1.1.2 and 1.1.3

The main objective of these lessons is for students to be able to fully describe the key elements of the graph of a function. To fully describe the graph of a function, students should respond to these graph investigation questions:

<table>
<thead>
<tr>
<th>Graph Investigation Question</th>
<th>Sample Summary Statement</th>
</tr>
</thead>
<tbody>
<tr>
<td>What is the shape of the graph?</td>
<td><em>The graph is a line/curve.</em></td>
</tr>
<tr>
<td>Is the function increasing or decreasing (reading left to right)?</td>
<td><em>As x gets bigger, y gets bigger, so the function is increasing.</em></td>
</tr>
<tr>
<td>What are the x- and y-intercepts?</td>
<td><em>The graph crosses the x-axis at (2, 0) and the y-axis at (0, -3).</em></td>
</tr>
<tr>
<td>Are there any limitations on the inputs (domain) of the equation?</td>
<td><em>Only positive values of x are possible. Zero is also possible.</em></td>
</tr>
<tr>
<td>Are there any limitations on the outputs (range) of the equation? (Is there a maximum or minimum y-value?)</td>
<td><em>The smallest y-value is 0. There is no maximum y-value.</em></td>
</tr>
<tr>
<td>Should the points be connected?</td>
<td><em>The given situation only makes sense for integer inputs, so the points should not be connected.</em></td>
</tr>
</tbody>
</table>

The more formal concepts of function and domain and range are addressed in Lessons 1.2.2 and 1.2.3.

For more information, see the Math Notes boxes in Lesson 1.1.2. Student responses to the Learning Log in Lesson 1.1.3 (problem 1-27) can also be helpful.

**Example 1**

For the situation below, make an $x \rightarrow y$ table, draw a graph, and describe the graph.

At the farmer’s market, apples cost $0.50 each.

Note that the smallest possible number for the $x \rightarrow y$ table is $x = 0$. You cannot buy a negative number of apples.

<table>
<thead>
<tr>
<th>$x$ (number of apples)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$ (cost)</td>
<td>0</td>
<td>0.50</td>
<td>1.00</td>
<td>2.00</td>
<td>3.00</td>
<td>5.00</td>
</tr>
</tbody>
</table>

The graph is a discrete set of linear points because you can only buy whole numbers of apples. It starts at (0, 0) and increases from left to right. The inputs are limited to positive integers, and the outputs are 0 and multiples of $0.50$.  

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Example 2

For the equation \( y = 2^x - 2 \), make an \( x \rightarrow y \) table, draw a graph, and fully describe the graph.

At this point there is no way to know how many points are sufficient for the \( x \rightarrow y \) table. Add more points as necessary until you are convinced of shape and location.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -2 )</th>
<th>( -1 )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>(-1.75)</td>
<td>(-1.5)</td>
<td>(-1)</td>
<td>(0)</td>
<td>(2)</td>
<td>(6)</td>
</tr>
</tbody>
</table>

Be careful with substitution using negative exponents when calculating values. The negative exponent moves across the fraction bar to become positive so \( 2^{-2} = \frac{1}{2^2} \).

For example if \( x = -2 \), \( y = 2^{-2} - 2 = \frac{1}{4} - 2 = -\frac{7}{4} = -1.75 \).

The graph is a curve. From left to right, the function increases. The \( x \)-intercept is \((1, 0)\). The \( y \)-intercept is \((0, -1)\). The points on the graph are connected. There are no limitations on inputs to the function. Outputs can be any value greater than \(-2\).

Problems

For each equation or situation, make an \( x \rightarrow y \) table, draw a graph, and describe the graph.

1. \( y = \frac{x^4}{2} \) 
2. Gasoline costs $4.00 per gallon. How much does it cost to buy \( x \) gallons of gas?
3. \( y = 2^x + 1 \) 
4. My science experiment starts with 5 bacteria and each hour the amount doubles. How many bacteria are there after \( x \) hours?
5. \( y = 5 - 2x \) 
6. The product of two numbers is 12.
7. \( y = (0.5)^x \) 
8. My tomato plant was 5 cm tall when planted and grows 2 cm per week. How tall is my tomato plant after \( x \) weeks?
9. \( y = x^2 - 4 \)
Answers

1. Line; intercepts (–4, 0) and (0, 2); increasing function. Inputs can be any real number. Outputs are any real number. The points are connected.

2. A ray (proportional graph); intercept and starting point (0, 0); increasing function. Inputs can be any non-negative number. Outputs are greater than or equal to 0. The points are connected.

3. Curve; intercept (0, 2); increasing function. Inputs can be any number. Outputs are greater than 1. The points are connected.

4. Curve; intercept and starting point (0, 5); increasing function. Inputs can be any non-negative number. Outputs are greater than or equal to 5. The graph should be several disconnected points (but there are so many it will look connected).

5. Line; intercepts (2.5, 0) and (0, 5); decreasing function. Inputs can be any non-negative number. Outputs are any real number. The points are connected.

6. Inverse variation; no intercepts, decreasing function. Inputs can be any number except 0. Outputs are any number except 0. The points are connected except at x = 0.

7. Curve; intercept (0, 1); decreasing function. Inputs can be any real number. Outputs are greater than 0. The points are connected.

8. Ray; intercept and starting point (0, 5); increasing function; Inputs can be any non-negative number. Outputs are greater than or equal to 5 (but probably also less then or equal to 180).

9. U-shape; intercepts (2, 0), (–2, 0) and (0, –4); decreasing for x < 0, increasing for x > 0. Minimum value at (0, –4). Inputs can be any real number. Outputs are greater than or equal to –4. The points are connected.
A relationship between the input values (usually $x$) and the output values (usually $y$) is called a function if for each input value, there is no more than one output value. Functions can be represented with an illustration of a “function (input–output) machine”, as shown in Lesson 1.2.3 of the textbook and in the diagram in Example 1 below. Note: $f(x) = 2x + 1$ is equivalent to $y = 2x + 1$.

The set of all possible inputs is called the domain, while the set of all possible outputs is called the range.

For additional information about functions, function notation, and domain and range, see the Math Notes box in Lesson 1.2.

**Example 1**

The inputs of a function are “$x$”s and the outputs are “$f(x)$”s. Numbers are input into the function machine labeled $f$ one at a time, and then the function performs the indicated operation on each input to determine its corresponding output. For example, when $x = 3$ is put into the function machine $f$ at right, the function multiplies the 3 by 2 and then adds 1 to get the corresponding output, which is 7. The notation $f(3) = 7$ shows that the function named $f$ connects the input 3 with the corresponding output 7. This also means the point $(3, 7)$ lies on the graph of the function.

**Example 2**

a. If $f(x) = \sqrt{x - 2}$ then $f(11) =$ ? $f(11) = \sqrt{11 - 2}$
   $f(11) = \sqrt{9}$
   $f(11) = 3$

b. If $g(x) = 3 - x^2$ then $g(5) =$ ?
   $g(5) = 3 - (5)^2$
   $g(5) = 3 - 25$
   $g(5) = -22$

c. If $f(x) = \frac{x+3}{2x-5}$ then $f(2) =$ ?
   $f(2) = \frac{2+3}{2\cdot2-5}$
   $f(2) = \frac{5}{1}$
   $f(2) = -5$
Example 3

A relationship in which each input has only one output is called a **function**.

\[
g(x) \quad \text{g(x) is a function; each input (x) has only one output (y).} \\
g(-2) = 1, \ g(0) = 3, \ g(4) = -1, \ \text{and so on.} \\
f(x) \quad \text{f(x) is not a function: each input greater than -3 has two y-values associated with it.} \\
f(1) = 2 \ \text{and } f(1) = -2.
\]

Example 4

The set of all possible inputs is called the **domain**, while the set of all possible outputs of is called the **range**.

In Example 3 above, the domain of \(g(x)\) is \(-2 \leq x \leq 4\), or “all numbers between -2 and 4”. The range of \(g(x)\) is \(-1 \leq g(x) \leq 3\) or “all numbers between -1 and 3”.

The domain of \(f(x)\) in Example 3 above is \(x \geq -3\) or “any real number greater than or equal to 3,” since the graph starts at \(x = -3\) and continues forever to the right. Since the graph of \(f(x)\) extends in both the positive and negative \(y\) (vertical) directions forever, the range is “all real numbers”.

Example 5

For the graph at right, since the graph extends forever horizontally in both directions, the domain is “all real numbers”. The \(y\)-values start at \(y = 1\) and increase, so the range is \(y \geq 1\) or “all numbers greater or equal to 1”.
Problems

Determine the outputs for the following function machines and the given inputs.

1. \(x = 2\)
   \[f(x) = -2x + 4\]
   \(f(x) = ?\)

2. \(x = -6\)
   \[f(x) = |x - 2|\]
   \(f(x) = ?\)

3. \(x = 9\)
   \[f(x) = \sqrt{x} + 1\]
   \(f(x) = ?\)

4. \(f(x) = (5 - x)^2\)
   \(f(8) = ?\)

5. \(g(x) = x^2 - 5\)
   \(g(-3) = ?\)

6. \(f(x) = \frac{2x + 7}{x^2 - 9}\)
   \(f(3) = ?\)

7. \(h(x) = 5 - \sqrt{x}\)
   \(h(9) = ?\)

8. \(h(x) = \sqrt{5 - x}\)
   \(h(9) = ?\)

9. \(f(x) = -x^2\)
   \(f(4) = ?\)

Determine if each graph below represents a function. Then state its domain and range.

10. 

11. 

12. 

13. 

14. 

15. 

**Answers**

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( f(2) = 0 )</td>
<td>2.</td>
</tr>
<tr>
<td>4.</td>
<td>( f(8) = 9 )</td>
<td>5.</td>
</tr>
<tr>
<td>7.</td>
<td>( f(9) = 2 )</td>
<td>8.</td>
</tr>
<tr>
<td>10.</td>
<td>Yes, each input has one output; domain is all numbers, range is (-1 \leq y \leq 3)</td>
<td>11.</td>
</tr>
<tr>
<td>12.</td>
<td>Yes; domain all numbers, range is (-3 \leq y \leq 3)</td>
<td></td>
</tr>
<tr>
<td>13.</td>
<td>No; ( x = -1 ) has two outputs; domain is (-4, -3, -1, 0, 1, 2, 3, 4), range is (-4, -3, -2, -1, 0, 1, 2)</td>
<td>14.</td>
</tr>
<tr>
<td>15.</td>
<td>No, many inputs have two outputs; domain is (-2 \leq x \leq 4) range is (-2 \leq y \leq 4)</td>
<td></td>
</tr>
</tbody>
</table>
Laws of Exponents

In general, to simplify an expression that contains exponents means to eliminate parentheses and negative exponents if possible. The basic laws of exponents are listed below.

1. $x^a \cdot x^b = x^{a+b}$  
   Examples: $x^3 \cdot x^4 = x^{3+4} = x^7$  
               $2^7 \cdot 2^4 = 2^{7+4} = 2^{11}$

2. $\frac{x^a}{x^b} = x^{a-b}$  
   Examples: $\frac{x^{10}}{x^4} = x^{10-4} = x^6$  
               $\frac{2^4}{2^7} = 2^{4-7} = 2^{-3}$ or $\frac{1}{2^3}$

3. $(x^a)^b = x^{ab}$  
   Examples: $(x^4)^3 = x^{4\cdot3} = x^{12}$  
               $(2x^3)^5 = 2^{15} \cdot x^{3\cdot5} = 2^5 \cdot x^{15} = 32x^{15}$

4. $x^0 = 1$  
   Examples: $2^0 = 1$  
               $(-3)^0 = 1$  
               $(\frac{1}{4})^0 = 1$

5. $x^{-n} = \frac{1}{x^n}$  
   Examples: $x^{-3} = \frac{1}{x^3}$  
               $y^{-4} = \frac{1}{y^4}$  
               $\frac{1}{4^{-2}} = \frac{1}{\frac{1}{16}} = \frac{16}{1}$

In all expressions with fractions we assume the denominator does not equal zero.

For additional information, see the Math Notes box in Lesson 1.3.2. For additional examples and practice, see the Checkpoint 4 materials.

**Example 1**

$$\left(2xy^3\right)\left(5x^2y^4\right)$$

Reorder: $2 \cdot 5 \cdot x \cdot x^2 \cdot y^3 \cdot y^4$

Using law (1): $10x^3y^7$

**Example 2**

$$\frac{14x^2y^{12}}{7x^5y^7}$$

Separate: $\frac{14}{7} \cdot \frac{x^2}{x^5} \cdot \frac{y^{12}}{y^7}$

Using laws (2) and (5): $2x^{-3}y^5 = \frac{2y^5}{x^3}$
Example 3
\[(3x^2y^4)^3\]
Using law (3): \[3^3 \cdot (x^3)^3 \cdot (y^3)^3\]
Using law (3) again: \[27x^6y^{12}\]

Example 4
\[(2x^3)^{-2}\]
Using law (5): \[\frac{1}{(2x^3)^2}\]
Using law (3): \[\frac{1}{2^2 \cdot (x^3)^2}\]
Using law (3) again: \[\frac{1}{4x^6}\]

Example 5
Simplify: \[\frac{10x^7y^3}{15x^{-2}y^3}\]
Separate: \[\left(\frac{10}{15}\right) \cdot \left(\frac{x^7}{x^{-2}}\right) \cdot \left(\frac{y^3}{y^3}\right)\]
Using law (2): \[\frac{2}{3}x^9y^0\]
Using law (4): \[\frac{2}{3}x^9 \cdot 1 = \frac{2}{3}x^9 = \frac{2x^9}{3}\]

Problems

Simplify each expression. Final answers should contain no parentheses or negative exponents.

1. \(y^5 \cdot y^7\)
2. \(b^4 \cdot b^3 \cdot b^2\)
3. \(8^6 \cdot 8^{-2}\)
4. \((y^5)^2\)
5. \((3a)^4\)
6. \(\frac{m^8}{m^3}\)
7. \(\frac{12m^6}{6m^{-3}}\)
8. \((x^3y^2)^3\)
9. \(\frac{(y^4)^2}{(y^3)^2}\)
10. \(\frac{15x^2y^5}{3x^4y^5}\)
11. \((4c^4)(ac^3)(3a^{-5}c)\)
12. \((7x^3y^5)^2\)
13. \((4xy^2)(2y)^3\)
14. \(\left(\frac{4}{x^2}\right)^3\)
15. \(\frac{(2a^7)(3a^2)}{6a^3}\)
16. \(\left(\frac{5m^2n}{m^3}\right)^3\)
17. \((3a^2x^3)^2(2ax^4)^3\)
18. \(\left(\frac{x^3y}{y^4}\right)^4\)
19. \(\left(\frac{6x^8y^2}{12x^5y^7}\right)^2\)
20. \(\frac{(2x^5y^3)(4xy^4)^2}{8x^7y^{12}}\)
21. \(x^{-3}\)
22. \(2x^{-3}\)
23. \((2x)^{-3}\)
24. \((2x^3)^0\)
25. \(5^{-2} \cdot 3\)
26. \(\left(\frac{2a^3}{3}\right)^{-2}\)
Answers

1. \( y^{12} \)  
2. \( b^9 \)  
3. \( 8^4 \)  
4. \( y^{10} \)  
5. \( 81a^4 \)  
6. \( m^5 \)  
7. \( 2m^{11} \)  
8. \( x^9y^6 \)  
9. \( y^2 \)  
10. \( \frac{5}{x^2} \)  
11. \( 12a^6c^8 \)  
12. \( 49x^6y^{10} \)  
13. \( 32xy^5 \)  
14. \( \frac{64}{x^6} \)  
15. \( a^6 \)  
16. \( \frac{125n^3}{m^6} \)  
17. \( 72a^7x^{18} \)  
18. \( \frac{x^{12}}{y^{12}} \)  
19. \( \frac{x^{10}}{4y^{10}} \)  
20. \( 16x^{10}y^5 \)  
21. \( \frac{1}{x^3} \)  
22. \( \frac{2}{x^3} \)  
23. \( \frac{1}{8x^3} \)  
24. \( 1 \)  
25. \( \frac{3}{25} \)  
26. \( \frac{9}{4x^2} \)

Scientific Notation

Scientific notation is a way of writing very large and very small numbers compactly. A number is said to be in scientific notation when it is written as the product of two factors as described below.

- The first factor is less than 10 and greater than or equal to 1.
- The second factor has a base of 10 and an integer exponent.
- The factors are separated by a multiplication sign.
- A positive exponent indicates a number whose absolute value is greater than 1.
- A negative exponent indicates a number whose absolute value is less than 1.

<table>
<thead>
<tr>
<th>Scientific Notation</th>
<th>Standard Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 5.32 \times 10^{11} )</td>
<td>532,000,000,000</td>
</tr>
<tr>
<td>( 2.61 \times 10^{-15} )</td>
<td>0.00000000000000261</td>
</tr>
</tbody>
</table>
It is important to note that the exponent does not necessarily mean to use that number of zeros.

The number $5.32 \times 10^{11}$ means $5.32 \times 100,000,000,000$. Thus, two of the eleven decimal places in the standard form of the number are the 3 and the 2 in 5.32. Standard form in this case is $532,000,000,000$. In this example you are moving the decimal point eleven places to the right to write the standard form of the number.

The number $2.61 \times 10^{-15}$ means $2.61 \times 0.0000000000001$. You are moving the decimal point to the left 15 places to write the standard form. Here the standard form is 0.00000000000000261.

For additional information, see the Math Notes box in Lesson 1.3.1.

**Example 1**

Write each number in standard form.

$7.84 \times 10^8 \Rightarrow 784,000,000$ and $3.72 \times 10^{-3} \Rightarrow 0.00372$

When taking a number in standard form and writing it in scientific notation, remember there is only *one* digit to the left of the decimal point allowed.

**Example 2**

Write each number in scientific notation.

$52,050,000 \Rightarrow 5.205 \times 10^7$ and $0.000372 \Rightarrow 3.72 \times 10^{-4}$

The exponent denotes the number of places you moved the decimal point in the standard form. In the first example above, the decimal point is at the end of the number and it was moved 7 places. In the second example above, the exponent is negative because the original number is very small, that is, less than 1.
Problems

Write each number in standard form.

1. $7.85 \times 10^{11}$  
2. $1.235 \times 10^9$  
3. $1.2305 \times 10^3$  
4. $3.89 \times 10^{-7}$  
5. $5.28 \times 10^{-4}$

Write each number in scientific notation.

6. $391,000,000,000$  
7. $0.0000842$  
8. $123056.7$  
9. $0.000000502$  
10. $25.7$  
11. $0.035$  
12. $5,600,000$  
13. $1346.8$  
14. $0.000000000006$  
15. $634,700,000,000,000$

Note: On your scientific calculator, displays like $4.357 \times 10^{12}$ (or $4.357E12$) and $3.65 \times 10^{-3}$ (or $3.65E−3$) are numbers expressed in scientific notation. The first number means $4.375 \times 10^{12}$ and the second means $3.65 \times 10^{-3}$. The calculator does this because there is not enough room on its display window to show the entire number.

Answers

1. $785,000,000,000$  
2. $1,235,000,000$  
3. $1230.5$  
4. $0.000000389$  
5. $0.000528$  
6. $3.91 \times 10^{11}$  
7. $8.42 \times 10^{-5}$  
8. $1.230567 \times 10^5$  
9. $5.02 \times 10^{-7}$  
10. $2.57 \times 10^1$  
11. $3.5 \times 10^{-2}$  
12. $5.6 \times 10^6$  
13. $1.3468 \times 10^3$  
14. $6.0 \times 10^{-12}$  
15. $6.347 \times 10^{14}$
Students used the equation $y = mx + b$ to graph lines and describe patterns in previous courses. Lesson 2.1.1 is a review of writing linear equations. When the equation of a line is written in $y = mx + b$ form, the coefficient $m$ represents the slope of the line. Slope indicates the direction of the line and its steepness. The constant term $b$ is the $y$-intercept, written $(0, b)$, and indicates where the line crosses the $y$-axis.

For additional information about slope, see the Math Notes box in Lesson 2.1.4.

**Example 1**

If $m$ is positive, the line goes upward from left to right. If $m$ is negative, the line goes downward from left to right. If $m = 0$ then the line is horizontal. The value of $b$ indicates the $y$-intercept.

\[
y = \frac{1}{2}x - 1 \quad y = -\frac{1}{2}x - 1 \quad y = 0x + 3 \text{ or } y = 3
\]

**Example 2**

When $m = 1$, as in $y = x$ (or $y = 1x + 0$), the line goes upward by one unit each time it goes over one unit to the right. Steeper lines have a greater $m$ value, that is, $m > 1$ or $m < -1$. Flatter lines have an $m$-value that is between $-1$ and 1, often in the form of a fraction. All three examples below have $b = 2$ (a $y$-intercept of 2).

\[
y = x + 2 \quad y = -3x + 2 \quad y = \frac{1}{3}x + 2
\]

(steeper and in the downward direction) (less steep)
Example 3

If a line is drawn on a set of axes, a slope triangle can be drawn between any two convenient points (usually where grid lines cross), as shown in the graph at right. Count the vertical distance (labeled $\Delta y$) and the horizontal distance (labeled $\Delta x$) on the dashed sides of the slope triangle. Write the distances in a ratio: $\text{slope} = m = \frac{\Delta y}{\Delta x} = \frac{2}{3}$. The symbol $\Delta$ means change. The order in the fraction is important: the numerator (top of the fraction) must be the vertical distance and the denominator (bottom of the fraction) must be the horizontal distance. The slope of a line is constant, so the slope ratio is the same for any two points on the line.

Parallel lines have the same steepness and direction, so they have the same slope, as shown in the graph at right.

If $\Delta y = 0$, then the line is horizontal and has a slope of zero, that is, $m = 0$. If $\Delta x = 0$, then the line is vertical and its slope is undefined, so we say that it has no slope.

Example 4

When the vertical and horizontal distances are not easy to determine, you can find the slope by drawing a generic slope triangle and using it to find the lengths of the vertical $\Delta y$ and horizontal ($\Delta x$) segments. The figure at right shows how to find the slope of the line that passes through the points $(-21, 9)$ and $(19, -15)$. First graph the points on unscaled axes by approximating where they are located, and then draw a slope triangle. Next find the distance along the vertical side by noting that it is 9 units from point B to the $x$-axis then 15 units from the $x$-axis to point C, so $\Delta y$ is 24. Then find the distance from point A to the $y$-axis (21) and the distance from the $y$-axis to point B (19). $\Delta x$ is 40. This slope is negative because the line goes downward from left to right, so the slope is $m = \frac{\Delta y}{\Delta x} = \frac{-24}{40} = -\frac{3}{5}$.

Example 5

The equation of a vertical line is $x = \text{a number}$. For example, the graph at right shows the line $x = 3$. Every point on the line has an $x$-coordinate of 3.

The slope of the line is $m = \frac{\Delta y}{\Delta x} = \frac{\text{any number}}{0}$ and since division by zero is not possible, the slope is undefined.
Problems

Is the slope of each line negative, positive, or zero?

1.  
2.  
3.  

Identify the slope in each equation. State whether the graph of the line is steeper or flatter than \( y = x \) or \( y = -x \), whether it goes up or down from left to right, or if it is horizontal or vertical.

4. \( y = 3x + 2 \)  
5. \( y = -\frac{1}{2}x + 4 \)  
6. \( y = \frac{1}{3}x - 4 \)
7. \( 4x - 3 = y \)  
8. \( y = -2 + \frac{1}{2}x \)  
9. \( 3 + 2y = 8x \)
10. \( y = 2 \)  
11. \( x = 5 \)  
12. \( 6x + 3y = 8 \)

Without graphing, determine the slope of each line based on the given information.

13. \( \Delta y = 27 \) and \( \Delta x = -8 \)  
14. \( \Delta x = 15 \) and \( \Delta y = 3 \)  
15. \( \Delta y = 7 \) and \( \Delta x = 0 \)
16. horizontal \( \Delta = 6 \)  
17. between \( (5, 28) \) and \( (64, 12) \)  
18. between \( (-3, 2) \) and \( (5, -7) \)

Answers

1. zero  
2. negative  
3. positive  
4. slope = 3, steeper, up  
5. slope = \( -\frac{1}{2} \), flatter, down  
6. slope = \( \frac{1}{3} \), flatter, up  
7. slope = 4, steeper, up  
8. slope = \( \frac{1}{2} \), flatter, up  
9. slope = 4, steeper, up  
10. slope = 0, horizontal  
11. slope is undefined, vertical  
12. slope = -2, steeper, down  
13. \( -\frac{27}{8} \)  
14. \( \frac{3}{15} = \frac{1}{5} \)  
15. undefined  
16. 0  
17. \( -\frac{16}{59} \)  
18. \( -\frac{9}{8} \)
Dimensional analysis (or unit analysis) means using the rules for multiplying and simplifying fractions to solve problems involving different units.

Unit conversion equations are written as a fraction (equivalent to multiplying by “one”) so that the unwanted units are removed during simplification and the desired units remain.

For information about conversion factors, metric prefixes, and common abbreviations, see the Math Notes box in Lesson 2.2.4. For additional information about dimensional analysis, see the Math Notes box in Lesson 2.3.1.

Example 1
A driveway is \(15 \frac{3}{4}\) feet long. How long is this in inches?

Use: \(12\) inches = \(1\) foot

We want the “foot” units to cancel so the fraction we want is: \(\frac{12 \text{ inches}}{1 \text{ foot}}\).

\[
15 \frac{3}{4} \text{ feet} = \frac{63 \text{ feet}}{4} \cdot \frac{12 \text{ inches}}{1 \text{ foot}} = \frac{756 \text{ inches}}{4} = 189 \text{ inches}
\]

Example 2
An automobile web site on the Internet advertises that the new Neptune Stratus averages \(15\) kilometers per liter of petroleum. What is the equivalent in miles per gallon?

Use: 1 gallon = 3.79 liters 1 mile = 1.61 kilometers

We want the “kilometer” and “liter” units to cancel. Use the fractions: \(\frac{1 \text{ mile}}{1.61 \text{ kilometers}}\) and \(\frac{3.79 \text{ liters}}{1 \text{ gallon}}\).

\[
15 \text{ kilometers} \cdot \frac{1 \text{ mile}}{1.61 \text{ kilometers}} \cdot \frac{3.79 \text{ liters}}{1 \text{ gallon}} = 35.3 \text{ miles per gallon}
\]

Example 3
A container is strong enough not to break under a weight of 40 pounds per square inch \(\left(\frac{40 \text{ pounds}}{(\text{inch})^2}\right)\).

What is the equivalent in grams per square centimeter \(\left(\frac{\text{grams}}{\text{centimeter}^2}\right)\)?

Use: 1 kilogram = 1000 grams = 2.2 pounds 1 inch = 2.54 centimeters

We want “pounds” and “inches” unit to cancel. Use the fractions: \(\frac{1000 \text{ grams}}{2.2 \text{ pounds}}\) and \(\frac{1 \text{ inch}}{2.54 \text{ centimeter}}\).

Notice that to cancel the \((\text{inch})^2\) we will need to multiply by the second fraction twice.

\[
\frac{40 \text{ pounds}}{(\text{inch})^2} \cdot \frac{1000 \text{ grams}}{2.2 \text{ pounds}} \cdot \frac{1 \text{ inch}}{2.54 \text{ cm}} \cdot \frac{1 \text{ inch}}{2.54 \text{ cm}} = 2800 \frac{\text{grams}}{\text{centimeter}^2}
\]
Use unit analysis and the following conversion equations to solve each problem.

<table>
<thead>
<tr>
<th>1 hour = 60 minutes</th>
<th>1 week = 7 days</th>
<th>1 day = 24 hours</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year = 365 days</td>
<td>1 liter = 1000 milliliters</td>
<td>1 liter = 1000 centimeters</td>
</tr>
<tr>
<td>1 milliliter = 20 drops</td>
<td>1 kilometer = 0.625 miles</td>
<td>1 mile = 5280 feet</td>
</tr>
<tr>
<td>1 gallon = 3.79 liters</td>
<td>1 inch = 2.54 centimeters</td>
<td>1 gallon = 128 fluid ounces</td>
</tr>
<tr>
<td>1 gallon = 4 quarts</td>
<td>1 meter = 100 centimeters</td>
<td>1 foot = 12 inches</td>
</tr>
<tr>
<td>1 yard = 3 feet</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Problems**

1. A 10-kilometer race is how many miles?
2. One year is how many hours?
3. The distance to the moon is about 250,000 miles. How many feet is this?
4. Traveling 50 miles per hour is the same as how many feet per second?
5. Two hundred fluid ounces is how many gallons?
6. Five hundred feet is the same as how many meters?
7. If tile costs $8.50 per square foot, how much does it cost per square inch?
8. If carpet is advertised at $20 per square yard, what is the cost per square foot?
9. A soda can holds 355 milliliters. How many drops is this? How many gallons is this?
10. A worm moved 8 inches in 5 seconds. How many miles per hour is this?
11. A chemistry experiment calls for 2 drops of acid for 100 milliliters of solution. How much acid should be used for one gallon of solution?
12. If I buy $30 of gasoline at $2.75 per gallon and my car gets 34.2 miles per gallon, how far will I travel on my $30?
13. A swimming pool holds 10,000 gallons of water. How many cubic meters is this?
14. John’s go-cart went 10 laps around a 1320-foot track in 12 minutes. How fast was he traveling in kilometers per hour?
15. In October 2007, one euro was worth $1.42. The price of gasoline in Switzerland was 0.85 euros per liter. What was the cost in dollars per gallon?
## Answers

1. 6.25 miles  
2. 8760 hours  
3. 1,320,000,000 feet  
4. \( \approx 73.3 \text{ feet/second} \)  
5. 1.5625 gallons  
6. \( \approx 152.4 \text{ meters} \)  
7. \( \approx \$0.06 \text{ per sq. inch} \)  
8. \( \$2.22 \text{ per ft}^2 \)  
9. 7100 drops \( \approx \) 0.09 gallons  
10. \( \approx 0.09 \text{ miles/hour} \)  
11. 75.8 drops = 4 ml  
12. 373 miles  
13. 3.79 meters\(^3\)  
14. 20 \( \frac{\text{kilometers}}{\text{hour}} \)  
15. \( \approx \$4.57 \text{ per gallon} \)
WRITING THE EQUATION OF A LINE GIVEN
THE SLOPE AND A POINT

In earlier work students used substitution in equations like \( y = 2x + 3 \) to find \( x \) and \( y \) pairs that make the equation true. Students recorded those pairs in a table, and then used them as coordinates to graph a line. Every point \((x, y)\) that makes the equation true (or is a solution to the equation) lies on the graph of the line.

Students then used the patterns they saw in the tables and graphs to recognize and write equations in the form of \( y = mx + b \). The “\( b \)” represents the \( y \)-intercept of the line, the “\( m \)” represents the slope, while \( x \) and \( y \) represent the coordinates of any point on the line. Each line has a specific value for \( m \) and a specific value for \( b \), but there are infinitely many \((x, y)\) coordinate pairs that are solutions to a linear equation.

The slope of the line is the same between any two points on that line. We can use this information to write equations without creating tables or graphs.

For additional information, see the Math Notes boxes in Lessons 2.2.2, 2.2.3, and 2.3.2.

Example 1

What is the equation of the line with a slope of 2 that passes through the point \((10, 17)\)?

Write the general equation of a line. \( y = mx + b \)

Substitute the values we know: \( m, x, \) and \( y \).

\[ 17 = 2(10) + b \]
\[ 17 = 20 + b \]
\[ -3 = b \]

Write the complete equation using the values of \( m = 2 \) and \( b = -3 \).

\[ y = 2x - 3 \]

Problems

Write the equation of the line with the given slope that passes through the given point.

1. slope = 5, \((3, 13)\) 2. slope = \(-\frac{5}{3}\), \((3, -1)\) 3. slope = -4, \((-2, 9)\)
4. slope = \(\frac{3}{2}\), \((6, 8)\) 5. slope = 3, \((-7, -23)\) 6. slope = 2, \((\frac{5}{2}, -2)\)
Answers

1. $y = 5x - 2$

2. $y = -\frac{5}{3}x + 4$

3. $y = -4x + 1$

4. $y = \frac{3}{2}x - 1$

5. $y = 3x - 2$

6. $y = 2x - 7$
WRITING THE EQUATION OF A LINE GIVEN TWO POINTS 2.3.2

Students now have all the tools they need to find the equation of a line passing through two given points. Recall that the equation of a line requires a slope and a $y$-intercept in $y = mx + b$. Students can write the equation of a line from two points by creating a slope triangle and calculating $\frac{\Delta y}{\Delta x}$ as explained in Lessons 2.1.2 through 2.1.4.

For additional information, see the Math Notes box in Lesson 2.3.2. For additional examples and more practice, see the Checkpoint 5 materials.

Example 1

Write the equation of the line that passes through the points $(1, 9)$ and $(-2, -3)$.

Position the two points approximately where they belong on coordinate axes—you do not need to be precise. Draw a generic slope triangle.

Calculate \( \text{slope} = \frac{\Delta y}{\Delta x} = \frac{12}{3} = 4 \) using the given values of the two points.

Write the general equation of a line. Substitute \( m \) and either one of the points into the equation. For example, use \((x, y) = (1, 9)\) and \( m = 4 \).

Solve for \( b \). Write the complete equation.

\[ 5 = b \quad \frac{12}{3} \quad y = 4x + 5 \]

Example 2

Write the equation of the line that passes through the points $(8, 3)$ and $(4, 6)$.

Draw a generic slope triangle located approximately on coordinate axes. Approximate the locations of the given points.

Calculate \( m = \frac{\Delta y}{\Delta x} = -\frac{3}{4} \). The slope is negative since the line goes down from left to right.

Substitute \( m \) and either one of the points, for example $(8, 3)$, into the general equation for a line.

Solve for \( b \). Write the complete equation.

\[ 9 = b \quad \frac{-3}{4} \quad y = -\frac{3}{4}x + 9 \]
Problems

Write the equation of the line containing each pair of points.

1. (1, 1) and (0, 4)  
   2. (5, 4) and (1, 1)  
   3. (1, 3) and (−5, −15)  
   4. (−2, 3) and (3, 5)  
   5. (2, −1) and (3, −3)  
   6. (4, 5) and (−2, −4)  
   7. (1, −4) and (−2, 5)  
   8. (−3, −2) and (5, −2)  
   9. (−4, 1) and (5, −2)

Answers

1. \(y = −3x + 4\)  
   2. \(y = \frac{3}{4}x + \frac{1}{4}\)  
   3. \(y = 3x\)  
   4. \(y = \frac{2}{3}x + 3\frac{4}{5}\)  
   5. \(y = −2x + 3\)  
   6. \(y = \frac{3}{2}x − 1\)  
   7. \(y = −3x − 1\)  
   8. \(y = −2\)  
   9. \(y = −\frac{1}{3}x − \frac{1}{3}\)
TRANSFORMATIONS AND SYMMETRY  3.1.1, 3.1.2, and 3.1.4

Studying transformations of geometric shapes builds a foundation for a key idea in geometry: congruence. The students explore three rigid transformations: translations, reflections, and rotations. These explorations are done with tracing paper as well as with dynamic tools on the computer or other device. Students apply one or more of these motions to a figure, creating its image in a new position without changing its size or shape. Rigid transformations also lead to studying symmetry in shapes. Symmetry will help with describing and classifying geometric shapes.

See the Math Notes box in Lesson 3.1.4 for more information about rigid transformations.

Example 1

Decide which rigid transformation was used on each pair of shapes below. Some may be a combination of transformations.

Part (a): The parallelogram is reflected (flipped) across an invisible vertical line. (Imagine a mirror running vertically between the two figures. One figure would be the reflection of the other.) Reflecting a shape once changes its orientation. For example, the two sides of the figure at left slant upwards to the right, whereas in its reflection at right, they slant upwards to the left. Likewise, the angles in the figure at left “switch positions” in the figure at right.

Part (b): The shape is translated (or slid) to the right and down. The orientation remains the same, with all sides slanting the same.

Part (c): This is a combination of transformations. First the triangle is reflected (flipped) across an invisible horizontal line. Then it is translated (slid) to the right.
Part (d): The pentagon has been rotated (turned) clockwise to create the second figure. Imagine tracing the first figure on tracing paper, then holding the tracing paper with a pin at one point below the first pentagon, then turning the paper to the right 90°. The second pentagon would be the result. Some students might see this as a reflection across a diagonal line. The pentagon itself could be, but with the added dot, the entire shape cannot be a reflection. If it had been reflected, the dot would have to be on the corner below the one shown in the rotated figure.

Part (e): The triangles are rotations of each other (90° again).

Part (f): This is another combination transformation. The triangle is rotated (the shortest side becomes horizontal instead of vertical) and reflected.

Example 2

What will the figure at right look like if it is first reflected across line \( l \) and then the result is reflected across line \( m \)?

The first reflection is the new figure shown between the two lines. If we were to join each vertex (corner) of the original figure to its corresponding vertex on the second figure, those line segments would be perpendicular to line \( l \) and the vertices of (and all the other points in) the reflection would be the same distance away from \( l \) as they are in the original figure. One way to draw the reflection is to use tracing paper to trace the figure and the line \( l \). Then turn the tracing paper over, so that line \( l \) is on top of itself. This will show the position of the reflection. Transfer the figure to your paper by tracing it. Repeat this process with line \( m \) to form the third figure by tracing.

As students discovered in class, reflecting twice like this across two intersecting lines produces a rotation of the figure about the point P. Put the tracing paper back over the original figure to line \( l \). Put a pin or the point of a pen or pencil on the tracing paper at point P (the intersection of the lines of reflection) and rotate the tracing paper until the original figure will fit perfectly on top of the last figure.
Example 3

The shape at right is trapezoid ABCD. Translate the trapezoid 7 units to the right and 4 units up. Label the new trapezoid A’B’C’D’ and give the coordinates of its vertices. Is it possible to translate the original trapezoid in such a way to create A”B”C”D” so that it is a reflection of ABCD? If so, what would be the reflecting line? Will this always be possible for any figure?

Translating (or sliding) the trapezoid 7 units to the right and 4 units up gives a new trapezoid A’(2, 2), B’(4, 2), C’(5, 0), and D’(1, 0). If we go back to trapezoid ABCD, we now wonder if we can translate it in such a way that we can make it look as if it were a reflection rather than a translation. Since the trapezoid is symmetrical, it is possible to do so. We can slide the trapezoid horizontally left or right. In either case, the resulting figure would look like a reflection.

This will not always work. It works here because we started with an isosceles trapezoid, which has a line of symmetry itself. Students explored which polygons have lines of symmetry, and which have rotational symmetry as well. Again, they used tracing paper as well as technology to investigate these properties.

Exploring these transformations and symmetrical properties of shapes helps to improve students’ visualization skills. These skills are often neglected or taken for granted, but much of mathematics requires students to visualize pictures, problems, or situations. That is why we ask students to “visualize” or “imagine” what something might look like as well as practice creating transformations of figures.
Problems

Perform the indicated transformation on each polygon below to create a new figure. You may want to use tracing paper to see how the figure moves.

1. Rotate Figure A 90° clockwise about the origin.

2. Reflect Figure B across line l.

3. Translate Figure C 6 units left.

4. Rotate Figure D 270° clockwise about the origin (0, 0).

For problems 5 through 20, refer to the figures below.
State the new coordinates after each transformation.

5. Translate Figure A left 2 units and down 3 units.

6. Translate Figure B right 3 units and down 5 units.

7. Translate Figure C left 1 unit and up 2 units.

8. Reflect Figure A across the x-axis.

9. Reflect Figure B across the x-axis.

10. Reflect Figure C across the x-axis.

11. Reflect Figure A across the y-axis.

12. Reflect Figure B across the y-axis.

13. Reflect Figure C across the y-axis.

14. Rotate Figure A 90° counterclockwise about the origin.

15. Rotate Figure B 90° counterclockwise about the origin.

16. Rotate Figure C 90° counterclockwise about the origin.

17. Rotate Figure A 180° counterclockwise about the origin.

18. Rotate Figure C 180° counterclockwise about the origin.

19. Rotate Figure B 270° counterclockwise about the origin.

20. Rotate Figure C 90° clockwise about the origin.

21. Plot the points A(3, 3), B(6, 1), and C(3, -4). Translate the triangle 8 units to the left and 1 unit up to create ΔA′B′C′. What are the coordinates of the new triangle?

22. How can you translate ΔABC in the last problem to put point A″ at (4, -5)?

23. Reflect Figure Z across line l, and then reflect the new figure across line m. What are these two reflections equivalent to?
For each shape below, (i) draw all lines of symmetry, and (ii) describe its rotational symmetry if it exists.

24. 

25. 

26. 

27. 

Answers

1.

2.

3.

4.
5. \((-1, -3) \ (1, 1) \ (3, -1)\)  
6. \((-2, -3) \ (2, -3) \ (3, 0)\)

7. \((-5, 4) \ (3, 4) \ (-3, -1)\)  
8. \((1, 0) \ (3, -4) \ (5, -2)\)

9. \((-5, -2) \ (-1, -2) \ (0, -5)\)  
10. \((-4, -2) \ (4, -2) \ (-2, 3)\)

11. \((-1, 0) \ (-3, 4) \ (-5, 2)\)  
12. \((5, 2) \ (1, 2) \ (0, 5)\)

13. \((4, 2) \ (-4, 2) \ (2, -3)\)  
14. \((0, 1) \ (-4, 3) \ (-2, 5)\)

15. \((-2, -5) \ (-2, -1) \ (-5, 0)\)  
16. \((-2, -4) \ (-2, 4) \ (3, -2)\)

17. \((-1, 0) \ (-3, -4) \ (-5, -2)\)  
18. \((4, -2) \ (-4, -2) \ (2, 3)\)

19. \((2, 5) \ (2, 1) \ (5, 0)\)  
20. \((2, 4) \ (2, -4) \ (-3, 2)\)

21. \(A'(5, -4) \ B'(-2, 2) \ C'(-5, -3)\)

22. Translate it 1 unit right and 8 units down.

23. The two reflections are the same as rotating \(Z\) about point \(X\).

24. This has 180° rotational symmetry.

25. No rotational symmetry.

26. A circle has infinitely many lines of symmetry, every one of them illustrates reflection symmetry. It also has rotational symmetry for every possible degree measure.

27. This irregular shape has no lines of symmetry and does not have rotational symmetry, nor reflection symmetry.
SLOPES OF PARALLEL AND PERPENDICULAR LINES 3.1.3

Parallel lines have the same slope. The most commonly used equation of a line is **slope-intercept** form or $y = mx + b$, where $m$ represents the slope and $b$ is the $y$-intercept.

For perpendicular lines, the product of the slopes equals $-1$.

See the Math Notes box in Lesson 3.1.6 for more information about slopes of parallel and perpendicular lines.

**Example 1**

Write the equation of the line that is parallel to $y = \frac{1}{2}x - 5$ and passes through the point $(4, 10)$.

Any line parallel to the line $y = \frac{1}{2}x - 5$, which has a slope of $\frac{1}{2}$, must also have a slope of $\frac{1}{2}$. Therefore, the equation must look like $y = \frac{1}{2}x + b$.

Substituting the given point in place of $x$ and $y$ we have: $10 = \frac{1}{2}(4) + b$. Solving, we find that $b = 8$. The equation is $y = \frac{1}{2}x + 8$.

**Example 2**

Write the equation of the line that is perpendicular to $y = \frac{1}{2}x - 5$, passing through the point $(4, 10)$.

Since the original line has a slope of $\frac{1}{2}$, a perpendicular line must have a slope of $-2$.

Therefore, the equation must look like $y = -2x + b$.

Substituting the given point in place of $x$ and $y$ we have: $10 = -2(4) + b$. Solving, we find that $b = 18$. The equation is $y = -2x + b$. 
Problems

Write an equation for each of the lines described below.

1. The line with a slope of \( \frac{1}{3} \), passing through the point \((-2, 5)\).
2. The line parallel to \( y = \frac{2}{3} x + 5 \), passing through the point \((3, 2)\).
3. The line parallel to \( y = -\frac{3}{4} x - 2 \), passing through the point \((-4, 2)\).
4. The line parallel to the line determined by the points \((-3, -2)\) and \((2, 4)\), passing through the point \((0, -1)\).
5. The line parallel to \( y = 7 \), passing through the point \((-2, 5)\).
6. The line through the point \((0, 0)\), and parallel to \( y = 2x - 3 \).
7. The line perpendicular to \( y = \frac{2}{3} x + 5 \), passing through the point \((3, 2)\).
8. The line perpendicular to \( y = -\frac{3}{4} x + 1 \), passing through the point \((-4, 2)\).
9. The line perpendicular to \( y = \frac{2}{3} x - 2 \), passing through the point \((0, 3)\).
10. The line perpendicular to \( y = 7 \), passing through the point \((-2, 5)\).

Answers

1. \( y = \frac{1}{3} x + 5 \frac{2}{3} \)
2. \( y = \frac{2}{3} x \)
3. \( y = -\frac{3}{4} x + 1 \)
4. \( y = \frac{6}{3} x - 1 \)
5. \( y = 5 \)
6. \( y = 2x \)
7. \( y = -\frac{3}{2} x + 6 \frac{1}{2} \)
8. \( y = \frac{4}{3} x + \frac{22}{3} \)
9. \( y = -\frac{3}{2} x + 3 \)
10. \( x = -2 \)
Two ways to find the area of a rectangle are: as a product of the (height) \cdot (base) or as the sum of the areas of individual pieces of the rectangle. For a given rectangle these two areas must be the same, so **area as a product = area as a sum**. Algebra tiles, and later, area models, help multiply expressions in a visual, concrete manner.

For additional information, see the Math Notes boxes in Lessons 3.2.1 and 3.2.2. For additional examples and practice, see the Checkpoint 6B materials.

**Example 1: Using Algebra Tiles**

The algebra tile pieces \( x^2 + 6x + 8 \) are arranged into a rectangle as shown at right. The area of the rectangle can be written as the **product** of its base and height or as the **sum** of its parts.

\[
\begin{align*}
\text{area as a product} & = (x + 4)(x + 2) \\
\text{area as a sum} & = x^2 + 6x + 8
\end{align*}
\]

**Example 2: Using an Area Model**

An area model allows us to organize the problem in the same way as the first example without needing to draw the individual tiles. It does not have to be drawn accurately or to scale.

Multiply \((x - 3)(2x + 1)\).

\[
\begin{align*}
\text{area as a product} & = (x - 3)(2x + 1) \\
\text{area as a sum} & = 2x^2 - 5x - 3
\end{align*}
\]
Problems

Write an equation showing area as a product equals area as a sum.

1. 

2. 

3. 

4. 

5. 

6. 

Multiply.

7. \((3x + 2)(2x + 7)\)  
8. \((2x - 1)(3x + 1)\)  
9. \((2x)(x - 1)\)  
10. \((2y - 1)(4y + 7)\)  
11. \((y - 4)(y + 4)\)  
12. \((y)(x - 1)\)  
13. \((3x - 1)(x + 2)\)  
14. \((2y - 5)(y + 4)\)  
15. \((3y)(x - y)\)  
16. \((3x - 5)(3x + 5)\)  
17. \((4x + 1)^2\)  
18. \((x + y)(x + 2)\)  
19. \((2y - 3)^2\)  
20. \((x - 1)(x + y + 1)\)  
21. \((x + 2)(x + y - 2)\)

Answers

1. \((x + 1)(x + 3) = x^2 + 4x + 3\)  
2. \((x + 2)(2x + 1) = 2x^2 + 5x + 2\)  
3. \((x + 2)(2x + 3) = 2x^2 + 7x + 6\)  
4. \((x - 5)(x + 3) = x^2 - 2x - 15\)  
5. \(6(3y - 2x) = 18y - 12x\)  
6. \((x + 4)(3y - 2) = 3xy - 2x + 12y - 8\)  
7. \(6x^2 + 25x + 14\)  
8. \(6x^2 - x - 1\)  
9. \(2x^2 - 2x\)  
10. \(8y^2 + 10y - 7\)  
11. \(y^2 - 16\)  
12. \(xy - y\)  
13. \(3x^2 + 5x - 2\)  
14. \(2y^2 + 3y - 20\)  
15. \(3xy - 3y^2\)  
16. \(9x^2 - 25\)  
17. \(16x^2 + 8x + 1\)  
18. \(x^2 + 2x + xy + 2y\)  
19. \(4y^2 - 12y + 9\)  
20. \(x^2 + xy - y - 1\)  
21. \(x^2 + xy + 2y - 4\)
MULTIPLE METHODS FOR SOLVING EQUATIONS  3.3.1 – 3.3.3

Equations may be solved in a variety of ways. These sections in the textbook use three methods that allow students to solve complex equations in more efficient ways. The methods are called Rewriting, Looking Inside, and Undoing.

For additional information, see the Math Notes box in Lesson 3.3.3.

Example 1  Rewriting

Solve: \( \frac{x}{2} + \frac{x}{5} = 3 \)

This problem can be rewritten without fractions by using Fraction Busters. Begin by determining the common denominator of the fractions.

\[
10 \cdot \left( \frac{x}{2} + \frac{x}{5} \right) = 10(3) \\
5x + 2x = 30 \\
7x = 30 \\
x = \frac{30}{7} \approx 4.29
\]

Example 2  Looking Inside

Solve: \( \sqrt{x} + 1 = 6 \)

This problem has a square root and a sum, so looking inside can solve this.

\[
\begin{align*}
\sqrt{x} + 1 &= 6 \\
\sqrt{x} &= 5 \\
x &= 25
\end{align*}
\]

Example 3  Undoing

Solve: \( \frac{2}{5}x + 1 = 7 \)

Undoing the addition and then the multiplication by a fraction can solve this problem.

\[
\begin{align*}
\frac{2}{5}x + 1 - 1 &= 7 - 1 \\
\frac{2}{5}x &= 6 \\
\frac{3}{2} \left( \frac{2}{5}x \right) &= \frac{3}{2} (6) \\
x &= 9
\end{align*}
\]

Example 4  Rewriting

Solve \((x - 3)(x + 6) = x^2 - 12x - 6\)

Rewriting the left side of the equation using an area model can solve this problem.

\[
\begin{align*}
(x - 3)(x + 6) &= x^2 - 12x - 6 \\
x^2 - 3x + 6x - 18 &= x^2 - 12x - 6 \\
-x^2 - x^2 &= -12x - 6 \\
3x - 18 &= -12x - 6 \\
+18 &+ 18 \\
15x &= 12 \\
\frac{15x}{15} &= \frac{12}{15} \\
x &= \frac{12}{15} = \frac{4}{5}
\end{align*}
\]
Example 5  Rewriting and Looking Inside to Solve Exponential Equations

Solve: \(10^{-5x+7} = 10^x\)

Rewriting so that the bases are the same and then looking inside the exponents can solve this problem. Once the equations are rewritten such that the bases are the same, then the exponents need to be the same.

Rewriting:

Looking Inside:

\[
\begin{align*}
10^{-5x+7} &= 10^x \\
\Rightarrow 10^{-5x+7} &= 10^{2x} \\
-5x + 7 &= 2x \\
7 &= 7x \\
x &= 1
\end{align*}
\]

Problems

Solve each equation. Find all solutions.

1. \(\frac{3x}{2} + \frac{1}{2} = 5\)  
2. \((x - 1)(x + 3) = x^2 - 2x + 13\)  
3. \(\sqrt{2x} + 5 = 10\)

4. \(625 = 5^{-6x+5}\)  
5. \(3(2x - 7) = -21\)  
6. \(2^{4x-1} = 32\)

7. \(\sqrt{x} - 3 = 7\)  
8. \((x + 1)^2 = 81\)  
9. \(\frac{x}{2} - \frac{x}{5} = 3\)

10. \(3^{x+3} = 27\)  
11. \(2\sqrt{x} - 3 = 8\)  
12. \(0.04(x - 1) = 0.16\)

13. \(400 + 300x = 1300\)  
14. \(10 - (x + 7) = 5\)  
15. \(\frac{m}{3} - \frac{2m}{5} = \frac{1}{5}\)

16. \(x^2 + 5 = 4\)  
17. \(x(3x - 1) + 8 = (3x + 1)(x - 5)\)  
18. \(0.2x - 0.4 = 1.2\)

19. \((y - 1)^2 = 9\)  
20. \(20,000 - (3000x) = 10,000\)

Answers

1. \(x = 3\)  
2. \(x = 4\)  
3. \(x = \frac{95}{2} = 47.5\)

4. \(x = -\frac{1}{6}\)  
5. \(x = 0\)  
6. \(x = 1.5\)

7. \(x = 100\)  
8. \(x = 8\) or \(-10\)  
9. \(x = 10\)

10. \(x = 0\)  
11. \(x = 19\)  
12. \(x = 5\)

13. \(x = 3\)  
14. \(y = -2\)  
15. \(m = -3\)

16. no solution  
17. \(x = -1\)  
18. \(x = 8\)

19. \(y = 4\) or \(-2\)  
20. \(x = \frac{10}{3}\)
A line of best fit can model an association between two variables. In this lesson, lines of best fit are estimated by “eyeballing” them on a scatterplot.

Once a line of best fit is created, the slope and y-intercept can be determined. In statistical analyses, the slope is often written as the amount of change expected in the dependent variable ($\Delta y$), when the independent variable is changed by one unit ($\Delta x = 1$). The y-intercept is the predicted value of the dependent variable when the value of the independent variable is zero. In statistical scatterplots, the vertical axis is often not drawn at the origin, so the y-intercept can be someplace other than where the line of best fit crosses the vertical axis of the scatterplot.

Associations between two variables are often described by their form, direction, strength of association, and outliers. Form refers to the shape of the pattern in the scatterplot: linear, some type of curve, or perhaps no pattern at all. Direction refers to whether the pattern is in general positive (increasing from left to right) or negative (decreasing from left to right). For linear relationships, the slope can be used to describe the steepness in addition to the direction. The strength of the relationship indicates how closely the data are to the line of best fit. Outliers are data points far removed from the bulk of the data.

For additional information, see the Math Notes box in Lessons 4.1.2 and review the Checkpoint 7 materials. (Note that the following problems do not use a graphing calculator, while the problems in Checkpoint 7 assume a graphing calculator is available. Graphing calculators are introduced for statistical calculations in Lesson 4.1.4.)

**Example**

It seems reasonable that there would be a relationship between the amount of time a student spends studying and their GPA. Suppose you were interested in predicting a student’s GPA based on the hours they study per week. You were able to randomly select 12 students and obtain this information from each student.

<table>
<thead>
<tr>
<th>Hours</th>
<th>4</th>
<th>5</th>
<th>11</th>
<th>1</th>
<th>15</th>
<th>2</th>
<th>10</th>
<th>6</th>
<th>7</th>
<th>0</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>GPA</td>
<td>2.9</td>
<td>3.3</td>
<td>3.9</td>
<td>2.2</td>
<td>4.1</td>
<td>1.8</td>
<td>4.6</td>
<td>2.9</td>
<td>2.2</td>
<td>3</td>
<td>3.3</td>
<td>4.5</td>
</tr>
</tbody>
</table>

a. Without a calculator, make a scatterplot.

*Example continues on next page →*
Example continued from previous page.

b. Estimate a line of best fit (use a ruler and “eyeball” it), and determine its equation.

If it is reasonable, draw the line of best fit through a point where grid lines intersect or a data point. That way, the equation will be easier to write. In this problem, the data points (1, 2.2) and (11, 3.9) seem to be on a line that “fits” the data.

Using those two points, and techniques from Lesson 2.3.2, the equation for the estimated line of best fit is

\[ y = 2.03 + 0.17x \]

where \( y \) is the predicted GPA and \( x \) is the number of hours per week the student studies. Note that the variables and their units were defined.

c. Interpret the slope and \( y \)-intercept in context.

The slope indicates that a student’s GPA is expected to increase by 0.17 points for every additional hour of studying per week. The \( y \)-intercept predicts that students who do not study at all will have a GPA of 2.03.

d. Describe the association.

The form is linear; it does not appear to be curved nor simply a collection of randomly scattered points. The direction is positive; in general, students who study more also have higher GPAs. A student’s GPA is expected to increase by 0.17 points for every additional hour of studying per week. The strength is moderate: it is strong enough to easily see its form, but there is scatter about the line. There do not appear to be any outliers.

In summary one could say there is a moderately strong linear relationship between study hours and GPA for students with no outliers in our data.

Problems

1. It seems reasonable that the power of a car is related to its gas mileage. Suppose a random sample of 10 cars of various models is selected and the engine power and city gas mileage is recorded for each one.

<table>
<thead>
<tr>
<th>Power (hp)</th>
<th>197</th>
<th>170</th>
<th>166</th>
<th>230</th>
<th>381</th>
<th>438</th>
<th>170</th>
<th>326</th>
<th>451</th>
<th>290</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mileage (mpg)</td>
<td>16</td>
<td>24</td>
<td>19</td>
<td>15</td>
<td>13</td>
<td>20</td>
<td>21</td>
<td>11</td>
<td>10</td>
<td>15</td>
</tr>
</tbody>
</table>

a. Create a scatterplot of the data.

b. Estimate a line of best fit and determine its equation.

c. Interpret the slope and \( y \)-intercept in context.

d. Describe the association.
2. Many people believe that students who are strong in music are also strong in mathematics. The principal at University High School wonders if that same connection exists between music students and English students. The principal went through the records for the past year and found 10 students who were enrolled in both Advanced Placement Music and Advanced Placement English. He compared their final exam scores.

   a. Create a scatterplot of the data.
   b. Estimate a line of best fit and determine its equation.
   c. Interpret the slope and y-intercept in context.
   d. Describe the association.

<table>
<thead>
<tr>
<th>Final Exam Scores</th>
</tr>
</thead>
<tbody>
<tr>
<td>AP Music</td>
</tr>
<tr>
<td>88</td>
</tr>
<tr>
<td>74</td>
</tr>
<tr>
<td>82</td>
</tr>
<tr>
<td>64</td>
</tr>
<tr>
<td>97</td>
</tr>
<tr>
<td>90</td>
</tr>
<tr>
<td>82</td>
</tr>
<tr>
<td>72</td>
</tr>
<tr>
<td>78</td>
</tr>
<tr>
<td>62</td>
</tr>
</tbody>
</table>

**Answers**

1. a. See scatterplot at right.
   b. Answers can vary. If you ignore the possible outlier at (438, 20), a reasonable line could pass through the points (170, 21) and (290, 15). The line of best fit has equation $y = 29.5 - 0.05x$.
   c. For every increase of 1 horsepower, the gas mileage is expected to decrease by 0.05 mpg. The y-intercept means that a 0 horsepower car would get 29.5 mpg. This does not make sense because the y-axis is far from the data and thus this is extrapolation. (Prediction models of all types are unreliable when you extrapolate them.)
   d. When the outlier at (438, 20) is removed, there appears to be a strong, negative, linear relationship. If the outlier is not removed, the association is more moderate (not as strong). For every increase of 1 horsepower, gas mileage is expected to decrease by about 0.05 mpg.
2. a. See plot at right.

b. Answers can vary. A reasonable line could pass through the points (64, 90) and (97, 68). The line of best fit has equation $y = 132.7 - 0.67x$.

c. An increase of 1 point in the music score results in a predicted decrease of 0.67 points in the English score. The y-intercept would mean that a student who scored 0 on the music test is expected to score 133 on the English test. This does not make sense. The y-axis is far outside of the data and represents an extrapolation. (Prediction models of all types are unreliable when you extrapolate them.)

d. There is a lot of scatter around the line of best fit. The linear association is weak. An increase of 1 point in the music score results in a predicted decrease of 0.67 points in the English score. There are no apparent outliers.
Residuals and Upper and Lower Bounds  

4.1.2 and 4.1.3

Residuals are a measure of how far the actual data points are from the line of best fit. Residuals are measured in the vertical (y) direction from the data point to the line. A single residual is calculated by:

\[ \text{residual} = \text{actual } y\text{-value} - \text{predicted } y\text{-value}. \]

A positive residual means that the actual value is greater than the predicted value; a negative residual means that the actual value is less than the predicted value.

A prediction made from a line of best fit gives no indication of the variability in the original data. Upper and lower boundary lines are parallel lines above and below the line of best fit. They give upper and lower limits (a “margin of error”) to predictions made by the line of best fit. For example, predicting a test score of 87 is useful. But a prediction of 87 ± 1 is very different from a prediction of 87 ± 10. The upper and lower boundary lines help us put limits like these on predictions.

The most commonly used techniques for finding upper and lower boundary lines are beyond the scope of this course. However, the bounds can be reasonably approximated by finding the residual with the greatest distance, then adding and subtracting that distance from the prediction.

For additional information, see the Math Notes box in Lesson 4.1.4. For additional examples and more practice on the topics from this chapter, see the Checkpoint 7 materials. Note that the problems below do not use a graphing calculator, while the problems in Checkpoint 7 assume a graphing calculator is available. Graphing calculators are introduced for statistical calculations in Lesson 4.1.4.

Example

It seems reasonable that there would be a relationship between the amount of time a student spends studying and their GPA. Suppose you were interested in predicting a student’s GPA based on the hours they study per week. You were able to randomly select 12 students and obtain this information from each student. An estimated line of best fit could be

\[ y = 2.03 + 0.17x. \]

(Note that this is the same data as in the example in Lesson 4.1.1 of this Parent Guide with Extra Practice.)

<table>
<thead>
<tr>
<th>Hours</th>
<th>4</th>
<th>5</th>
<th>11</th>
<th>1</th>
<th>15</th>
<th>2</th>
<th>10</th>
<th>6</th>
<th>7</th>
<th>0</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>GPA</td>
<td>2.9</td>
<td>3.3</td>
<td>3.9</td>
<td>2.2</td>
<td>4.1</td>
<td>1.8</td>
<td>4.6</td>
<td>2.9</td>
<td>2.2</td>
<td>3</td>
<td>3.3</td>
<td>4.5</td>
</tr>
</tbody>
</table>

a. Find the residual for the student who studied 10 hours, and interpret it in context. Would a student prefer a positive or negative residual?

*Example continues on next page* →
Example continued from previous page.

From the best-fit line, a student who studies 10 hours is predicted to earn a GPA of 
\[ y = 2.03 + 0.17(10) = 3.73. \] 
This student actually received a 4.6 GPA. The residual is:

\[
\text{residual} = \text{actual } y\text{-value} - \text{predicted } y\text{-value} \\
\text{residual} = 4.6 - 3.73 = 0.87
\]

The student who studied 10 hours earned a GPA that was 0.87 points higher than predicted. A student would prefer a positive residual because it means that he/she earned a higher GPA than was predicted from the number of hours spent studying.

b. Write the equations for the upper and lower boundary lines, and show the boundary lines on a scatterplot.

Looking at the scatterplot, the largest residual (the largest vertical distance, \( \Delta y \)) seems to be at the point (7, 2.2). The predicted GPA for a student who studies 7 hours is
\[ y = 2.03 + 0.17(7) = 3.22. \] 
The residual for this point is:
\[
\text{residual} = \text{actual } y\text{-value} - \text{expected } y\text{-value} \\
\text{residual} = 2.2 - 3.22 = -1.02
\]

The boundary lines are parallel to the best-fit line, at a distance of 1.02 above and below it. Parallel lines have the same slope as the best-fit line. The \( y\)-intercepts of the boundary lines will be 1.02 more or less than the \( y\)-intercept of the best-fit line.

Therefore, the upper boundary line is:
\[ y = (2.03 + 1.02) + 0.17x \text{ or } y = 3.05 + 0.17x \]
and the lower boundary line is:
\[ y = (2.03 - 1.02) + 0.17x \text{ or } y = 1.01 + 0.17x. \]

c. Predict the upper and lower bound of the GPA for a student who studies 8 hours per week. Is your prediction useful?

The lower bound is \( y = 1.01 + 0.17(8) = 2.37 \), and the upper bound is 
\[ y = 3.03 + 0.17(8) = 4.41. \] 
We predict that a student who studies 8 hours per week has a GPA between 2.37 and 4.41. Due to the large amount of variability in the data collected, this is a large range. The prediction is not particularly useful.
Problems

1. It seems reasonable that the power of a car is related to its gas mileage. Suppose a random sample of 10 car models is selected and the engine horsepower and city gas mileage is recorded for each one. An outlier was removed and a line of best fit was estimated to be \( y = 29.5 - 0.05x \). (Note that this is the same data as in Problem 1 in Lesson 4.1.1 of this Parent Guide with Extra Practice.)

<table>
<thead>
<tr>
<th>Power (hp)</th>
<th>197</th>
<th>170</th>
<th>166</th>
<th>230</th>
<th>381</th>
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<td>19</td>
<td>15</td>
<td>13</td>
<td>21</td>
<td>11</td>
<td>10</td>
<td>15</td>
</tr>
</tbody>
</table>

a. Find the residual for the vehicle that had 381 horsepower, and interpret it in context. Would a car owner prefer a positive or negative residual?

b. Remove the outlier from the data set. Write the equations for the upper and lower boundary lines, and show the boundary lines on a scatterplot.

c. Predict the upper and lower bounds of the gas mileage for a car with 300 horsepower. Is your prediction useful?

2. Many people believe that students who are strong in music are also strong in mathematics. The principal at University High School wonders if that same connection exists between music students and English students. The principal went through the records for the past year and found 10 students who were enrolled in both Advanced Placement Music and Advanced Placement English. He compared their final exam scores. A line of best fit was estimated to be \( y = 132.7 - 0.67x \). (Note that this is the same data as in Problem 2 in Lesson 4.1.1 of this Parent Guide with Extra Practice.)

<table>
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<tr>
<td>82</td>
</tr>
<tr>
<td>72</td>
</tr>
<tr>
<td>78</td>
</tr>
<tr>
<td>62</td>
</tr>
</tbody>
</table>

a. The principal checked the records of a student who just entered the school. She had a perfect score of 100 on the English final and her residual was 30 points. What was her predicted English score? What was her music score?

b. Write the equations for the upper and lower boundary lines, and show the boundary lines on a scatterplot. Do not add the new student from part (a) to your scatterplot.

c. Predict the upper and lower bounds of the AP English score for a student with a perfect score of 100 on the music final. Is your prediction useful?
Chapter 4

Answers

1. a. From the best-fit line, a vehicle with 381 horsepower is predicted to have a mileage of $y = 29.5 - 0.05(381) \approx 10.5$. But this vehicle actually got 13 mpg. The residual is:

\[
\text{residual} = \text{actual } y\text{-value} - \text{predicted } y\text{-value}
\]

\[
\text{residual} = 13 - 10.5 = 2.5
\]

The vehicle with 381 horsepower got 2.5 mpg more than was predicted. A car owner would prefer a positive residual—a positive residual means the car is getting more miles per gallon than was predicted from its power.

b. Looking at the scatterplot without the outlier, the largest residual (the largest vertical distance, $\Delta y$) seems to be at the point (197, 16). The predicted mileage for a car with 197 horsepower is $y = 29.5 - 0.05(197) = 19.7$ mpg. The residual for this point is:

\[
\text{residual} = \text{actual } y\text{-value} - \text{predicted } y\text{-value}
\]

\[
\text{residual} = 16 - 19.7 = -3.7
\]

The boundary lines are parallel to the best-fit line, but a distance of 3.7 above it and below it. Parallel lines have the same slope as the best-fit line. The $y$-intercepts of the boundary lines will be 3.7 more or less than the $y$-intercept of the best-fit line.

Therefore, the upper boundary line is: $y = (29.5 + 3.7) - 0.05x$ or $y = 33.2 - 0.05x$, and the lower boundary line is: $y = (29.5 - 3.7) - 0.05x$ or $y = 25.8 - 0.05x$.

c. The upper bound is $y = 33.2 - 0.05(300) = 18.2$, and the lower bound is $y = 25.8 - 0.05(300) = 10.8$. We predict that a car with 300 horsepower will have gas mileage between 10.8 and 18.2 mpg. The prediction could be useful: it indicates that this is a car with a low miles per gallon ratio.
2.  
   a.  residual = actual y-value – predicted y-value
      
      \[30 = 100 - \text{predicted y-value}\]
      
      predicted y-value = 100 – 30 = 70
      
      From the best-fit line we can find the music score: 70 = 132.7 – 0.67x or \(x \approx 94\).
      Her predicted English score was 70 and her music score was 94.
      
   b.  Looking at the scatterplot, the largest residual (the largest vertical distance, \(\Delta y\)) seems to be at the point (90, 90). The predicted English score for a student with 90 on the music test is
      
      \(y = 132.7 - 0.67(90) = 72.4\). The residual for this point is: 90 – 72.4 = 17.6.
      Therefore, the upper boundary line is:
      
      \(y = (132.7 + 17.6) - 0.67x\) or \(y = 150.3 - 0.67x\),
      and the lower boundary line is:
      
      \(y = (132.7 - 17.6) - 0.67x\) or \(y = 115.1 - 0.67x\).
      
   c.  The upper bound is \(y = 150.3 - 0.67(100) = 83.3\),
      and the lower bound is
      
      \(y = 115.1 - 0.67(100) = 48.1\). We predict that a student who scores 100 on the AP Music final exam will earn a score between 48 and 83 on the AP English final exam. This range is much too large to be useful. There is a lot of variability in the data the principal collected, making the margin of error on the prediction very large.
LEAST SQUARES REGRESSION LINE

A **least squares regression line** (LSRL) is a line of best fit that minimizes the sum of the square of the residuals. Students can investigate this using the Least Squares eTool available in your eBook or at studenthelp.cpm.org.

For additional information, see the Math Notes box in Lesson 4.2.1 and the Checkpoint 7 materials.

**CALCULATORS**
Statistical calculators or software can find the LSRL and residuals with ease. Students are welcome to use a TI-83/84+ calculator if they have one available.

An eTool is also available in your eBook or at studenthelp.cpm.org for students who do not have a graphing calculator.

**TI-83/84+ NOTES**
More detailed instructions for using a TI-83+/84+ calculator can be found in your eBook or at studenthelp.cpm.org. Very briefly:

For part (a) of the example:
- Enter the data points in two lists.
- For data tables in which a checksum is given, the checksum is used to verify that data has been entered into the calculator correctly. Use 1-Var Stats and verify that $\Sigma x$ on the calculator screen matches the “checksum” given at the bottom of the table in certain problems. $\Sigma x$ stands for “the sum of x” or the sum of all the values in the list.
- Use 1-Var Stats to find the maximum and the minimum of both lists, then set the window appropriately.
- Use $2^{nd}$ [STAT PLOT] to set up a graph, and then press $\text{GRAPH}$.

For part (b) of the example:
- Find the LSRL with … “CALC” “8:LinReg(a+bx)” $2^{nd}$ [L1] , $2^{nd}$ [L2] , $\text{VARS}$ “Y-VARS” “1:Function” “1:Y_1” $\text{ENTER}$.
- Use $2^{nd}$ [WINDOW] to set the scale of the graph, then press $\text{GRAPH}$.

For part (c) of the example:
Each time you perform a LSRL regression, the calculator will automatically store the residuals in a list named “RESID”. You can display the residuals in List 3 as follows.
- Highlight the label for List 3 and press $\text{ENTER}$.
- Press $2^{nd}$ [LIST] “RESID” and then $\text{ENTER}$ to copy the list.
Example

It seems reasonable that there would be a relationship between the amount of time a student spends studying and their GPA. Suppose you were interested in predicting a student’s GPA based on the hours they study per week. You were able to randomly select 12 students and obtain this information from each student.

<table>
<thead>
<tr>
<th>Hours</th>
<th>4</th>
<th>5</th>
<th>11</th>
<th>1</th>
<th>15</th>
<th>2</th>
<th>10</th>
<th>6</th>
<th>7</th>
<th>0</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>GPA</td>
<td>2.9</td>
<td>3.3</td>
<td>3.9</td>
<td>2.2</td>
<td>4.1</td>
<td>1.8</td>
<td>4.6</td>
<td>2.9</td>
<td>2.2</td>
<td>3</td>
<td>3.3</td>
<td>4.5</td>
</tr>
</tbody>
</table>

a. Use your calculator to graph the scatterplot. Choose a reasonable viewing window on your calculator, then make a sketch of the scatterplot on your paper and label the axes.

Since the “hours” range from 0 to 15, an appropriate window might have an x-axis scale from 0 to 16 with an interval of 2. The GPA data ranges from 1.8 to 4.6, so an appropriate y-axis might range from 0 to 5 with an interval of 1. Note that the axes are labeled with both the variable name and with the units.

b. Use your calculator to find the equation of the LSRL.

From the calculator, \( y = 2.249 + 0.152x \). It is essential when giving an LSRL equation to define the variables and units. In this case, \( x \) is the numbers of hours spent studying per week, and \( y \) is the predicted GPA for that student.

c. Use your calculator to create a residual list. What is the residual for the student who spent 10 hours studying? Interpret the residual in context.

A partial residual is shown at right (in List L3):

From the residual list, the residual for the students that studied 10 hours is 0.83 grade points. That means the student earned a GPA that was 0.83 points higher than was predicted by the least squares regression line.
Problems

1. It seems reasonable that the power of a car is related to its gas mileage. Suppose a random sample of 9 car models is selected and the engine power and city gas mileage is recorded for each one.

<table>
<thead>
<tr>
<th>Power (hp)</th>
<th>197</th>
<th>170</th>
<th>166</th>
<th>230</th>
<th>381</th>
<th>170</th>
<th>326</th>
<th>451</th>
<th>290</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mileage (mpg)</td>
<td>16</td>
<td>24</td>
<td>19</td>
<td>15</td>
<td>13</td>
<td>21</td>
<td>11</td>
<td>10</td>
<td>15</td>
</tr>
</tbody>
</table>

a. Use your calculator to graph the scatterplot. Choose a reasonable viewing window on your calculator, then make a sketch of the scatterplot on your paper and label the axes.

b. Use your calculator to find the equation of the LSRL. What do you predict the gas mileage will be for a car with 400 horsepower?

c. Use your calculator to create a residual list. What is the residual for the 230 horsepower car?

2. Many people believe that students who are strong in music are also strong in mathematics. But the principal at University High School wonders if that same connection exists between music students and English students. The principal went through the records for the past year and found 10 students who were enrolled in both Advanced Placement Music and Advanced Placement English. He compared their final exam scores.

<table>
<thead>
<tr>
<th>Final Exam Scores</th>
</tr>
</thead>
<tbody>
<tr>
<td>AP Music</td>
</tr>
<tr>
<td>88</td>
</tr>
<tr>
<td>74</td>
</tr>
<tr>
<td>82</td>
</tr>
<tr>
<td>64</td>
</tr>
<tr>
<td>97</td>
</tr>
<tr>
<td>90</td>
</tr>
<tr>
<td>82</td>
</tr>
<tr>
<td>72</td>
</tr>
<tr>
<td>78</td>
</tr>
<tr>
<td>62</td>
</tr>
</tbody>
</table>

checksum 789 checksum 820

a. Use your calculator to graph the scatterplot. Choose a reasonable viewing window on your calculator, then make a sketch of the scatterplot on your paper and label the axes. Did you remember to use the checksum to verify you entered the data correctly?

b. Use your calculator to find the equation of the LSRL.

c. Use your calculator to create a residual list. Would a student prefer a positive or a negative residual in this situation?
Answers

1. a. See scatterplot at right.
   
   b. \( y = 26.11 - 0.0382x \). \( y \) is the predicted gas mileage (in mpg) and \( x \) is the power (in hp). A car with 400 horsepower is predicted to have gas mileage of \( 26.11 - 0.0382(400) = 10.8 \text{ mpg} \).
   
   c. The car with 230 horsepower had residual of \(-2.3 \text{ mpg}\). That means the car had gas mileage that was 2.3 mpg less than was predicted by the LSRL.

2. a. See scatterplot at right.
   
   b. \( y = 112.1 - 0.3813x \) where \( y \) is the predicted AP English final exam score, and \( x \) is the AP Music final exam score.
   
   c. Residual list follows in List L3 below. A student would prefer a positive residual. That means their AP English score is higher than predicted by the LSRL.

\[
\begin{array}{ccc}
L1 & L2 & L3 \\
88 & 63 & 12.132 \\
74 & 86 & 5.1821 \\
64 & 90 & 2.3103 \\
87 & 68 & 7.0986 \\
90 & 90 & 12.238 \\
82 & 78 & -2.818 \\
\end{array}
\]

\( L3(1) = -15.5299673... \)
RESIDUAL PLOTS AND CORRELATION  
4.2.1, 4.2.2, and 4.2.4

A **residual plot** is created in order to analyze the appropriateness of a best-fit model. See the Math Notes box in Lesson 4.2.2 for more on residual plots.

The **correlation coefficient**, \( r \), is a measure of how much or how little the data is scattered around the LSRL; it is a measure of the strength of a linear association. \( R \)-squared can be interpreted as the percentage of the change in the dependent variable (\( y \)) that can be accounted for or explained by the change in the independent variable (\( x \)). See the Math Notes box in Lesson 4.2.4.

For additional information, see the Checkpoint 7 materials.

**CALCULATORS**
Statistical calculators or software can make statistical computations with ease. See the previous section for additional information about using a TI-83/84+ calculator. More detailed instructions for using a TI-83+/84+ calculator can be found in your eBook or at studenthelp.cpm.org.

When you calculate the LSRL, the calculator reports the correlation coefficient on the same screen as it reports the slope and \( y \)-intercept. If your TI calculator does not calculate \( r \), press `[2nd] [CATALOG] “DiagnosticOn” [ENTER] [ENTER]` and try again.
Example

It seems reasonable that there would be a relationship between the amount of time a student spends studying and their GPA. Suppose you were interested in predicting a student’s GPA based on the hours they study per week. You were able to randomly select 12 students and obtain this information from each student.

<table>
<thead>
<tr>
<th>Hours</th>
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<tbody>
<tr>
<td>4</td>
<td>2.9</td>
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<tr>
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</tr>
<tr>
<td>1</td>
<td>2.2</td>
</tr>
<tr>
<td>15</td>
<td>4.1</td>
</tr>
<tr>
<td>2</td>
<td>1.8</td>
</tr>
<tr>
<td>10</td>
<td>4.6</td>
</tr>
<tr>
<td>6</td>
<td>2.9</td>
</tr>
<tr>
<td>7</td>
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</tr>
</tbody>
</table>

a. Graph the data and the LSRL.
   See scatterplot at right.

b. Create and interpret a residual plot.
   See residual plot below right.
   
   There is no apparent pattern in the residual plot—it looks like randomly scattered points—which means a linear model (instead of a curve) is the most appropriate way to model the relationship.

c. Find the coefficient of correlation $r$ and $R^2$-squared. Interpret the meaning of $R^2$-squared in context.
   From the calculator, $r \approx 0.7338$, $R^2$-squared $\approx 53.8\%$.
   About 54% of the variability in GPA can be explained by a linear relationship with study hours per week.

d. Fully describe the association between GPA and hours studied.
   For a description of how to fully describe an association, see the narrative in Checkpoint 8 at the back of the textbook. When describing an association, the form direction, strength, and outliers should be described.

   From the calculator, the least squares regression line is $y = 2.249 + 0.152x$, where $y$ is the predicted GPA, and $x$ is the number of hours studied.

   The form in the scatterplot appears to be linear; it does not appear to be curved nor simply a collection of randomly scattered points. The residual plot shows random scatter, confirming that a linear model was an appropriate choice.

   The direction is positive; students who study more have higher GPAs. From the slope, a student’s GPA is expected to increase by 0.15 points for every additional hour of studying per week.

   The strength is moderate: it is strong enough to easily see its form, but there is scatter about the line. The correlation coefficient is 0.73, confirming a moderately strong linear association. From $R^2$-squared, 54% of the variability in GPA can be explained by the variability in study hours per week.

   There are no apparent outliers.
Problems

1. It seems reasonable that the power of a car is related to its gas mileage. Suppose a random sample of 9 car models is selected and the engine power and city gas mileage is recorded for each one.

| Power (hp) | 197 | 170 | 166 | 230 | 381 | 170 | 326 | 451 | 290 |
| Mileage (mpg) | 16 | 24 | 19 | 15 | 13 | 21 | 11 | 10 | 15 |

a. Graph the data and the LSRL.
b. Create and interpret a residual plot.
c. Find the coefficient of correlation $r$ and $R$-squared. Interpret the meaning of $R$-squared in context.
d. Fully describe the association.

2. Many people believe that students who are strong in music are also strong in mathematics. But the principal at University High School wonders if that same connection exists between music students and English students. The principal went through the records for the past year and found 10 students who were enrolled in both Advanced Placement Music and Advanced Placement English. He compared their final exam scores.

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<tr>
<td>72</td>
</tr>
<tr>
<td>78</td>
</tr>
<tr>
<td>62</td>
</tr>
</tbody>
</table>

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a. Graph the data and the LSRL.
b. Create and interpret a residual plot.
c. Find the coefficient of correlation $r$ and $R$-squared. Interpret the meaning of $R$-squared in context.
d. Fully describe the association.
Answers

1. a. See scatterplot at right.

   b. See residual plot below right.

   There is an apparent U-shaped pattern (points are above the line on the left and right ends, but below the line in the middle) in the residual plot, which suggests a curved model (instead of a line) would be the most appropriate way to model the relationship. Nonetheless, by visual examination of the scatterplot and LSRL, a linear model is not completely inappropriate. We will continue with the analysis of the linear model.

c. \( r \approx -0.8602 \), \( R\)-squared \( \approx 0.7400 \). About 74% of the variability in gasoline mileage can be explained by a linear relationship with horsepower.

d. From the calculator, the least squares regression line is \( y = 26.11 - 0.0382x \), where \( y \) is the predicted mileage (in mpg) and \( x \) is the power (in hp). The residual plot indicates a curved model might fit the data better, but by observation of the scatterplot, the LSRL models the data well enough to proceed. The association is negative: the model predicts that for every increase of one horsepower, the mileage is expected to decrease by 0.04 mpg. The association is fairly strong with a correlation coefficient of nearly \(-0.9\). About 74% of the variability in gasoline mileage can be explained by a linear relationship with power. There are no apparent outliers. (However, if you have been following this problem since Lesson 4.1.1 in this Parent Guide with Extra Practice, you will recall that the data set initially had an apparent outlier at 438 horsepower and 20 mpg. That point was not considered in creating the best-fit line, and subsequently the data point was dropped from the analysis. If that data point is included, the correlation coefficient will be closer to 0.)
2.  
   a.  See scatterplot at right.
   
   b.  See residual plot below right.

   There is no apparent pattern in the residual plot—it looks like randomly scattered points—which means a linear model (instead of a curve) is the most appropriate way to model the relationship.

   c.  From the calculator, \( r \approx -0.3733 \), \( R\text{-squared} \approx 13.9\% \). About 14% of the variability in AP English scores can be explained by a linear relationship with AP Music scores.

   d.  The LSRL is \( y = 112.1 - 0.3813x \), where \( y \) is the predicted AP English score, and \( x \) is the AP Music score. The residual plot confirms that a linear model is appropriate. The association is negative: the model predicts that for every increase of one in the AP Music score, the AP English score will drop by 0.38 points. The association is very weak, with a correlation coefficient of approximately \(-0.37\). Since \( R\text{-squared} \) is approximately 14%, about 14% of the variability in AP English scores can be explained by a linear relationship with AP Music scores. There are no apparent outliers in the data collected.
Considering all of the statistical tools available to measure the correlation between two variables it is very tempting to believe that a strong association between two variables is evidence of causation. The fact that two variables move together in a predictable manner does not show that one causes the other. Cause can usually only be shown using carefully controlled experiments, and usually not by just observing the relationship between variables.

Example

A city is proposing a new bus route. A community activist who owns a residence along the route does some research which shows a strong association between the number of bus stops and the crime rate of a community. She presents her findings to the city council claiming, “Bus stops cause crime, so please vote no.” Another community activist who owns a business along the proposed bus route also does research and finds that bus stops are highly associated with the number of restaurants in a neighborhood. He presents his findings to the city council claiming, “Bus stops increase restaurants. Please vote yes.”

A city council member (who also teaches statistics) informs them that association is not the same as causation because of the many thousands of variables not included (lurking) in a study. The council member said, “Bus stops are generally an indication of large population centers so bus stops will be found along with other things associated with urban living. This does not mean that bus stops cause things found in cities.”

a. List three desirable variables that are likely associated with bus stops but not necessarily caused by them.
   numbers of museums, theaters, stadiums

b. List three undesirable variables that are likely associated with bus stops but not necessarily caused by them.
   quantity of litter, air pollution, noise
Problems

For each faux news headline, accept that the association is true and indicate a hidden or lurking variable that may be the actual cause of the relationship.

a. “BOOKS IN THE HOME ASSOCIATED WITH HIGHER TEST SCORES”
   “Students buying truckloads of books before taking national exams.”

b. “CHESS CLUB BOASTS HIGHEST IVY LEAGUE ADMISSION RATES”
   “Club president now flooded with membership applications.”

c. “STUDENTS WHO LIVE OFF CAMPUS AT HIGHER RISK FOR AUTO ACCIDENTS”
   “State Police Warn: Play it safe, live on campus.”

Answers

a. A person who does a lot of reading would likely have a larger number of books in their home and higher test scores. So reading regularly is likely the cause. Just buying books does not cause higher test scores.

b. Chess is a game of strategy and an extracurricular activity. Those who play regularly may be more inclined to apply to Ivy League schools than non-chess players.

c. Students who live off campus probably commute to school and may spend more time driving to and from school than students who live on campus. The additional hours on the road may be the cause of more accidents.
In Lessons 5.1.1 through 5.1.3 students are introduced to patterns that grow by multiplying in everyday situations. They investigate these situations using tables and graphs. These situations can be modeled using **exponential functions**. If the function is increasing, it represents **exponential growth**. If the function is decreasing, it represents **exponential decay**. The graphs of these functions are curved and have a horizontal asymptote (a line that the graph of a curve approaches).

**Example 1**

Three friends decide to start a chain letter. They will each send the letter to four other friends. Those who receive the letter will continue the chain. Create a table and graph to model the number of letters that are sent each round. Assume that everyone maintains the same letter-sending schedule so that all letters in a round are sent at the same time. Does this situation involve exponential growth or decay? During which round will over 10,000 letters be sent?

Solution: If three friends (round 0) each send four letters, twelve letters will be sent (round 1). If those twelve friends each send four letters, 48 letters will be sent (round 2). The pattern is multiplication by 4. This can be modeled using the table and graph shown at right.

<table>
<thead>
<tr>
<th>round #</th>
<th># of letters</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>48</td>
</tr>
<tr>
<td>3</td>
<td>192</td>
</tr>
<tr>
<td>4</td>
<td>768</td>
</tr>
<tr>
<td>5</td>
<td>3072</td>
</tr>
<tr>
<td>…</td>
<td>…</td>
</tr>
</tbody>
</table>

This situation involves exponential growth because the number of letters is increasing as the round number increases.

If the table were continued, round 6 would have 12,288 letters sent, so round 6 would exceed 10,000 letters being sent.
Example 2

Bryson inherited $1,000,000. He wants to make it last as long as possible, so he decides that he will only spend 25% of the remaining amount each year. Create a table and graph to model the amount of money Bryson has left after each year has passed. Does this situation involve exponential growth or decay? When will Bryson have less than $1000 left?

Solution: Bryson will keep 75% (100% – 25%) of the remaining amount each year, so 0.75 can be used as the multiplier to determine how much he will have left after each year has passed.

This situation represents exponential decay because the amount of money is decreasing as the amount of time increases.

<table>
<thead>
<tr>
<th>year</th>
<th>money ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1,000,000</td>
</tr>
<tr>
<td>1</td>
<td>750,000</td>
</tr>
<tr>
<td>2</td>
<td>562,500</td>
</tr>
<tr>
<td>3</td>
<td>421,875</td>
</tr>
<tr>
<td>4</td>
<td>316,406</td>
</tr>
<tr>
<td>5</td>
<td>237,304</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

If the table is continued, Bryson will have less than $1000 after 25 years.

Problems

Complete each table below. What is the multiplier? Then decide if the table represents exponential growth or decay.

1. \[
\begin{array}{c|c}
  x & y \\
  0 & 5200 \\
  1 & 52 \\
  2 & 5.2 \\
  3 &  \\
  4 &  \\
  5 &  \\
  6 & \\
\end{array}
\]

2. \[
\begin{array}{c|c}
  x & y \\
  -2 & 1.481 \\
  -1 & 8 \\
  0 & 43.2 \\
  1 &  \\
  2 &  \\
  3 & 4 \\
\end{array}
\]

3. \[
\begin{array}{c|c}
  x & y \\
  5 &  \\
  6 &  \\
  7 &  \\
  8 & 1.9 \\
  9 & 6.65 \\
  10 & 23.275 \\
  11 &  \\
\end{array}
\]

Create a table and graph to model each situation below. Does each situation represent exponential growth or decay?

4. A nonnative species of fish are taking over a lake in Florida. Each pair of fish creates 30 new fish each year. Scientists predict that there are currently 50 nonnative fish in the lake. Create a table and graph to model this situation. How long will it take for the nonnative fish population to exceed 1,000,000?

5. Researchers were testing out a new antibiotic on a certain bacteria. They started with 45,000 bacteria. After one treatment with antibiotics, 40,500 bacteria were left. After a second treatment, 36,450 bacteria were left. Create a table and graph to model this situation. How many treatments of antibiotics will it take to kill at least half of the bacteria?
Answers

1. \[
\begin{array}{c|c}
  x & y \\
  \hline
  0 & 5200 \\
  1 & 520 \\
  2 & 52 \\
  3 & 5.2 \\
  4 & 0.52 \\
  5 & 0.052 \\
  6 & 0.0052 \\
\end{array}
\]
  multiplier: 0.1, decay

2. \[
\begin{array}{c|c}
  x & y \\
  \hline
  -2 & 0.2743 \\
  -1 & 1.481 \\
  0 & 8 \\
  1 & 43.2 \\
  2 & 233.28 \\
  3 & 1259.712 \\
  \ldots & \ldots \\
\end{array}
\]
  multiplier: 5.4, growth

3. \[
\begin{array}{c|c}
  x & y \\
  \hline
  5 & 0.0443 \\
  6 & 0.1551 \\
  7 & 0.5429 \\
  8 & 1.9 \\
  9 & 6.65 \\
  10 & 23.275 \\
  11 & 81.463 \\
\end{array}
\]
  multiplier: 3.5, growth

4. See table and graph at right.

   The nonnative fish population will exceed 1,000,000 between 3 and 4 years from now.

5. See table and graph at right.

   It will take 7 rounds of antibiotics to have less then half of the original 45,000 bacteria (22,500).
In these lessons, students learn multiple representations for sequences: as a list of numbers, as a table, as a graph, and as an equation. Read more about writing equations for sequences in the Math Notes box in Lesson 5.3.3. In an arithmetic sequence, the terms of the sequence are generated by adding (or subtracting) a constant value to the previous term. In a geometric sequence, the terms of the sequence are generated by multiplying (or dividing) a constant value by the previous term.

In addition to the ways to write explicit equations for sequences, equations for sequences can also be written recursively. An explicit formula tells exactly how to find any specific term in a sequence. A recursive formula names the first term (or any other term) and how to get from one term to the next. For an explanation of recursive sequences, see the Math Notes box in Lesson 5.3.3.

Example 1

Consider the two sequences:

A: –8, –5, –2, 1, …
B: 256, 128, 64, …

a. For each sequence, is it arithmetic, geometric, or neither? How can you tell? Explain completely.

b. What are the “zeroth term” and the generator for each sequence?

To determine the type of sequence for A and B above, we have to look at the pattern of growth for each sequence.

A: \[-8, \quad \\vdash \quad -5, \quad \\vdash \quad -2, \quad \\vdash \quad 1, \quad \vdash \quad ...
+3 \quad +3 \quad +3\]

Sequence A is made (generated) by adding three to each term to get the next term. When each term has a common difference (in this case, “+3”) the sequence is arithmetic.

Sequence B, however, is different. The terms do not have a common difference. Instead, these terms have a common ratio (multiplier). A sequence with a common ratio is a geometric sequence.

B: 256, 128, 64, …

\[-128 \quad \vdash \quad -64 \quad \vdash \quad ...
\cdot \frac{1}{2} \quad \cdot \frac{1}{2}\]

The first term for sequence A is –8, and has a generator or common difference of +3. Therefore the zeroth term is –11 (because \(-11 + 3 = -8\)).

For sequence B, the first term is 256 with a generator or common ratio of \(\frac{1}{2}\). Therefore the zeroth term is 512, because \(512 \cdot \frac{1}{2} = 256\).
Example 2

Peachy Orchard Developers are preparing land to create a large subdivision of single-family homes. They have already built 15 houses on the site. Peachy Orchard plans to build six new homes every month. Write a sequence for the number of houses built, then write an equation for the sequence. Fully describe a graph of this sequence.

The sequence is 21, 27, 33, 39, …. Note that all sequences begin with the first term, in this case it is when the number of months \( n = 1 \).

The common difference is \( m = 6 \), and the zeroth term is \( b = 15 \). The equation can be written \( t(n) = mn + b = 6n + 15 \). Note that for a sequence, \( t(n) \) is used instead of \( y \). \( t(n) \) indicates the equation is for a discrete sequence (separate points), as opposed to a continuous function (connected points). Students compare sequences to functions in Lesson 5.3.3.

The equation could also have been written as \( a_n = 6n + 15 \).

The graph of the sequence is shown at right. There are no \( x \)- or \( y \)-intercepts. There is no point at \((0, 15)\) because sequences are written starting with the first term where \( n = 1 \). The domain consists of integers (whole numbers) greater than or equal to one. The range consists of the \( y \)-values of the points that result from the equation \( t(n) = 6n + 15 \) when \( n \geq 1 \). There are no asymptotes. The graph is linear and is shown at right. This graph is discrete (separate points). (Note: The related function, \( y = 6x + 15 \), would have a domain of all real numbers and the graph would be a continuous connected line.)

Example 3  (An explicit formula)

List the first five terms of the arithmetic sequence. \( t(n) = 5n + 2 \)

\[

t(1) = 5(1) + 2 = 7 \\
t(2) = 5(2) + 2 = 12 \\
t(3) = 5(3) + 2 = 17 \\
t(4) = 5(4) + 2 = 22 \\
t(5) = 5(5) + 2 = 27
\]

The sequence is: 7, 12, 17, 22, 27, …
Example 4  (A recursive formula)

List the first five terms of the arithmetic sequence. \( t(1) = 3 \)

\( t(n+1) = t(n) - 5 \)

\( t(1) = 3 \)

If the term you are on is \( t(n) \), then the next term is \( t(n+1) \).

For example, if \( n = 1 \), \( t(n) \) is \( t(1) \), or the first term.

Then \( t(n+1) \) is \( t(1+1) \) or \( t(2) \), the second term.

The sequence is: 3, –2, –7, –12, –17, …

Example 5

Write an explicit and a recursive formula for the sequence: –2, 1, 4, 7, …

Explicit: \( m = 3, b = -5 \), so the equation is: \( t(n) = mn + b = 3n - 5 \)

Recursive: \( t(1) = -2, t(n+1) = t(n) + 3 \)

Problems

List the first five terms of each sequence, then state whether the sequence is arithmetic, geometric, or neither. (Hint: Substitute the numbers 1-5 for \( n \) to find the first five terms of the sequence.)

1. \( t(n) = 5n + 2 \)
2. \( s(n) = 3 - 8n \)
3. \( u(n) = 9n - n^2 \)
4. \( t(n) = (-4)^n \)
5. \( s(n) = \left( \frac{1}{4} \right)^n \)
6. \( u(n) = n(n+1) \)
7. \( t(n) = 8 \)
8. \( s(n) = \frac{3}{4} n + 1 \)

List the first five terms of each arithmetic sequence. Note: Problems 9 through 12 are given as explicit equations, while problems 13 through 16 are given as recursive equations.

9. \( t(n) = 5n - 2 \)
10. \( t(n) = -3n + 5 \)
11. \( t(n) = -15 + \frac{1}{2} n \)
12. \( t(n) = 5 + 3(n-1) \)
13. \( t(1) = 5, t(n+1) = t(n) + 3 \)
14. \( t(1) = 5, t(n+1) = t(n) - 3 \)
15. \( t(1) = -3, t(n+1) = t(n) + 6 \)
16. \( t(1) = \frac{1}{2}, t(a+1) = t(n) + \frac{1}{2} \)

Find the 30th term of each arithmetic sequence.

17. \( t(n) = 5n - 2 \)
18. \( t(n) = -15 + \frac{1}{2} n \)
19. \( t(31) = 53, d = 5 \)
20. \( t(29) = 25, t(n+1) = t(n) - 3 \)
For each arithmetic sequence, write an explicit and a recursive formula.

21. 4, 8, 12, 16, 20, …
22. –2, 5, 12, 19, 26, …
23. 27, 15, 3, –9, –21, …
24. 3, 3\(\frac{1}{3}\), 3\(\frac{2}{3}\), 4, 4\(\frac{1}{3}\), …

Note: Sequences are graphed using points of the form: (term number, term value).
For example, the sequence 4, 9, 16, 25, 36, … would be graphed by plotting the points (1, 4), (2, 9), (3, 16), (4, 25), (5, 36), …. Sequences are graphed as points and not connected.

25. Graph the sequences from problems 1 and 2 and determine the slope of each line that would contain the points.

26. How do the slopes of the lines found in the previous problem relate to the sequences?

Graph the following two sequences on the same set of axes.

27. \(t(n) = -6n + 20\)
28. 1, 4, 16, 64, …

29. Fully describe the graph of the sequence \(t(n) = -4n + 18\).

30. Find the missing terms for this arithmetic sequence and an equation for \(t(n)\).

\[__, 15, 11, __, 3\]

31. For this sequence each term is \(\frac{1}{5}\) of the previous one. Work forward and backward to find the missing terms.

\[__, __, \frac{2}{5}, __, __\]

32. The 30\textsuperscript{th} term of a sequence is 42. If each term in the sequence is four greater than the previous number, what is the first term?

33. The microscopic length of a crystalline structure grows so that each day it is 1.005 times as long as the previous day. If on the third day the structure was 12.5 nm long, write a sequence for how long it was on the first five days. (nm stands for nanometer, or \(1 \times 10^{-9}\) meters.)

34. Davis loves to ride the mini-cars at the amusement park, but riders must be no more than 125 cm tall. If on his fourth birthday he is 94 cm tall and grows approximately 5.5 cm per year, at what age will he no longer be able to go on the mini-car ride?

35. Davis has $5.40 in his bank account on his fourth birthday. If his parents add $0.40 to his bank account every week, when will he have enough to buy the new Smokin’ Derby racecar set which retails for $24.99?
Answers

1. 7, 12, 17, 22, 27, arithmetic, common difference is 5.
3. 8, 14, 18, 20, 20, neither
5. \( \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64} \), geometric, common ratio is \( \frac{1}{2} \).
6. 2, 6, 12, 20, 30, neither
8. \( \frac{7}{4}, \frac{5}{2}, \frac{13}{4}, \frac{19}{4} \), arithmetic, common difference is \( \frac{3}{4} \).

If he must also cover tax, he will need another three or four weeks.

10. 2, −1, −4, −7, −10
12. 5, 8, 11, 14, 17
14. 5, 2, −1, −4, −7
16. \( \frac{1}{3}, \frac{5}{6}, 1 \frac{1}{3}, 1 \frac{5}{6}, 2 \frac{1}{3} \)

21. explicit: \( t(n) = 4n \)
23. explicit: \( t(n) = -12n + 39 \)
25. graph (1): points (1, 3), (2, 8), (3, 13), (4, 18), (5, 23) slope = 5
graph (2): points (1, −5), (2, −13), (3, −21), (4, −29), (5, −37) slope = −8

The slope of the line containing the points is the same as the common difference of the sequence.

27. See filled dots on graph at right.

29. This is a function, it represents an arithmetic sequence, the graph is discrete but the points are linear. The domain is the positive integers: 1, 2, 3, … . The range is the sequence itself: 14, 10, 6, 2, −2, …. There are no asymptotes.

30. 19 and 8; \( t(n) = 23 - 4n \)
32. \( 42 - 29(4) = -74 \)
34. \( t(n) = 5.5n + 94 \), so solve \( 5.5n + 94 \leq 125 \). \( n \approx 5.64 \). At \( \approx 4 + 5.64 = 9.64 \) he will be too tall. Davis can continue to go on the ride until he is about 9 \( \frac{1}{2} \) years old.
35. \( t(n) = 0.4n + 5.4 \), so solve \( 0.4n + 5.4 \geq 24.99 \). \( n = 48.975 \). In 49 weeks he will have $25. If he must also cover tax, he will need another three or four weeks.
To recognize if a sequence has a pattern of growth that is linear, exponential, or neither, look at the differences of the \( t(n) \)-values between successive \( n \)-values. If the difference is constant, the sequence has a pattern of growth that is linear. If the difference is not constant, look at the pattern in the \( t(n) \)-values. If a constant multiplier can be used to move from one \( t(n) \)-value to the next, then the sequence has a pattern of growth that is exponential.

**Examples**

Use each table to identify the pattern of growth as linear, exponential, or neither.

**Example 1**

\[
\begin{array}{cccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
 t(n) & -7 & -5 & -3 & -1 & 1 & 3 & 5 \\
\end{array}
\]

The difference in the \( t(n) \)-values is always a constant 2. The growth pattern is linear.

**Example 2**

\[
\begin{array}{cccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
 t(n) & 9 & 4 & 1 & 0 & 1 & 4 & 9 \\
\end{array}
\]

The first difference in \( t(n) \)-values is not constant, and there is not a constant multiplier in moving from one \( t(n) \)-value to the next. The growth pattern is neither linear nor exponential.

**Example 3**

\[
\begin{array}{cccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
 t(n) & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 & 2 & 4 & 8 \\
\end{array}
\]

The \( t(n) \)-values have a constant multiplier of 2. (Also the differences in \( t(n) \)-values have a constant multiplier of 2.) The growth pattern is exponential.
## Problems

Identify the pattern of growth as linear, exponential, or neither.

1. \[
\begin{array}{c|cccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 \\
  \hline
  t(n) & 14 & 10 & 6 & 2 & -2 & -6 \\
\end{array}
\]

2. \[
\begin{array}{c|cccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 \\
  \hline
  t(n) & \frac{1}{2} & 1 & 2 & 4 & 8 & 16 \\
\end{array}
\]

3. \[
\begin{array}{c|cccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 \\
  \hline
  t(n) & 21 & 12 & 5 & 0 & -3 & -4 \\
\end{array}
\]

4. \[
\begin{array}{c|cccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 \\
  \hline
  t(n) & -14 & -13 & -10 & -7 & -4 & -1 \\
\end{array}
\]

5. \[
\begin{array}{c|cccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 \\
  \hline
  t(n) & -14 & -9 & -4 & 1 & 6 & 11 \\
\end{array}
\]

6. \[
\begin{array}{c|cccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 \\
  \hline
  t(n) & -18 & -6 & -2 & 0 & 2 & 6 \\
\end{array}
\]

7. \[
\begin{array}{c|cccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 \\
  \hline
  t(n) & 4 & 8 & 16 & 32 & 64 & 128 \\
\end{array}
\]

8. \[
\begin{array}{c|cccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 \\
  \hline
  t(n) & \frac{1}{27} & \frac{1}{9} & \frac{1}{3} & 1 & 3 & 9 \\
\end{array}
\]

9. \[
\begin{array}{c|cccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 \\
  \hline
  t(n) & 30 & 20 & 12 & 6 & 2 & 0 \\
\end{array}
\]

10. \[
\begin{array}{c|cccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 \\
  \hline
  t(n) & 11 & 9 & 7 & 5 & 3 & 1 \\
\end{array}
\]

11. \[
\begin{array}{c|cccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 \\
  \hline
  t(n) & \frac{1}{9} & \frac{1}{3} & 1 & 3 & 9 & 27 \\
\end{array}
\]

12. \[
\begin{array}{c|cccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 \\
  \hline
  t(n) & -\frac{1}{27} & -\frac{1}{9} & -\frac{1}{3} & 0 & 3 & 9 \\
\end{array}
\]

13. \[
\begin{array}{c|cccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 \\
  \hline
  t(n) & 0 & 5 & 8 & 9 & 8 & 5 \\
\end{array}
\]

14. \[
\begin{array}{c|cccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 \\
  \hline
  t(n) & 3 & 0 & -1 & 0 & 3 & 8 \\
\end{array}
\]

15. \[
\begin{array}{c|cccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 \\
  \hline
  t(n) & 1 & 0 & -1 & -2 & -1 & 0 \\
\end{array}
\]

16. \[
\begin{array}{c|cccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 \\
  \hline
  t(n) & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 9 & 18 & 36 \\
\end{array}
\]
# Answers

1. linear  
2. exponential  
3. neither  
4. linear  
5. linear  
6. neither  
7. exponential  
8. exponential  
9. neither  
10. linear  
11. exponential  
12. neither  
13. neither  
14. neither  
15. neither  
16. exponential
In this lesson, students determine the multiplier in a situation and use it to write the equation of a geometric sequence. Read more about writing equations for sequences in the Math Notes box in Lesson 5.3.3.

In addition to the ways to write explicit equations for sequences, equations for sequences can also be written recursively. An explicit formula tells exactly how to find any specific term in the sequence. A recursive formula names the first term (or any other term) and how to get from one term to the next. For an explanation of recursive sequences, see the Math Notes box in Lesson 5.3.3.

Example 1

Determine the multiplier in each situation below.

a. A 64% increase.  
b. A 34% decrease.

Solution:
For part (a), any amount you start with is 100%. Since it is an increase, we add 100% + 64% = 164%, which is a multiplier of 1.64.

For part (b), any amount you start with is 100%. Since it is a decrease, we subtract 100% – 34% = 66%, which is a multiplier of 0.66.

List the first five terms of each geometric sequence.

Example 2 (An explicit formula)

\[ t(n) = 3 \cdot 2^{n-1} \]

- \[ t(1) = 3 \cdot 2^{1-1} = 3 \cdot 2^0 = 3 \]
- \[ t(2) = 3 \cdot 2^{2-1} = 3 \cdot 2^1 = 6 \]
- \[ t(3) = 3 \cdot 2^{3-1} = 3 \cdot 2^2 = 12 \]
- \[ t(4) = 3 \cdot 2^{4-1} = 3 \cdot 2^3 = 24 \]
- \[ t(5) = 3 \cdot 2^{5-1} = 3 \cdot 2^4 = 48 \]

The sequence is: 3, 6, 12, 24, 48, ...

Example 3 (A recursive formula)

\[ t(1) = 8, \ t(n + 1) = t(n) \cdot \frac{1}{2} \]

- \[ t(1) = 8 \]
- \[ t(2) = t(1) \cdot \frac{1}{2} = 8 \cdot \frac{1}{2} = 4 \]
- \[ t(3) = t(2) \cdot \frac{1}{2} = 4 \cdot \frac{1}{2} = 2 \]
- \[ t(4) = t(3) \cdot \frac{1}{2} = 2 \cdot \frac{1}{2} = 1 \]
- \[ t(5) = t(4) \cdot \frac{1}{2} = 1 \cdot \frac{1}{2} = \frac{1}{2} \]

The sequence is: 8, 4, 2, 1, \(\frac{1}{2}\), …
Example 4

Write an explicit and a recursive formula for the geometric sequence: 81, 27, 9, 3, ...  

Explicit: \( t(1) = 81, \quad b = \frac{1}{3} \) so \( a \) (the zeroth term) is found by \( a = 81 \div \frac{1}{3} = 243 \) and the answer is:

\[
t(n) = a \cdot b^n = 243 \cdot \left(\frac{1}{3}\right)^n, \quad \text{or alternatively, } a_n = 243 \cdot \frac{1}{3^n}.
\]

Recursive: \( t(1) = 81, \quad t(n+1) = t(n) \cdot \frac{1}{3} \)

Example 5

When Rosa tripped and fell into a muddy puddle at lunch (she was so embarrassed!), she knew exactly what would happen: within ten minutes, the two girls who saw her fall would each tell four people what they had seen. Within the next ten minutes, those eight students would each tell four more people. Rosa knew this would continue until everyone in the entire school was talking about her muddy experience. Write a sequence for the number of people who knew about Rosa’s mishap in ten-minute intervals, then write an equation for the sequence. Fully describe a graph of this sequence.

The multiplier is \( b = 4 \), and the zeroth term is \( a = 2 \). The equation can be written \( t(n) = ab^n = 2 \cdot 4^n \). The equation could also have been written as \( a_n = 2 \cdot 4^n \).

The sequence is: 8, 32, 128, 512, .... Note that the sequence is written starting with \( n = 1 \).

The graph of the sequence is to the right. There are no \( x \)- or \( y \)-intercepts. There is no point at \((0, 2)\) because sequences are written starting with the first term where \( n = 1 \). The domain consists of integers greater than or equal to 1. The range consists of the \( y \)-values of the points that follow the rule \( t(n) = 2(4)^n \) when \( n \geq 1 \). The graph is exponential and is shown at right. There is no symmetry. This graph is discrete (separate points). (Note: The related function, \( y = 2 \cdot 4^n \) would have a domain of all real numbers and the graph would be a connected curve.)
Problems

Determine the multiplier in each situation below.

1. A 44% increase.
2. A 93% decrease.
3. A 32% increase.
4. A 60% decrease.

List the first five terms of each geometric sequence.

5. \( t(n) = 5 \cdot 2^n \)
6. \( t(n) = -3 \cdot 3^n \)
7. \( t(n) = 40 \left( \frac{1}{2} \right)^{n-1} \)
8. \( t(n) = 6 \left( -\frac{1}{2} \right)^{n-1} \)
9. \( t(1) = 5, \ t(n + 1) = t(n) \cdot 3 \)
10. \( t(1) = 100, \ t(n + 1) = t(n) \cdot \frac{1}{2} \)
11. \( t(1) = -3, \ t(n + 1) = t(n) \cdot (-2) \)
12. \( t(1) = \frac{1}{3}, \ t(n + 1) = t(n) \cdot \frac{1}{2} \)

If \( r \) is the common ratio (multiplier), find the 15th term of each geometric sequence.

13. \( t(14) = 232, \ r = 2 \)
14. \( t(16) = 32, \ r = 2 \)
15. \( t(14) = 9, \ r = \frac{2}{3} \)
16. \( t(16) = 9, \ r = \frac{2}{3} \)

Write an explicit and a recursive equation for each geometric sequence.

17. 2, 10, 50, 250, 1250, …
18. 16, 4, 1, \( \frac{1}{4}, \ \frac{1}{16}, \ldots \)
19. 5, 15, 45, 135, 405, …
20. 3, −6, 12, −24, 48, …

21. Every year since 1548, the average height of a human male has increased slightly. The new height is 100.05% of what it was the previous year. If the average male’s height was 54 inches in 1548, what was the average height of a male in 2008?


Answers

1. 1.44
2. 0.07
3. 1.32
4. 0.40
5. 10, 20, 40, 80, 160
6. -9, -27, -81, -243, -729
7. 40, 20, 10, 5, \( \frac{5}{2} \)
8. 6, -3, \( \frac{3}{2} \), -\( \frac{3}{4} \), \( \frac{3}{8} \)
9. 5, 15, 45, 135, 405
10. 100, 50, 25, \( \frac{25}{2} \), \( \frac{25}{4} \)
11. -3, 6, -12, 24, -48
12. \( \frac{1}{3} \), \( \frac{1}{6} \), \( \frac{1}{12} \), \( \frac{1}{24} \), \( \frac{1}{48} \)
13. 464
14. 16
15. 6
16. \( \frac{27}{2} \)
17. explicit: \( t(n) = \frac{2}{5} \cdot 5^n \)
   recursive: \( t(1) = 2, \ t(n+1) = t(n) \cdot 5 \)
18. explicit: \( t(n) = 64 \cdot \left( \frac{1}{4} \right)^n \)
   recursive: \( t(1) = 16, \ t(n+1) = t(n) \cdot \frac{1}{4} \)
19. explicit: \( t(n) = \frac{5}{3} \cdot 3^n \)
   recursive: \( t(1) = 5, \ t(n+1) = t(n) \cdot 3 \)
20. explicit: \( t(n) = -\frac{3}{2} \cdot (-2)^n \)
   recursive: \( t(1) = 3, \ t(n+1) = t(n) \cdot (-2) \)
21. In 2008, \( 54 \cdot (1.0005)^{460} \approx 67.96 \) inches.
Rewriting MULTI-VARIABLE EQUATIONS 6.1.1

Rewriting equations with more than one variable uses the same process as solving an equation with one variable. The end result is often not a number, but rather an algebraic expression containing numbers and variables.

Example 1

Solve for $y$

$3x - 2y = 6$

Subtract $3x$

$-2y = -3x + 6$

Divide by $-2$

$y = \frac{-3x + 6}{-2}$

Simplify

$y = \frac{3}{2}x - 3$

Example 2

Solve for $y$

$7 + 2(x + y) = 11$

Subtract $7$

$2(x + y) = 4$

Distribute the $2$

$2x + 2y = 4$

Subtract $2x$

$2y = -2x + 4$

Divide by $2$

$y = \frac{-2x + 4}{2}$

Simplify

$y = -x + 2$

Example 3

Solve for $x$

$y = 3x - 4$

Add $4$

$y + 4 = 3x$

Divide by $3$

$\frac{y + 4}{3} = x$

Example 4

Solve for $t$

$I = prt$

Divide by $pr$

$\frac{I}{pr} = t$

Problems

Solve each equation for the specified variable.

1. Solve for $y$: $5x + 3y = 15$

2. Solve for $x$: $5x + 3y = 15$

3. Solve for $w$: $2l + 2w = P$

4. Solve for $m$: $4n = 3m - 1$

5. Solve for $a$: $2a + b = c$

6. Solve for $a$: $b - 2a = c$

7. Solve for $p$: $6 - 2(q - 3p) = 4p$

8. Solve for $x$: $y = \frac{1}{4}x + 1$

9. Solve for $r$: $4(r - 3s) = r - 5s$

Answers (Other equivalent forms are possible.)

1. $y = -\frac{5}{3}x + 5$

2. $x = -\frac{3}{5}y + 3$

3. $w = -l + \frac{P}{2}$

4. $m = \frac{4a + 1}{3}$

5. $a = \frac{c-b}{2}$

6. $a = \frac{c-b}{2}$ or $\frac{b+c}{2}$

7. $p = q - 3$

8. $x = 4y - 4$

9. $r = \frac{7s}{3}$
In these lessons, students translate written information, often modeling everyday situations, into algebraic symbols and linear equations. Students use “let” statements to specifically define the meaning of each of the variables they use in their equations. Either a single a equation or a system of equations can be used to represent a situation. For additional examples and more problems, see the Checkpoint 8B materials.

Example 1

The perimeter of a rectangle is 60 cm. The length is 4 times the width. Write one or more equations that model the relationships between the length and width.

Start by identifying what is unknown in the situation. Then define variables, using “let” statements, to represent the unknowns. When writing “let” statements, the units of measurement must also be identified. This is often done using parentheses, as shown in the “let” statements below. In this problem, length and width are unknown.

Let $w$ represent the width (cm) of the rectangle, and let $l$ represent the length (cm).

In this problem there are two variables. To be able to find unique solutions for these two variables, two unique equations need to be written.

From the first sentence and our knowledge about rectangles, the equation $P = 2l + 2w$ can be used to write the equation $60 = 2l + 2w$. From the sentence “the length is 4 times the width” we can write $l = 4w$.

A system of equations is two or more equations that use the same set of variables to represent a situation. The system of equations that represent the situation is:

Let $w$ represent the width (cm) of the rectangle, and let $l$ represent the length (cm).

\[
\begin{align*}
    l &= 4w \\
    2l + 2w &= 60
\end{align*}
\]

Note that students who took a CPM middle school course may recall a method called the 5-D Process. This 5-D Process is not reviewed in this course, but it is perfectly acceptable for students to use it to help write and solve equations for word problems.

Using a 5-D table:

<table>
<thead>
<tr>
<th>Define</th>
<th>Do</th>
<th>Decide</th>
</tr>
</thead>
<tbody>
<tr>
<td>Width</td>
<td>Length</td>
<td>Perimeter</td>
</tr>
<tr>
<td>Trial 1:</td>
<td>10</td>
<td>4(10)</td>
</tr>
<tr>
<td>Trial 2:</td>
<td>5</td>
<td>4(5)</td>
</tr>
<tr>
<td>$l$</td>
<td>4($l$)</td>
<td>2($4w$) + 2$w$ = 60</td>
</tr>
</tbody>
</table>
Example 2

Mike spent $11.19 on a bag containing red and blue candies. The bag weighed 11 pounds. The red candy costs $1.29 a pound and the blue candy costs $0.79 a pound. How much red candy did Mike have?

Start by identifying the unknowns. The question in the problem is a good place to look because it often asks for something that is unknown. In this problem, the amount of red candy and the amount of blue candy are unknown.

Let \( r \) represent the amount of red candy (lb), and \( b \) represent the amount of blue candy (lb).

Note how the units of measurement were defined. If we stated “\( r \) = red candy” it would be very easy to get confused as to whether \( r \) represented the weight of the candy or the cost of the candy.

From the statement “the bag weighed 11 pounds” we can write \( r + b = 11 \). Note that in this equation the units are lb + lb = lb, which makes sense.

The cost of the red candy will be $1.29/pound multiplied by its weight, or \( 1.29r \). Similarly, the cost of the blue candy will be \( 0.79b \). Thus \( 1.29r + 0.79b = 11.19 \).

Problems

Write an equation or a system of equations that models each situation. Do not solve your equations.

1. A rectangle is three times as long as it is wide. Its perimeter is 36 units. Find the length of each side.

2. A rectangle is twice as long as it is wide. Its area is 72 square units. Find the length of each side.

3. The sum of two consecutive odd integers is 76. What are the numbers?

4. Nancy started the year with $425 in the bank and is saving $25 a week. Seamus started the year with $875 and is spending $15 a week. When will they have the same amount of money in the bank?

5. Oliver earns $50 a day and $7.50 for each package he processes at Company A. His paycheck on his first day was $140. How many packages did he process?

6. Dustin has a collection of quarters and pennies. The total value is $4.65. There are 33 coins. How many quarters and pennies does he have?
7. A one-pound mixture of raisins and peanuts costs $7.50. The raisins cost $3.25 a pound and the peanuts cost $5.75 a pound. How much of each ingredient is in the mixture?

8. An adult ticket at an amusement park costs $24.95 and a child’s ticket costs $15.95. A group of 10 people paid $186.50 to enter the park. How many were adults?

9. Katy weighs 105 pounds and is gaining 2 pounds a month. James weighs 175 pounds and is losing 3 pounds a month. When will they weigh the same amount?

10. Harper Middle School has 125 fewer students than Holmes Junior High. When the two schools are merged there are 809 students. How many students attend each school?

Answers (Other equivalent forms are possible.)

<table>
<thead>
<tr>
<th>One Variable Equation</th>
<th>System of Equations</th>
<th>Let Statement</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $2w + 2(3w) = 36$</td>
<td>$l = 3w$</td>
<td>Let $l =$ length, $w =$ width</td>
</tr>
<tr>
<td></td>
<td>$2w + 2l = 36$</td>
<td></td>
</tr>
<tr>
<td>2. $w(2w) = 72$</td>
<td>$l = 2w$</td>
<td>Let $l =$ length, $w =$ width</td>
</tr>
<tr>
<td></td>
<td>$lw = 72$</td>
<td></td>
</tr>
<tr>
<td>3. $m + (m + 2) = 76$</td>
<td>$m + n = 76$</td>
<td>Let $m =$ the first odd integer and $n =$ the next consecutive odd integer</td>
</tr>
<tr>
<td></td>
<td>$n = m + 2$</td>
<td></td>
</tr>
<tr>
<td>4. $425 + 25x = 875 - 15x$</td>
<td>$y = 425 + 25x$</td>
<td>Let $x =$ the number of weeks and $y =$ the total money in the bank</td>
</tr>
<tr>
<td></td>
<td>$y = 875 - 15x$</td>
<td></td>
</tr>
<tr>
<td>5. $50 + 7.5p = 140$</td>
<td></td>
<td>Let $p =$ the number of packages Oliver processed</td>
</tr>
<tr>
<td>6. $0.25q + 0.01(33 - q) = 4.65$</td>
<td>$q + p = 33$</td>
<td>Let $q =$ number of quarters, $p =$ number of pennies</td>
</tr>
<tr>
<td></td>
<td>$0.25q + 0.01p = 4.65$</td>
<td></td>
</tr>
<tr>
<td>7. $3.25r + 5.75(1 - r) = 7.5$</td>
<td>$r + p = 1$</td>
<td>Let $r =$ weight of raisins and $p =$ weight of peanuts</td>
</tr>
<tr>
<td></td>
<td>$3.25r + 5.75p = 7.5(1)$</td>
<td></td>
</tr>
<tr>
<td>8. $24.95a + 15.95(10 - a) = 186.5$</td>
<td>$a + c = 10$</td>
<td>Let $a =$ number of adult tickets and $c =$ number of child’s tickets</td>
</tr>
<tr>
<td></td>
<td>$24.95a + 15.95c = 186.5$</td>
<td></td>
</tr>
<tr>
<td>9. $105 + 2m = 175 - 3m$</td>
<td>$w = 105 + 2m$</td>
<td>Let $m =$ the number of months and $w =$ the weight of each person</td>
</tr>
<tr>
<td></td>
<td>$w = 175 - 3m$</td>
<td></td>
</tr>
<tr>
<td>10. $x + (x - 125) = 809$</td>
<td>$x + y = 809$</td>
<td>Let $x =$ number of Holmes students and $y =$ number of Harper students</td>
</tr>
<tr>
<td></td>
<td>$y = x - 125$</td>
<td></td>
</tr>
</tbody>
</table>
A system of equations has two or more equations with two or more variables. The **Substitution Method** is used to change a two-variable system of equations to a one-variable equation. This method is useful when one of the variables is isolated in one of the equations in the system. Two substitutions must be made to find both the \( x \)- and \( y \)-values for a complete solution.

Students learned the **Equal Values Method** in Lesson 6.2.1. It is a simple use of the Substitution Method. The Equal Values Method is convenient to use when both equations are in \( y = \) form.

For additional information, see the Math Notes boxes in Lessons 6.2.1 and 6.2.3. For additional examples and more problems solving systems using multiple methods, see the Checkpoint 9 Materials.

### Example 1 (Equal Values Method)

Solve:

\[
\begin{align*}
y &= 2x \\
y &= 9 - x
\end{align*}
\]

Since \( y \) represents equal values in both equations, we can write \( 2x = 9 - x \).

Solving for \( x \):

\[
\begin{align*}
2x &= 9 - x \\
+x &+ x \\
3x &= 9 \\
x &= 3
\end{align*}
\]

Then, either equation can be used to solve for \( y \).

For example, use the first equation, \( y = 2x \), to solve for \( y \):

\[
\text{Since } x = 3, \text{ and } y = 2x:\n\begin{align*}
y &= 2(3) \\
y &= 6
\end{align*}
\]

The solution to this system of equation is \( x = 3 \) and \( y = 6 \). That is, the values \( x = 3 \) and \( y = 6 \) make both of the original equations true. When graphing, the point \((3, 6)\) is the intersection of the two lines.

Always check your solutions by substituting the \( x \)- and \( y \)-values back into *both* of the original equations to verify that you get true statements:

For \( y = 2x \) if \( x = 3 \) and \( y = 6 \):  
\[
\begin{align*}
6 &= 2(3) \\
6 &= \text{ is a true statement.}
\end{align*}
\]

For \( x + y = 9 \) if \( x = 3 \) and \( y = 6 \):

\[
\begin{align*}
3 + 6 &= 9 \\
9 &= \text{ is a true statement.}
\end{align*}
\]
Example 2 (Substitution Method)

Solve:
\[ y = 2x \]
\[ x + y = 9 \]

A similar system of equations as those in Example 1 can be solved using the Substitution Method. Since \( y = 2x \), we can replace the \( y \) in the second equation with \( 2x \):

Then, either equation can be used to solve for \( y \). For example, using the first equation to solve for \( y \):

\[ y = 2x \]
Since \( x = 3 \),
\[ y = 2(3) \]
\[ y = 6 \]

The solution to this system of equations is \( x = 3 \) and \( y = 6 \).

Always check your solution by substituting the \( x \)- and \( y \)-values back into both of the original equations to verify that you get true statements (see Example 1).

Example 3 (Substitution Method)

Look for a convenient substitution when using the Substitution Method; that is, look for a variable by itself on one side of the equation.

Solve:
\[ 4x + y = 8 \]
\[ x = 5 - y \]

Since \( x = 5 - y \), replace \( x \) in the first equation with \( 5 - y \).

Solve as usual.
\[ 4(5 - y) + y = 8 \]
\[ 20 - 4y + y = 8 \]

To solve for \( x \), use either of the original two equations. In this case, we will use \( x = 5 - y \).

\[ x = 5 - y \]
\[ 20 - 3y = 8 \]
Since \( y = 4 \),
\[ -3y = -12 \]
\[ x = 5 - (4) \]
\[ y = 4 \]
\[ x = 1 \]

The solution is \( (1, 4) \). Always check your solution by substituting the solution back into both of the original equations to verify that you get true statements.
Example 4

Sometimes you need to solve one of the equations for \(x\) or \(y\) to use the Substitution Method.

Solve: \(\begin{align*}
y - 2x &= -7 \\
3x - 4y &= 8
\end{align*}\)

Rewrite the first equation in \(y = \) form to get \(y = 2x - 7\).

Then substitute \(2x - 7\) for \(y\) in the second equation.

Use either of the original equations to solve for \(y\).

The solution is \((4, 1)\). Always check your solution by substituting the solution back into both of the original equations to verify that you get true statements.

Check:

\(\begin{align*}
y - 2x &= -7 \\
3x - 4y &= 8
\end{align*}\)

Then substitute \(2x - 7\) for \(y\) in the second equation.

Use either of the original equations to solve for \(y\).

The solution is \((4, 1)\). Always check your solution by substituting the solution back into both of the original equations to verify that you get true statements.

\(\begin{align*}
y &\quad 2x = -7 \\
\quad 3x - 4y &= 8
\end{align*}\)

\(\begin{align*}
y &\quad 2x = -7 \\
\quad 3x - 4y &= 8
\end{align*}\)

The solution is \((4, 1)\). Always check your solution by substituting the solution back into both of the original equations to verify that you get true statements.

Check:

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\end{align*}\)

\(\begin{align*}
y &\quad 2x = -7 \\
\quad 3x - 4y &= 8
\end{align*}\)

Problems

1. \(y = -3x\)
   \(4x + y = 2\)

2. \(y = 7x - 5\)
   \(2x + y = 13\)

3. \(x = -5y - 4\)
   \(x - 4y = 23\)

4. \(x + y = 10\)
   \(y = x - 4\)

5. \(y = 5 - x\)
   \(4x + 2y = 10\)

6. \(3x + 5y = 23\)
   \(y = x + 3\)

7. \(y = -x - 2\)
   \(2x + 3y = -9\)

8. \(x = \frac{1}{2}y + \frac{1}{2}\)
   \(2x + y = -1\)

9. \(x = 2y + 4\)
   \(y - 2x = 16\)

Answers

1. \((2, -6)\)

2. \((2, 9)\)

3. \((11, -3)\)

4. \((7, 3)\)

5. \((0, 5)\)

6. \((1, 4)\)

7. \((3, -5)\)

8. \((0, -1)\)

9. \((-12, -8)\)
Another method for solving systems of equations is the **Elimination Method**. This method is particularly convenient when both equations are in standard form (that is, \( ax + by = c \)). To solve this type of system, we can rewrite the equations by focusing on the coefficients. (The coefficient is the number in front of the variable.)

See problem 6-80 in the textbook for an additional explanation of the Elimination Method.

For additional information, see the Math Notes boxes in Lesson 6.3.3. For additional examples and more problems solving systems using multiple methods, see the Checkpoint 9 Materials.

---

**Example 1**

Solve: 

\[
\begin{align*}
\begin{align*}
x - y &= 2 \\
2x + y &= 1
\end{align*}
\end{align*}
\]

Recall that you are permitted to add the same expression to both sides of an equation. Since \( x - y \) is equivalent to 2 (from the first equation), you are permitted to add \( x - y \) to one side of the second equation, and 2 to the other side, then solve.

\[
\begin{align*}
2x + y &= 1 \\
+ (x - y + 2) &
\end{align*}
\]

\[
\frac{3x}{3} = 3
\]

\[
x = 1
\]

Note that this was an effective way to eliminate \( y \) and solve for \( x \) because \( -y \) and \( y \) eliminated each other.

Now substitute the value of \( x \) in either of the original equations to solve for \( y \).

\[
\begin{align*}
2x + y &= 1 \\
2(1) + y &= 1 \\
2 + y &= 1
\end{align*}
\]

\[
y = -1
\]

The solution is \((1, -1)\), since \( x = 1 \) and \( y = -1 \) make both of the original equations true. Always check your solution by substituting the \( x \)- and \( y \)-values back into both of the original equations to verify that you get true statements.
Example 2

Solve: \[3x + 6y = 24\]
\[3x + y = -1\]

Notice that both equations contain a 3x term. We can rewrite \(3x + y = -1\) by multiplying both sides by -1, resulting in \(-3x + (-y) = 1\). Now the two equations have terms that are opposites: 3x and -3x. This will be useful in the next step because \(-3x + 3x = 0\).

Since \(-3x + (-y)\) is equivalent to 1, we can add \(-3x + (-y)\) to one side of the equation and add 1 to the other side.

\[
\begin{align*}
3x + 6y &= 24 \\
-3x + (-y) + 1 &= 1
\end{align*}
\]

Notice how the two opposite terms, 3x and -3x, eliminated each other, allowing us to solve for \(y\).

Then substitute the value of \(y\) into either of the original equations to solve for \(x\).

The solution is \((-2, 5)\). Always check your solution by substituting the solution back into both of the original equations to verify that you get true statements.

Example 3

Solve: \[x + 3y = 7\]
\[4x - 7y = -10\]

To use the Elimination Method, one of the terms in one of the equations needs to be opposite of the corresponding term in the other equation. For example, in the system above, there are no terms that are opposite. However, if the first equation is multiplied by -4, then the equations will have 4x and -4x. The first equation now looks like this: \(-4(x + 3y = 7) \rightarrow -4x + (-12y) = -28\). When multiplying, be sure to multiply all the terms on both sides of the equation. With the first equation rewritten, the system of equations now looks like this:

\[
\begin{align*}
-4x + (-12y) &= -28 \\
4x - 7y &= -10
\end{align*}
\]

Since \(4x - 7\) is equivalent to \(-10\), they can be added either side of the first equation:

\[
\begin{align*}
-4x + (-12y) &= -28 \\
+4x - 7y &= -10 \\
\hline
-19y &= -38
\end{align*}
\]

Now any of the equations can be used to solve for \(x\):

The solution to the system of equations is \((1, 2)\).
Example 4

If multiplying one equation by a number will not make it possible to eliminate a variable, multiply both equations by different numbers to get coefficients that are the same or opposites.

Solve: \[8x - 7y = 5\] \[3x - 5y = 9\]

One possibility is to multiply the first equation by 3 and the second equation by -8. The resulting terms \(24x\) and \(-24x\) will be opposites.

The system of equations is now:

\[
\begin{align*}
24x - 21y &= 15 \\
-24x + 40y &= -72
\end{align*}
\]

This system can be solved by adding equivalent expressions (from the second equation) to the first equation:

\[
\begin{align*}
24x - 21y &= 15 \\
+ (24x + 40y) &= -72 \\
19y &= -57 \\
y &= -3
\end{align*}
\]

Then, solving for \(x\), the solution is \((-2, -3)\).

Example 5

Not all systems of equations have a solution. If solving the system results in an equation that is not true, then there is **no solution**. The graph of a system of two linear equations that has no solutions is two parallel lines; there is no point of intersection. See the Math Notes box in Lesson 6.4.1 for additional information.

Solve: \[y = 7 - 3x\] \[3x + y = 10\]

Replace \(y\) in the second equation with \(7 - 3x\).

The resulting equation is never true.

There is no solution to this system of equations.

Example 6

There may also be **infinitely many solutions**. If solving the system results in an equation that is always true, then there are infinitely many solutions (all of the points on both lines when graphed). This graph would appear as a single line for the two equations.

Solve: \[y = 4 - 2x\] \[-4x - 2y = -8\]

Substitute \(4 - 2x\) in the second equation for \(y\).

This statement is always true. There are infinitely many solutions to this system of equations.
SUMMARY OF METHODS TO SOLVE SYSTEMS

<table>
<thead>
<tr>
<th>Method</th>
<th>This Method is Most Efficient When</th>
<th>Example</th>
</tr>
</thead>
</table>
| Equal Values                   | Both equations in $y = \text{form.}$                | $y = x - 2$
|                                |                                                     | $y = -2x + 1$      |
| Substitution                   | One variable is alone on one side of one equation. | $y = -3x - 1$
|                                |                                                     | $3x + 6y = 24$     |
| Elimination: Add to            | Equations in standard form with opposite coefficients. | $x + 2y = 21$
| eliminate one variable.       |                                                     | $3x - 2y = 7$      |
| Elimination: Multiply one      | Equations in standard form. One equation can be multiplied to create opposite terms. | $x + 2y = 3$
| one equation to eliminate      |                                                     | $3x + 2y = 7$      |
| Elimination: Multiply both     | When nothing else works. In this case you could multiply the first equation by 3 and the second equation by -2, then add to eliminate the opposite terms. | $2x - 5y = 3$
| equations to eliminate one     |                                                     | $3x + 2y = 7$      |
| variable.                      |                                                     |                    |

Problems

1. $2x + y = 6$
   $-2x + y = 2$

2. $-4x + 5y = 0$
   $-6x + 5y = -10$

3. $2x - 3y = -9$
   $x + y = -2$

4. $y - x = 4$
   $2y + x = 8$

5. $2x - y = 4$
   $\frac{1}{2}x + y = 1$

6. $-4x + 6y = -20$
   $2x - 3y = 10$

7. $6x - 2y = -16$
   $4x + y = 1$

8. $6x - y = 4$
   $6x + 3y = -16$

9. $2x - 2y = 5$
   $2x - 3y = 3$

10. $y - 2x = 6$
    $y - 2x = -4$

11. $4x - 4y = 14$
    $2x - 4y = 8$

12. $3x + 2y = 12$
    $5x - 3y = -37$

Answers

1. (1, 4)

2. (5, 4)

3. (−3, 1)

4. (0, 4)

5. (2, 0)

6. infinitely many solutions

7. (−1, 5)

8. (−4/6, −5)

9. (4.5, 2)

10. no solution

11. (3, −1/2)

12. (−2, 9)
Two triangles are congruent if there is a sequence of rigid transformations (reflections, rotations, and translations) that carries one triangle onto the other. In these lessons, students find shortcuts that enable them to prove triangles congruent in the least amount of steps, by developing five triangle congruence conditions.

The triangle congruence conditions are $\text{SSS} \cong$, $\text{ASA} \cong$, $\text{AAS} \cong$, $\text{SAS} \cong$, and $\text{HL} \cong$, illustrated below. Note: “S” stands for “side” and “A” stands for “angle.” $\text{HL} \cong$ is only used with right triangles. The “H” stands for “hypotenuse” and the “L” stands for leg. The pattern appears to be “SSA” but this arrangement is NOT one of our conjectures, since it is only true for right triangles.

See the Math Notes boxes in Lessons 7.1.4 and 7.1.7. For additional examples and more practice, see the Checkpoint 10 Materials.
Example 1

Use the triangle congruence conditions to decide whether or not each pair of triangles must be congruent. Base each decision on the markings, not on appearances. Justify each answer.

a. The triangles are congruent by SAS $\cong$.

b. The triangles are congruent by SSS $\cong$.

c. The triangles are congruent by AAS $\cong$.

d. The triangles are not necessarily congruent. The first triangle displays an ASA arrangement, while the second triangle displays an AAS arrangement. The triangles could still be congruent, but based on the markings, we cannot conclude that they definitely are congruent.

e. The triangles are right triangles and congruent by HL $\cong$.

f. The triangles are congruent by ASA $\cong$. 
Example 2

Using the information given in the diagrams below, decide if any triangles are congruent. If you claim the triangles are congruent, create a flowchart justifying your answer.

a. 

b.

In part (a), \( \triangle ABD \cong \triangle CBD \) by the SAS \( \cong \) conjecture. Note: If you only see “SA,” observe that \( BD \) is congruent to itself.

In part (b), the triangles are not necessarily congruent; they could be congruent, but since we only have information about angles, we cannot conclude anything else. Note: They are similar by AA \( \sim \).

Problems

Briefly explain if each of the following pairs of triangles are congruent or not. If so, state the triangle congruence condition that supports your conclusion.

1. 
2. 
3. 
4. 
5. 
6.
7. 

8. 

9. 

10. 

11. 

12. 

13. 

14. 

15. 

16. 

17. 

18. 

19. 

20. 

21. 

22. 

23. 

24. 

25. 

Chapter 7
In each diagram below, are any triangles congruent? If so, prove it.

26.  
27.  
28.  

29.  
30.  
31.  

Complete a proof for each problem below.

32. Given: $\overline{TR}$ and $\overline{MN}$ bisect each other.  
   Prove: $\triangle NTP \cong \triangle MRP$

33. Given: $\overline{CD}$ bisects $\angle ACB$; $\angle 1 \equiv \angle 2$.  
   Prove: $\triangle CDA \cong \triangle CDB$

34. Given: $\overline{AB} \parallel \overline{CD}$, $\angle B \equiv \angle D$, $\overline{AB} \equiv \overline{CD}$  
   Prove: $\triangle ABD \cong \triangle CDE$

35. Given: $\overline{PG} \equiv \overline{SG}$, $\overline{TP} \equiv \overline{TS}$  
   Prove: $\triangle TPG \cong \triangle TSG$

36. Given: $\overline{OE} \perp \overline{MP}$, $\overline{OE}$ bisects $\angle MOP$  
   Prove: $\triangle MOE \cong \triangle POE$

37. Given: $\overline{AD} \parallel \overline{BC}$, $\overline{DC} \parallel \overline{BA}$  
   Prove: $\triangle ADB \cong \triangle CBD$
38. Given: \( \overline{AC} \) bisects \( \overline{DE}, \angle A \equiv \angle C \)
Prove: \( \triangle ADB \cong \triangle CEB \)

39. Given: \( \overline{PQ} \perp \overline{RS}, \angle R \equiv \angle S \)
Prove: \( \triangle PQR \cong \triangle PQS \)

40. Given: \( \angle S \equiv \angle R, \overline{PQ} \) bisects \( \angle SQR \)
Prove: \( \triangle SPQ \cong \triangle RPQ \)

41. Given: \( \overline{TU} \equiv \overline{GY}, \overline{KY} \parallel \overline{HU}, \overline{KT} \perp \overline{TG}, \overline{HG} \perp \overline{TG} \)
Prove: \( \angle K \equiv \angle H \)

42. Given: \( \overline{MQ} \parallel \overline{WL}, \overline{MQ} \equiv \overline{WL} \)
Prove: \( \overline{ML} \parallel \overline{WQ} \)

Consider the diagram at right.

43. Is \( \triangle BCD \equiv \triangle EDC \)? Prove it!

44. Is \( \overline{AB} \equiv \overline{DC} \)? Prove it!

45. Is \( \overline{AB} \equiv \overline{ED} \)? Prove it!
Answers

1. $\triangle ABC \cong \triangle DEF$ by ASA $\cong$.
2. $\triangle GIH \cong \triangle LJK$ by SAS $\cong$.
3. $\triangle PNM \cong \triangle PNO$ by SSS $\cong$.
4. $\overline{QS} \cong \overline{QS}$, so $\triangle QRS \cong \triangle QTS$ by HL $\cong$.
5. Not necessarily congruent.
6. $\triangle ABC \cong \triangle DFE$ by ASA $\cong$ or AAS $\cong$.
7. $\overline{GI} \cong \overline{GI}$, so $\triangle GHI \cong \triangle JIG$ by SSS $\cong$.
8. Alternate interior angles $\cong$ used twice, so $\triangle KLN \cong \triangle NMK$ by ASA $\cong$.
10. Vertical angles and/or alternate interior angles $\cong$, so $\triangle TUX \cong \triangle VWX$ by ASA $\cong$.
11. No, the length of each hypotenuse is different.
12. Pythagorean Theorem, so $\triangle EGH \cong \triangle IHG$ by SSS $\cong$.
13. Sum of angles of triangle = 180º, but since the equal angles do not correspond, the triangles are not congruent.
14. $AF + FC = FC + CD$, so $\triangle ABC \cong \triangle DEF$ by SSS $\cong$.
15. $\overline{XZ} \cong \overline{XZ}$, so $\triangle WXZ \cong \triangle YXZ$ by AAS $\cong$.
16. $\triangle ABC \cong \triangle EDC$ by AAS $\cong$.
17. $\triangle PQS \cong \triangle PRS$ by AAS $\cong$, with $\overline{PS} \cong \overline{PS}$ (segment congruent to itself).
18. $\triangle VXW \cong \triangle ZXY$ by ASA $\cong$, with $\angle VXW \cong \angle ZXY$ because vertical angles are $\cong$.
20. $\triangle KLB \cong \triangle EBL$ by HL $\cong$, with $\overline{BL} \cong \overline{BL}$ (segment congruent to itself).
21. Vertical angles $\cong$ at $O$, so $\triangle POQ \cong \triangle ROS$ by SAS $\cong$.
22. Not necessarily congruent.
23. $\triangle TEA \cong \triangle SAE$ by SSS $\cong$, with $\overline{EA} \cong \overline{EA}$ (segment congruent to itself).
25. $\triangle KRS \cong \triangle ISR$ by HL $\cong$. 
26. \( \angle BAD \equiv \angle BCD \) Given  
\( \overline{BD} \equiv \overline{BD} \) Side congruent to itself  
\( \angle BDC \equiv \angle BDA \) Right \( \angle \)'s are \( \equiv \)  
\( \triangle ABD \equiv \triangle CBD \) AAS \( \equiv \)

27.  
\( \angle B \equiv \angle E \) Given  
\( \overline{BC} \equiv \overline{CE} \) Vertical \( \angle \)'s are \( \equiv \)  
\( \angle BCA \equiv \angle BCD \)  
\( \triangle ABC \equiv \triangle DEC \) ASA \( \equiv \)

28. \( AC \equiv CD \) Given  
\( \angle BCD \equiv \angle BCA \) Right \( \angle \)'s are \( \equiv \)  
\( \overline{BC} \equiv \overline{BC} \) Side congruent to itself  
\( \triangle ABC \equiv \triangle DBC \) SAS \( \equiv \)

29.  
\( AD \equiv BC \) Given  
\( CA \equiv CA \) Side congruent to itself  
\( BA \equiv CD \)  
\( \triangle ABC \equiv \triangle CDA \) SSS \( \equiv \)

30. Not necessarily. Counterexample:

31. \( BC \equiv EF \) Given  
\( AC \equiv FD \) Given  
\( \triangle ABC \equiv \triangle DEF \) HL \( \equiv \)

32. \( \overline{NP} \equiv \overline{MP} \) and \( \overline{TP} \equiv \overline{RP} \) by definition of bisector. \( \angle NPT \equiv \angle MPR \) because vertical angles are equal. So, \( \triangle NTP \equiv \triangle MRP \) by SAS \( \equiv \).

33. \( \angle ACD \equiv \angle BCD \) by definition of angle bisector. \( \overline{CD} \equiv \overline{CD} \) (segment congruent to itself) so \( \triangle CDA \equiv \triangle CDB \) by ASA \( \equiv \).

34. \( \angle A \equiv \angle C \) since alternate interior angles of parallel lines congruent so \( \triangle ABF \equiv \triangle CDE \) by ASA \( \equiv \).

35. \( \overline{TG} \equiv \overline{TG} \) (segment congruent to itself) so \( \triangle TPG \equiv \triangle TSG \) by SSS \( \equiv \).
36. \( \angle MEO \cong \angle PEO \) because perpendicular lines form \( \cong \) right angles \( \angle MOE \cong \angle POE \) by angle bisector and \( \overline{OE} \cong \overline{OE} \) (segment congruent to itself). So, \( \triangle MOE \cong \triangle POE \) by ASA \( \cong \).

37. \( \angle CDB \cong \angle ABD \) and \( \angle ADB \cong \angle CBD \) since parallel lines give congruent alternate interior angles. \( \overline{DB} \cong \overline{DB} \) (segment congruent to itself) so \( \triangle ADB \cong \triangle CBD \) by ASA \( \cong \).

38. \( \overline{DB} \cong \overline{EB} \) by definition of bisector. \( \angle DBA \cong \angle EBC \) since vertical angles are congruent. So \( \triangle ADB \cong \triangle EBC \) by AAS \( \cong \).

39. \( \angle RQP \cong \angle SQP \) since perpendicular lines form congruent right angles. \( \overline{PQ} \cong \overline{PQ} \) (segment is congruent to itself) so \( \triangle PQR \cong \triangle PQS \) by AAS \( \cong \).

40. \( \angle SQP \cong \angle RQP \) by angle bisector and \( \overline{PQ} \cong \overline{PQ} \) (segment congruent to itself), so \( \triangle SPQ \cong \triangle RPQ \) by AAS \( \cong \).

41. \( \angle KYT \cong \angle HUG \) because parallel lines form congruent alternate exterior angles. \( TY + YU = YU + GU \) so \( TY \cong GU \) by subtraction. \( \angle T \cong \angle G \) since perpendicular lines form congruent right angles. So \( \triangle KTY \cong \triangle HGU \) by ASA. Therefore, \( \angle K \cong \angle H \) since congruent triangles have congruent parts.

42. \( \angle MQL \cong \angle WLQ \) since parallel lines form congruent alternate interior angles. \( \overline{QL} \cong \overline{QL} \) (segment congruent to itself) so \( \triangle MQL \cong \triangle WLQ \) by SAS so \( \angle WQL \cong \angle MLQ \) since congruent triangles have congruent parts. So \( \overline{ML} \parallel \overline{WQ} \) since congruent alternate interior angles are formed by parallel lines.

43. Not necessarily congruent.

44. Not necessarily congruent.
Now that students are familiar with many of the properties of various triangles, quadrilaterals, and special quadrilaterals, they can apply their algebra skills and knowledge of the coordinate grid to study **coordinate geometry**. In this section, polygons are plotted on coordinate axes. Using familiar ideas, such as the Pythagorean Theorem and slope, students can prove whether or not quadrilaterals have special properties.

See the Math Notes boxes in Lessons 7.2.2 and 7.2.3.

**Example 1**

On a set of coordinate axes, plot the points $A(-3, -1), B(1, -4), C(5, -1)$, and $D(1, 2)$ and connect them in the order given. Is this quadrilateral a rhombus? Justify your answer.

To show that this quadrilateral is a rhombus, we must show that all four sides are the same length (definition of a rhombus). When we want to determine the length of a segment on the coordinate graph, we use the Pythagorean Theorem. To begin, plot the points on a graph.

Although the shape appears to be a parallelogram, and possibly a rhombus, we cannot base our decision on appearances. To use the Pythagorean Theorem, we outline a **slope triangle**, creating a right triangle with $DC$ as the hypotenuse. The lengths of the legs of this right triangle are 3 and 4 units. Using the Pythagorean Theorem,

\[3^2 + 4^2 = (DC)^2\]
\[9 + 16 = (DC)^2\]
\[25 = (DC)^2\]
\[DC = 5\]

Similarly, we can draw slope triangles for the other three sides of the quadrilateral and use the Pythagorean Theorem again. In each case, we find the lengths are all 5 units. Therefore, since all four sides have the same length, the polygon is a rhombus.
Example 2

On a set of coordinate axes, plot the points \(A(-4, 1), B(1, 3), C(8, -1), \) and \(D(4, -3)\), and connect them in the order given. Is this quadrilateral a parallelogram? Justify your answer.

When we plot the points, the quadrilateral appears to be a parallelogram, but we cannot base our decision on appearances. To prove it is a parallelogram, we must show that the opposite sides are parallel. On a graph, we show that lines are parallel by showing that they have the same slope.

We can use slope triangles to find the slope of each side.

- Slope of \(BC = \frac{\Delta y}{\Delta x} = \frac{4}{7}\)
- Slope of \(AD = \frac{\Delta y}{\Delta x} = \frac{1}{2}\)
- Slope of \(BA = \frac{\Delta y}{\Delta x} = \frac{2}{3}\)
- Slope of \(DC = \frac{\Delta y}{\Delta x} = \frac{1}{2}\)

Although the values for the slopes of the opposite sides are close, they are not equal. Therefore this quadrilateral is not a parallelogram.

Example 3

On a set of coordinate axes, plot the points \(A(-1, 5), B(6, 1), C(-3, -1)\). Determine the midpoints of \(AC\) and \(AB\). Then connect the midpoints to draw a midsegment of the triangle. Show that the midsegment is parallel to, and half the length of \(BC\).

To determine the midpoints of \(AC\) and \(AB\), draw slope triangles.

The midpoint of a segment is found by adding half of the change in \(x\) (\(\frac{1}{2} \Delta x\)) and half the change in \(y\) (\(\frac{1}{2} \Delta y\)) to the coordinates of the leftmost endpoint.

For \(AC\), \(\Delta x = 2\) and \(\Delta y = 6\), so \(\frac{1}{2} \Delta x = 1\) and \(\frac{1}{2} \Delta y = 3\).
Point \(C(-3, -1)\) is the leftmost point, so the midpoint of \(AC\) is \((-3 + 1, -1 + 3) = (-2, 2)\).

Repeat this process to determine that the midpoint of \(AB\):
\(\Delta x = 7\) and \(\Delta y = -4\), so \(\frac{1}{2} \Delta x = 3.5\) and \(\frac{1}{2} \Delta y = -2\).
Point \(A\) is the leftmost point, so the midpoint of \(AB\) is \((-1 + 3.5, 5 + (-2)) = (2.5, 3)\).

To show that the midsegment is parallel to \(BC\), we need to show that they have the same slope. The slope of the midsegment is \(\frac{1}{2} = \frac{3}{2}\). The slope of \(BC\) is also \(\frac{3}{2}\).

Use your slope triangles and the Pythagorean Theorem to determine the length of the midsegment and the length of \(BC\).

Midsegment: \(4.5^2 + 1^2 = c^2\) \(21.25 = c^2\) \(c = 4.61\)
\(BC: 9^2 + 2^2 = c^2\) \(85 = c^2\) \(c = 9.22\) \(4.61\) is half of \(9.22\)
Problems

1. If $ABCD$ is a rectangle, and $A(1, 2), B(5, 2),$ and $C(5, 5)$, what is the coordinate of $D$?

2. If $P(2, 1)$ and $Q(6, 1)$ are the endpoints of the base of an isosceles right triangle, what is the $x$-coordinate of the third vertex?

3. The three points $S(-1, -1), A(1, 4),$ and $M(2, -1)$ are vertices of a parallelogram. What are the coordinates of three possible points for the other vertex?

4. Graph the following lines on the same set of axes. These lines enclose a shape. What is the name of that shape? Justify your answer.

   \[
   \begin{align*}
   y &= \frac{3}{5}x + 7 \\
   y &= 0.6x \\
   y &= -\frac{10}{6}x - 1 \\
   y &= -\frac{5}{3}x + 9
   \end{align*}
   \]

5. If $W(-4, -5), X(1, 0), Y(-1, 2),$ and $Z(-6, -3)$, what shape is $WXYZ$? Justify your answer.

6. If $\overline{DT}$ has endpoints $D(2, 2)$ and $T(6, 4)$, what is the equation of the line containing the perpendicular bisector of $\overline{DT}$?

Answers

1. $(1, 5)$

2. $(4, 4)$

3. $(4, 4), (0, -6),$ or $(-2, 4)$

4. Since the slopes of opposites side are equal, this is a parallelogram. Additionally, since the slopes of intersecting lines are negative reciprocals of each other, they are perpendicular. This means the angles are all right angles, so the figure is a rectangle.

5. The slopes are: $WX = 1$, $XY = -1$, $YZ = 1$, and $ZW = -1$. This shows that $WXYZ$ is a rectangle.

6. $y = -2x + 11$

   The midpoint of $\overline{DT}$ is $(4, 3)$. The slope of $\overline{DT}$ is $\frac{1}{2}$, so the perpendicular slope is $-2$. Start with $3 = -2(4) + b$ and solve for $b$. 
In these sections, students generalize what they have learned about geometric sequences to investigate exponential functions. Students study exponential functions of the form \( y = ab^x \). Students look at multiple representations of exponential functions, including graphs, tables, equations, and context. They learn how to move from one representation to another. Students learn that the value of \( a \) is the “starting value” of the function—\( a \) is the \( y \)-intercept or the value of the function at \( x = 0 \). \( b \) is the growth (multiplier). If \( b > 1 \) then the function increases; if \( b \) is a fraction between 0 and 1 (that is, \( 0 < b < 1 \)), then the function decreases (decays). In this course, values of \( b \leq 0 \) are not considered.

For additional information, see the Math Notes box in Lesson 8.1.6. For additional examples and more practice, see the Checkpoint 11 materials.

Example 1

Most homes appreciate in value, at varying rates, depending on the home’s location, size, and other factors. But, if a home is used as a rental, the Internal Revenue Service allows the owner to assume that it will depreciate in value. Suppose a house that costs $150,000 is used as a rental property, and depreciates at a rate of 8% per year. What is the multiplier that will give the value of the house after one year? What is the value of the house after one year? What is the value after ten years? When will the house be worth half of its purchase price? Create a graph to represent this situation.

Solution: Unlike interest, which increases the value of the house, depreciation takes value away. After one year, the value of the house is \( 150000 - 0.08(150000) \) which is the same as \( 150000(0.92) \). Therefore the multiplier is 0.92. After one year, the value of the house is \( 150000(0.92) = 138,000 \). After ten years, the value of the house will be \( 150000(0.92)^{10} = 65,158.27 \). This last equation is an exponential function in the form \( y = ab^x \), where \( y \) is the value of the house \( x \) and is the number of years. \( a = 150000 \) is the starting value (at 0 years), and \( b = 0.92 \) is the multiplier or growth factor (in this case, decay) each year.

To find when the house will be worth half of its purchase price, we need to determine when the value of the house reaches $75,000. We just found that after ten years, the value is below $75,000, so this situation occurs in less than ten years. To help answer this question, list the house’s values in a table to see the depreciation.

Example continues on next page →
Example continued from previous page.

From the table or graph, you can see that the house will be worth half its purchase price after 8 years.

Note: You can write the equation $75000 = 150000 \cdot 0.92^x$, but you will not have the mathematics skills to solve this equation for $x$ until a future course. However, you can use the equation to find a more exact value: try different values for $x$ in the equation, so the $y$-value gets closer and closer to $75,000$. At about 8.313 years the house’s value is close to $75,000.

<table>
<thead>
<tr>
<th># Years</th>
<th>House’s value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>138000</td>
</tr>
<tr>
<td>2</td>
<td>126960</td>
</tr>
<tr>
<td>3</td>
<td>116803.20</td>
</tr>
<tr>
<td>4</td>
<td>107458.94</td>
</tr>
<tr>
<td>5</td>
<td>98862.23</td>
</tr>
<tr>
<td>6</td>
<td>90953.25</td>
</tr>
<tr>
<td>7</td>
<td>83676.99</td>
</tr>
<tr>
<td>8</td>
<td>76982.83</td>
</tr>
<tr>
<td>9</td>
<td>70824.20</td>
</tr>
</tbody>
</table>

Example 2

Write an equation that represents the function in this table.

<table>
<thead>
<tr>
<th>Week</th>
<th>Weight of Bacterial Culture (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>756.00</td>
</tr>
<tr>
<td>2</td>
<td>793.80</td>
</tr>
<tr>
<td>3</td>
<td>833.49</td>
</tr>
</tbody>
</table>

The exponential function will have the form $y = ab^x$, where $y$ is the weight of the bacterial culture, and $x$ is the number of weeks. The multiplier, $b$, for the weight of the bacterial culture is 1.05 (because $793.80 \div 756 = 1.05$ and $833.49 \div 793.80 = 1.05$, etc.). The starting point, $a$ is not given because we are not given the weight at Week 0. However, since the growth is 1.05 every week, we know that $(1.05) \cdot (weight \ at \ Week \ 0) = 756.00g$. The weight at Week 0 is 720g, thus $a = 720$. We can now write the equation:

$$y = 720 \cdot 1.05^x,$$

where $y$ is the weight of the bacterial culture (g), and $x$ is the time (weeks).
Example 3

LuAnn has $500 to open a savings account. She can open an account at Fredrico’s Bank, which pays 7% interest, compounded monthly, or Money First Bank, which pays 7.25%, compounded quarterly. LuAnn plans to leave the money in the account, untouched, for ten years. In which account should she place the money? Justify your answer.

Solution: The obvious answer is that she should put the money in the account that will pay her the most interest over the ten years, but which bank is that? At both banks the principal (the initial value) is $500. Fredrico’s Bank pays 7% compounded monthly, which means the interest rate is $0.07 \div 12 \approx 0.00583$ each month. If LuAnn puts her money into Fredrico’s Bank, after one month she will have:

$$500 + 500(0.00583) = 500(1.00583) \approx 502.92.$$  

To calculate the amount at the end of the second month, we must multiply by 1.00583 again, making the amount:

$$500(1.00583)^2 \approx 505.85.$$  

At the end of three months, the balance is:

$$500(1.00583)^3 \approx 508.80.$$  

This will happen every month for ten years, which is 120 months. At the end of 120 months, the balance will be:

$$500(1.00583)^{120} \approx 1004.43.$$  

Note that this last equation is an exponential function in the form $y = ab^x$, where $y$ is the amount of money in the account and $x$ is the number of months (in this case, 120 months). $a = 500$ is the starting value (at 0 months), and $b = 1.00583$ is the multiplier or growth rate for the account each month.

A similar calculation is performed for Money First Bank. Its interest rate is higher, 7.25%, but it is only calculated and compounded quarterly. (Quarterly means four times each year, or every three months.) Hence, every quarter the bank calculates $0.0725 \div 4 = 0.018125$ interest. At the end of the first quarter, LuAnn would have:

$$500(1.018125) \approx 509.06.$$  

At the end of ten years (40 quarters) LuAnn would have:

$$500(1.018125)^{40} \approx 1025.69.$$  

Note that this last equation is an exponential function in the form $y = ab^x$, where $y$ is the amount of money in the account and $x$ is the number of quarters (in this case, 40 quarters). $a = 500$ is the starting value (at 0 quarters), and $b = 1.018125$ is the multiplier or growth rate for the account each quarter.

Since Money First Bank would pay her approximately $21 more in interest than Fredrico’s Bank, she should put her money in Money First Bank.
Problems

1. In seven years, Seta’s son Stu is leaving home for college. Seta hopes to save $8000 to help pay for his first year. She has $5000 now and has found a bank that pays 7.75% interest, compounded daily. At this rate, will she have the money she needs for Stu’s first year of college? If not, how much more does she need?

2. Eight years ago, Rudi thought that he was making a sound investment by buying $1000 worth of Pro Sports Management stock. Unfortunately, his investment depreciated steadily, losing 15% of its value each year. How much is the stock worth now? Justify your answer.

3. Based on each table below, write the equation of the exponential function \( y = ab^x \).
   a.  
   \[
   \begin{array}{c|c}
   x & f(x) \\
   \hline
   0 & 1600 \\
   1 & 2000 \\
   2 & 2500 \\
   3 & 3125 \\
   \end{array}
   \]
   b.  
   \[
   \begin{array}{c|c}
   x & y \\
   \hline
   1 & 40 \\
   2 & 32 \\
   3 & 25.6 \\
   \end{array}
   \]

4. The new Bamo Super Ball has a rebound ratio of 0.97. If you dropped the ball from a height of 125 feet, how high will it bounce on the tenth bounce?

5. Based on each graph below, write the equation of the exponential function \( y = ab^x \).
   a.  
   ![Graph 1]
   b.  
   ![Graph 2]

6. Fredrico’s Bank will let you decide how often your interest will be computed, but with certain restrictions. If your interest is compounded yearly you can earn 8%. If your interest is compounded quarterly, you earn 7.875%. Monthly compounding earns a 7.75% interest rate, while weekly compounding earns a 7.625% interest rate. If your interest is compounded daily, you earn 7.5%. What is the best deal? Justify your answer.

7. Fully investigate the graph of the function \( y = \left( \frac{3}{4} \right)^x + 4 \). See Describing Functions (Lessons 1.1.2 and 1.1.3) in this Parent Guide with Extra Practice for information on how to fully describe the graph of a function.
Answers

1. Yes, she will have about $8601.02 by then. The daily rate is \(\frac{0.0775}{365} \approx 0.000212329\). Seven years is 2555 days, so we have \(\$5000(1.000212329)^{2555} \approx \$8601.02\).

2. The multiplier is 100% – 15% = 85%, or 0.85. \(1000(0.85)^8 \approx 272.49\) so Rudi’s investment is now only worth about $272.49.

3. a. \(y = 1600(1.25)^x\) \hspace{1cm} b. \(y = 50(0.8)^x\)

4. \(125(0.97)^{10} \approx 92.178\) or about 92 feet

5. a. \(y = 3\left(\frac{5}{3}\right)^x\) \hspace{1cm} b. \(y = 40\left(\frac{40}{120}\right)^x = 40\left(\frac{1}{3}\right)^x\)

6. The best way to do this problem is to choose any amount, and see how it grows over the course of one year. Taking $100, after one year, 8% compounded yearly will yield $108. 7.875% compounded quarterly yields $108.11. 7.75% compounded monthly yields $108.03. 7.625% compounded weekly yields $107.91. 7.5% compounded daily yields $107.79. Quarterly is the best.

7. This is a function that it is continuous and nonlinear (curved). It has a \(y\)-intercept of (0, 5), and no \(x\)-intercepts. The domain is all real values of \(x\), and the range is all real values of \(y > 4\). This function has a horizontal asymptote of \(y = 4\), and no vertical asymptotes. It is an exponential function.
CURVE FITTING  8.2.1 – 8.2.2

Students write equations of exponential functions the form $y = ab^x$ that pass through two given points. (Equations of this form have an asymptote at $y = 0$.)

For additional information, see the Math Notes box in Lesson 8.2.2.

Example 1

Write an equation for an exponential function that passes through the points $(0, 8)$ and $(4, \frac{1}{2})$.

Solution: Substitute the $x$- and $y$-coordinates of each pair of points into the general equation. Then solve the resulting system of two equations to determine $a$ and $b$.

$y = ab^x$

Using $(x, y) = (8, 0)$:  
$8 = ab^0$  
Since $b^0 = 1$, we know $a = 8$.

Using $(x, y) = (4, \frac{1}{2})$:  
$\frac{1}{2} = ab^4$

Substitute $a = 8$ from the first equation into $\frac{1}{2} = ab^4$ from the second equation and solve for $b$.

$\frac{1}{2} = 8b^4$

$\frac{1}{16} = b^4$

$\frac{1}{4} = \sqrt[4]{b^4}$

Since $a$ and $b$ have been determined, we can now write the equation $y = 8\left(\frac{1}{2}\right)^x$. 
Example 2

In the year 2000, Club Leopard was first introduced on the Internet. In 2004, it had 14,867 “leopards” (members). In 2007, the leopard population had risen to 22,610. Model this data with an exponential function and use the model to predict the leopard population in the year 2012.

Solution: We can call the year 2000 our time zero, or \( x = 0 \). Then 2004 is \( x = 4 \), and the year 2007 will be \( x = 7 \). This gives us two data points, \((4, 14867)\) and \((7, 22610)\).

To model with an exponential function we will use the equation \( y = ab^x \) and substitute both coordinate pairs to obtain a system of two equations.

Preparing to use the Equal Values Method to solve the system of equations, we rewrite both equations in “\( a = \)” form:

\[
\begin{align*}
14867 &= ab^4 \\
22610 &= ab^7
\end{align*}
\]

Then by the Equal Values Method,

\[
\frac{14867}{b^4} = \frac{22610}{b^7}
\]

From the equations above,

\[
a = \frac{14867}{b^4}
\]

Since \( b \approx 1.15 \),

\[
a = \frac{14867}{(1.15)^4}
\]

\[
a \approx 8500
\]

Since \( a \approx 8500 \) and \( b \approx 1.15 \) we can write the equation: \( y = 8500 \cdot 1.15^x \), where \( y \) represents the number of members, and \( x \) represents the number of years since 2000.

We will use the equation with \( x = 12 \) to predict the population in 2012.

\[
y = 8500(1.15)^{12}
\]

\[
y \approx 45477
\]

Assuming the trend continues to the year 2012 as it has in the past, we predict the population in 2012 to be 45,477.
Problems

For each of the following pairs of points, write the equation of an exponential function with an asymptote $y = 0$ that passes through them.

1. $(0, 6)$ and $(3, 48)$
2. $(1, 21)$ and $(2, 147)$
3. $(-1, 72.73)$ and $(3, 106.48)$
4. $(-2, 351.5625)$ and $(3, 115.2)$
5. On a cold wintry day the temperature outside hovered at 0°C. Karen made herself a cup of cocoa, and took it outside where she would be chopping some wood. However, she decided to conduct a mini science experiment instead of drinking her cocoa, so she placed a thermometer in the cocoa and left it sitting next to her as she worked. She wrote down the time and the reading on the thermometer as shown in the table below.

<table>
<thead>
<tr>
<th>Time since 1st reading</th>
<th>5</th>
<th>10</th>
<th>12</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temp (°C)</td>
<td>24.41°</td>
<td>8.51°</td>
<td>5.58°</td>
<td>2.97°</td>
</tr>
</tbody>
</table>

Write the equation of an exponential function of the form $y = ab^x$ that models this data.

Answers

1. $y = 6(2)^x$
2. $y = 3(7)^x$
3. $y = 80(1.1)^x$
4. $y = 225(0.8)^x$
5. Answers will vary, but should be close to $y = 70(0.81)^x$. 
SOLVING ONE-VARIABLE INEQUALITIES

To solve an inequality in one variable, first change it to an equation (a mathematical sentence with an “=” sign) and then solve. Place the solution, called a “boundary point”, on a number line. This point separates the number line into two regions. The boundary point is included in the solution for situations that involve ≥ or ≤, and excluded from situations that involve strictly > or <. On the number line boundary points that are included in the solutions are shown with a solid filled-in circle and excluded solutions are shown with an open circle. Next, choose a number from within each region separated by the boundary point, and check if the number is true or false in the original inequality. If it is true, then every number in that region is a solution to the inequality. If it is false, then no number in that region is a solution to the inequality.

For additional information, see the Math Notes boxes in Lessons 9.1.1 and 9.1.3.

Example 1

Solve: \(3x - (x + 2) \geq 0\)

Change to an equation and solve.

Place the solution (boundary point) on the number line. Because \(x = 1\) is also a solution to the inequality (≥), we use a filled-in dot.

Test a number on each side of the boundary point in the original inequality. Highlight the region containing numbers that make the inequality true.

The solution is \(x \geq 1\).

Example 2

Solve: \(-x + 6 > x + 2\)

Change to an equation and solve.

Place the solution (boundary point) on the number line. Because \(x = 2\) is not a solution to the original inequality (>), we use an open dot.

Test a number on each side of the boundary point in the original inequality. Highlight the region containing numbers that make the inequality true.

The solution is \(x < 2\).
Problems

Solve each inequality.

1. \(4x - 1 \geq 7\)  
2. \(2(x - 5) \leq 8\)  
3. \(3 - 2x < x + 6\)  
4. \(\frac{1}{2}x > 5\)  
5. \(3(x + 4) > 12\)  
6. \(2x - 7 \leq 5 - 4x\)  
7. \(3x + 2 < 11\)  
8. \(4(x - 6) \geq 20\)  
9. \(\frac{1}{4}x < 2\)  
10. \(12 - 3x > 2x + 1\)  
11. \(\frac{x - 5}{7} \leq -3\)  
12. \(3(5 - x) \geq 7x - 1\)  
13. \(3y - (2y + 2) \leq 7\)  
14. \(\frac{m + 2}{5} < \frac{2m}{3}\)  
15. \(\frac{m - 2}{3} \geq \frac{2m + 1}{7}\)

Answers

1. \(x \geq 2\)  
2. \(x \leq 9\)  
3. \(x > -1\)  
4. \(x > 10\)  
5. \(x > 0\)  
6. \(x \leq 2\)  
7. \(x < 3\)  
8. \(x \geq 11\)  
9. \(x < 8\)  
10. \(x < \frac{11}{5}\)  
11. \(x \leq -16\)  
12. \(x \leq 1.6\)  
13. \(y \leq 9\)  
14. \(m > \frac{6}{7}\)  
15. \(m \geq 17\)
SOLVING ABSOLUTE VALUE EQUATIONS AND INEQUALITIES  9.1.3

To solve an equation with absolute value, first break the problem into two equations since the quantity inside the absolute value can be positive or negative. Then solve each part separately. For additional information, see the Math Notes box in Lesson 9.2.2.

There are several methods for solving absolute value and other inequalities, but one method that works for all kinds of inequalities is to change the inequality to an equation, solve it, then place the solution(s) on a number line. The solution(s), called “boundary point(s),” divide the number line into regions. Check any point within each region in the original inequality. If that point makes the original inequality true, then all the points in that region are solutions. If that point makes the original inequality false, then none of the points in that region are solutions. The boundary points are included in (≥ or ≤) or excluded from (> or <) the solution depending on the inequality sign.

Solving an absolute value inequality is very similar to solving a linear inequality in one variable, except that there are often three solution regions on the number line instead of just two. For more on solving linear inequalities see the section “Solving One-Variable Inequalities (Lessons 9.1.1 and 9.1.2)” of this Parent Guide with Extra Practice.

Example 1

Solve $|2x + 3| = 11$

“Looking inside” the absolute value can solve this problem. (See “Multiple Methods for Solving Equations” (3.3.1 – 3.3.3) in this Parent Guide with Extra Practice.)

Since $|11| = |-11| = 11$, $2x + 3 = 11$ or $2x + 3 = -11$.

Solving both equations yields two answers: $2x + 3 = 11$ or $2x + 3 = -11$

$2x = 8$ or $2x = -14$

$x = 4$ or $x = -7$

Example 2

Solve $|2x - 3| = 7$

Separate into two equations. $2x - 3 = 7$ or $2x - 3 = -7$

Add 3. $2x = 10$ or $2x = -4$

Divide by 2. $x = 5$ or $x = -2$

Remember that you can always check your solutions in the original equation to make sure they are correct!
Example 3

Solve: \( |x - 3| \leq 5 \)

Change to an equation and solve.

\[ |x - 3| = 5 \]

\[ x - 3 = 5 \text{ or } x - 3 = -5 \]

\[ x = 8 \text{ or } x = -2 \]

Choose \( x = -3, x = 0, \) and \( x = 9 \) to test in the original inequality. \( x = -3 \) is false, \( x = 0 \) is true, and \( x = 9 \) is false.

The solution is all numbers greater than or equal to \(-2\) and less than or equal to \(8\), written as \(-2 \leq x \leq 8\).

Example 4

Solve: \( 3|2y + 1| + 1 > 28 \)

Change to an equation and solve.

\[ 3|2y + 1| + 1 = 28 \]

Get the absolute value by itself by subtracting 1 and then dividing by 3 on both sides. Then solve the absolute value equation.

\[ 2y + 1 = 9 \text{ or } 2y + 1 = -9 \]

\[ y = 4 \text{ or } y = -5 \]

Choose \( y = -6, y = 0, \) and \( y = 5 \) to test in the original inequality. \( y = -6 \) is true, \( y = 0 \) is false, and \( y = 5 \) is true.

The solution is all numbers less than \(-5\) or greater than \(4\), written as \(y < -5 \text{ or } y > 4\).

Problems

Solve each absolute value equation.

1. \( |x - 2| = 5 \)  
2. \( |3x + 2| = 11 \)  
3. \( |5 - x| = 9 \)  
4. \( |3 - 2x| = 7 \)  
5. \( |2x + 3| = -7 \)  
6. \( |4x + 1| = 10 \)

Solve each absolute value inequality.

7. \( |x + 4| \geq 7 \)  
8. \( |x| - 5 \leq 8 \)  
9. \( |x - 5| \leq 8 \)  
10. \( |4r - 2| > 8 \)  
11. \( |3x| \leq 12 \)  
12. \( |1 - 3x| \leq 13 \)  
13. \( |2x - 3| > 15 \)  
14. \( |5x| > -15 \)  
15. \( -2|x - 3| + 6 < -4 \)  
16. \( |4 - d| \leq 7 \)  
17. \( |x - 4| \leq 0 \)  
18. \( |2x + 1| - 2 < -3 \)
Answers

1. \( x = 7 \) or \(-3\)
2. \( x = 3 \) or \(-\frac{13}{3}\)
3. \( x = -4 \) or \(14\)
4. \( x = -2 \) or \(5\)
5. no solution
6. \( x = \frac{9}{4} \) or \(-\frac{11}{4}\)
7. \( x \geq 3 \) or \(x \leq -11\)
8. \(-13 \leq x \leq 13\)
9. \(-3 \leq x \leq 13\)
10. \( r < -\frac{3}{2} \) or \( r > \frac{5}{2} \)
11. \(-4 \leq x \leq 4\)
12. \(-4 < x < \frac{14}{3}\)
13. \( x < -6 \) or \(x > 9\)
14. all real numbers
15. \( x > 8 \) or \(x < -2\)
16. \(-3 \leq d \leq 11\)
17. \( x = 4\)
18. no solution
To graph the solutions to an inequality in two variables, first graph the corresponding equation. This graph is the boundary line (or curve), since all points that make the inequality true lie on one side or the other of the line. Before graphing the equation, decide whether the line or curve is part of the solution or not, that is, whether it is solid or dashed. If the inequality symbol is either \( \leq \) or \( \geq \), then the points on the boundary line are solutions to the inequality and the line must be solid. If the symbol is either < or >, then the points on the boundary line are not solutions to the inequality and the line is dashed.

Next, decide which side of the boundary line must be shaded to show the part of the graph that represents all \((x, y)\) coordinate pairs that make the inequality true. To do this, choose a point not on the boundary line. Substitute the \(x\)- and \(y\)-values of this point into the original inequality. If the inequality is true for the test point, then shade the region on the side of the boundary line that contains the test point. If the inequality is false for the test point, then shade the opposite region.

The shaded portion represents all of the \((x, y)\) coordinate pairs that are solutions to the original inequality.

Caution: If you need to rewrite the inequality in order to graph it, such as rewriting it in slope-intercept form, always use the original inequality to test a point, not the rearranged form.

For additional information, see the Math Notes box in Lesson 9.3.1.

**Example 1**

Graph the solutions to the inequality \( y > 3x - 2 \).

First, graph the line \( y = 3x - 2 \), but draw it dashed since > means the boundary line is not part of the solution. For example, the point \((0, -2)\) is on the boundary line, but it is not a solution to the inequality because \(-2 \not> 3(0) - 2\) or \(-2 \not> -2\).

Next, test a point that is not on the boundary line. For this example, use the point \((-2, 4)\).

\[ 4 > 3(-2) - 2, \text{ so } 4 > -8 \text{ which is a true statement.} \]

Since the inequality is true for this test point, shade the region containing the point \((-2, 4)\). All of the coordinate pairs that are solutions lie in the shaded region.
Example 2

Graph the solutions to the inequality \( y \leq 2^x - 6 \).

First, graph the exponential function \( y = 2^x - 6 \) and draw it as a solid curve, since \( \leq \) means that the points on the boundary curve are solutions to the inequality. For example, the point \((0, -5)\) is on the boundary curve. It is a solution to the inequality because \(-5 \leq 2^0 - 6\) or \(-5 \leq 1 - 6\).

Next, test a point not on the boundary curve. For this example use the point \((2, 2)\).
\[ 2 \leq 2^2 - 6 \text{, so } 2 \leq -2 \text{ which is a false statement.} \]

Since the inequality is false for this test point, shade below the region that does not contain this point. All of the coordinate pairs that are solutions lie in the shaded region.

Problems

Graph the solutions to each of the following inequalities on a separate set of axes.

1. \( y \leq 3x + 1 \)  
2. \( y \geq -2x + 3 \)  
3. \( y > 4x - 2 \)  
4. \( y < -3x - 5 \)  
5. \( y \leq 3 \)  
6. \( x > 1 \)  
7. \( y > \frac{2}{3}x + 8 \)  
8. \( y < -\frac{3}{2}x - 7 \)  
9. \( 3x + 2y \geq 7 \)  
10. \( -4x + 2y < 3 \)  
11. \( y \leq 2^x \)  
12. \( y > 2^x - 3 \)  
13. \( y \geq \left(\frac{1}{2}\right)^x - 2 \)  
14. \( y < 4\left(\frac{1}{2}\right)^x \)  
15. \( y \leq -(2)^x \)

Answers

1.  
2.  
3.  

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Chapter 9

4. 

5. 

6. 

7. 

8. 

9. 

10. 

11. 

12. 

13. 

14. 

15.
To graph the solutions to a system of inequalities, follow the same procedure outlined in the previous section but do it twice—once for each inequality. The solution to the system of inequalities is the overlap of the shading from the individual inequalities. When two boundary lines are graphed, there are often four regions. The region containing the coordinate pairs that make both of the inequalities true is the solution region.

Example 1

Graph the solutions to the system:

\[ y \leq \frac{1}{2}x + 2 \]
\[ y > -\frac{2}{3}x + 1 \]

Graph the lines \( y = \frac{1}{2}x + 2 \) and \( y = -\frac{2}{3}x + 1 \).
The first is solid and the second is dashed.

Test a point in the first inequality. For this example, \((-4, 5)\).

\[ 5 \leq \frac{1}{2}(-4) + 2 \quad \text{or} \quad 5 \leq 0 \]
This inequality is false, so shade the region of the first boundary line that does not contain the point \((-4, 5)\).

Test a point in the second inequality. For this example \((0, 0)\).

\[ 0 > -\frac{2}{3}(0) + 1 \quad \text{or} \quad 0 > 1 \]
This inequality is false, so shade the region of the second boundary line that does not contain the point \((0, 0)\).

The coordinate pair solutions are represented by the overlap of the two shaded regions shown by the darkest shading in the graph at right.
Example 2

Graph the solutions to the system: \( y \leq -x + 5 \)
\[ y \geq 2^x - 1 \]

Graph the line \( y = -x + 5 \) and the curve \( y = 2^x - 1 \) with a solid line and curve.

Test the point \((0, -4)\) in the first inequality.

\[ 0 \leq -(0) + 5 \quad \text{or} \quad 0 \leq 1 \]

This inequality is true, so shade the region containing the point \((0, -4)\).

Test the point \((0, 3)\) in the second inequality.

\[ 3 \geq 2^0 - 1 \quad \text{or} \quad 3 \geq 0 \]

This inequality is true, so shade the region containing the point \((0, 3)\).

The coordinate pair solutions are represented by the overlap of the two shaded regions shown by the darkest shading in the graph at right.

Problems

Graph the solutions to each of the following pairs of inequalities on the same set of axes.

1. \( y > 3x - 4 \)
   \[ y \leq -2x + 5 \]
2. \( y \geq -3x - 6 \)
   \[ y > 4x - 4 \]
3. \( y < -\frac{3}{2}x + 4 \)
   \[ y < \frac{1}{3}x + 3 \]
4. \( y < -\frac{3}{4}x - 1 \)
   \[ y > \frac{4}{3}x + 1 \]
5. \( y < 3 \)
   \[ y > \frac{1}{2}x + 2 \]
6. \( x \leq 3 \)
   \[ y < \frac{3}{4}x - 4 \]
7. \( y \leq 2x + 1 \)
   \[ y \geq 2^x - 4 \]
8. \( y < -x + 5 \)
   \[ y \geq 2^x + 1 \]
9. \( y < 2^x + 3 \)
   \[ y \geq \left(\frac{1}{2}\right)^x \]
Answers

1. 

2. 

3. 

4. 

5. 

6. 

7. 

8. 

9.
ASSOCIATION IN A TWO-WAY TABLE

Data based on measurements such as height, speed, and temperature is *numerical*. In Chapter 4 you described associations between two numerical variables and graphed data on scatterplots. Data can also be counts or percentages in categories such as gender, religion, or blood type. This kind of non-numerical data is called categorical. Categorical data cannot be graphed on a scatterplot, but there are other techniques to look for a relationship (an association) between categorical variables including Venn diagrams, bar charts, and two-way tables.

For additional information see the Math Notes box in Lesson 10.1.1.

**Example 1**

Is there an association between a voter’s political party and whether the voter supported the ballot proposition or not? The following data was collected:

<table>
<thead>
<tr>
<th></th>
<th>Republican</th>
<th>Democrat</th>
</tr>
</thead>
<tbody>
<tr>
<td>supported proposition</td>
<td>234</td>
<td>286</td>
</tr>
<tr>
<td>did not support proposition</td>
<td>162</td>
<td>198</td>
</tr>
<tr>
<td>undecided</td>
<td>54</td>
<td>66</td>
</tr>
</tbody>
</table>

Solution:

Determine the row and column totals:

<table>
<thead>
<tr>
<th></th>
<th>Republican</th>
<th>Democrat</th>
</tr>
</thead>
<tbody>
<tr>
<td>supported proposition</td>
<td>234</td>
<td>286</td>
</tr>
<tr>
<td>did not support proposition</td>
<td>162</td>
<td>198</td>
</tr>
<tr>
<td>undecided</td>
<td>54</td>
<td>66</td>
</tr>
<tr>
<td></td>
<td>450</td>
<td>550</td>
</tr>
</tbody>
</table>

Now calculate the percentages down the columns. Make a two-way relative frequency table:

<table>
<thead>
<tr>
<th></th>
<th>Republican</th>
<th>Democrat</th>
</tr>
</thead>
<tbody>
<tr>
<td>supported proposition</td>
<td>$\frac{234}{450} = 52%$</td>
<td>$\frac{286}{550} = 52%$</td>
</tr>
<tr>
<td>did not support proposition</td>
<td>$\frac{162}{450} = 36%$</td>
<td>$\frac{198}{550} = 36%$</td>
</tr>
<tr>
<td>undecided</td>
<td>$\frac{54}{450} = 12%$</td>
<td>$\frac{66}{550} = 12%$</td>
</tr>
<tr>
<td></td>
<td>100%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Compare the percentages across the rows. They are all the same, indicating that there is no association between political party and preference for the ballot proposition. 52% of voters supported the ballot proposition regardless of whether they were a Democrat or Republican.
Example 2

A survey of 155 recent high school graduates found that 130 had driver’s licenses and 58 had jobs. Twenty-one said they had neither a driver’s license nor a job. Is there an association between having a driver’s license and a job among the recent graduates?

Solution:

Make a two-way table to display the counts (the number of graduates) in each category.

<table>
<thead>
<tr>
<th></th>
<th>Job</th>
<th>No Job</th>
</tr>
</thead>
<tbody>
<tr>
<td>License</td>
<td>54</td>
<td>76</td>
</tr>
<tr>
<td>No Lic.</td>
<td>4</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>58</td>
<td>97</td>
</tr>
</tbody>
</table>

Add or subtract to fill in the remaining counts.

<table>
<thead>
<tr>
<th></th>
<th>Job</th>
<th>No Job</th>
</tr>
</thead>
<tbody>
<tr>
<td>License</td>
<td>130</td>
<td></td>
</tr>
<tr>
<td>No Lic.</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td></td>
<td>155</td>
<td></td>
</tr>
</tbody>
</table>

Investigate: Of the 58 graduates who have jobs, find the percentages of those who have, and do not have, a driver’s license. Then, of the 97 graduates who do not have jobs, find the percentages of those who have, and do not have, a driver’s license. Fill in the table at right.

<table>
<thead>
<tr>
<th></th>
<th>Job</th>
<th>No Job</th>
</tr>
</thead>
<tbody>
<tr>
<td>License</td>
<td>93%</td>
<td>78%</td>
</tr>
<tr>
<td>No Lic.</td>
<td>7%</td>
<td>22%</td>
</tr>
<tr>
<td></td>
<td>100%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Compare the percentages across the rows. In this case, the percentages across rows are quite different, indicating an association between having a job and a driver’s license. People with jobs tend have a higher percentage of licenses (93%) that those without a job (78%). There is an association between being employed and having a driver’s license.

Problems

1. In a survey of 200 pet owners, 76 claimed to not own a cat and 63 indicated they did not own a dog. Eighty-two responded that they own both a cat and a dog. Make a two-way table displaying the counts in each category. Is there an association between cat and dog ownership?

2. Is there an association between Parker’s pitching style and Brandon’s chance of getting a base hit when batting in softball games? Brandon batted 32 times when Parker was pitching. Brandon got a base hit eight times. Parker used his fastball 12 times, but Brandon got only three base hits when Parker used his fastball. Make a two-way table and decide whether there is an association between pitching style and getting a base hit.
Answers

1. There appears to be a weak association between cat and dog ownership. The probability of owning a cat is slightly lower if you own a dog. Only 60% of dog owners have a cat, while 67% of non-dog owners have a cat. That is, you are more likely to own a cat if you are not a dog owner. So there is an association. The association is weak because there is not a big difference.

Alternatively, it is also possible to set up the table as follows. The conclusion that there is a weak association is the same. You are less likely to own a dog if you own a cat.

2. There is no association between Parker’s pitch and Brandon’s ability to get a base hit. The probability Brandon gets a base hit is 25%, regardless of whether Parker uses his fastball pitch or not.

Alternatively, it is also possible to set up the table as follows. The conclusion is the same: there is no association. Parker used a fastball 37.5% of the time, regardless of whether Brandon got a base hit or not.
Data sets can be represented graphically using histograms and boxplots. For assistance with creating histograms and boxplots, see the Parent Guide with Extra Practice for CPM Core Connections Courses 1, 2, and 3, which is available at cpm.org.

Two distributions of data can be compared by examining their centers, shapes, spreads, and outliers. A distribution of a set of data is an organized method of showing all the possible values of the data set and how often they occur. Histograms and boxplots are examples of data distributions.

The center, or “typical” value, of a data distribution can be described by the median. If the distribution is symmetric and has no outliers, the mean can be used to describe the center. An outlier is a data value that is far away from the bulk of the data. Outliers shift the mean towards the outlier, making the data set skewed and the mean less representative of a typical value.

Common shapes of data distributions can be found in the Math Notes box in Lesson 10.1.3.

The spread of a distribution can be described using the interquartile range (IQR), described in the Math Notes boxes in Lesson 10.1.2, or the standard deviation, described in the Math Notes box in Lesson 10.1.4. Since the standard deviation is based upon the mean, it should be used only to describe the spread of distributions that are symmetric and without outliers.

Example 1

University professors are complaining that the English Literature classes at community colleges are not demanding enough. Specifically, the university professors claim that community college literature courses are not assigning enough novels to read. A community college statistics student collected the following data from 42 universities and community college literature courses in the state. Compare the number of novels read in the two types of colleges.

Number of novels assigned in community college literature courses:
13, 10, 15, 12, 14, 9, 11, 15, 12, 14, 9, 10, 13, 15, 12, 9, 11, 15, 12, 10, 15, 14  checksum 270

Number of novels assigned in university literature courses:
11, 8, 14, 13, 25, 11, 7, 13, 8, 16, 11, 10, 20, 7, 8, 13, 14, 16, 18, 10  checksum 253

Solution follows on next page →
Solution:

Any analysis of data distributions should begin with a graphical representation of the data. A bin width of two was chosen for the histograms used here. So that the distributions can be compared, both graphs have the same scale on the x-axis, and the graphs are stacked on top of each other. For assistance with using a TI-83+/84+ calculator, see your eBook or studenthelp.cpm.org.

Note: The checksum is used to verify that data has been entered into the graphing technology correctly. The sum of the data set, as determined by the statistical functions of the calculator, should match the given checksum value.

When comparing the distributions, the center, shape, spread, and outliers should be considered. Since neither of the distributions is symmetric and one of the distributions has an outlier, it would not be appropriate to use the means or standard deviations to compare their centers and spreads. The five number summaries (see the Math Notes box in Lesson 10.1.2) are shown to the right of each graph.

**Center:** Both types of colleges assign the same median of 12 novels.

**Shape:** The distribution for community colleges is skewed to the left, with a low of 8 to 9 novels and increasing to a peak at 14 to 15 novels. The distribution for universities is skewed to the right, with a peak at 10 to 11 novels.

**Spread:** The variability in the number of novels assigned at the community college level is much less than the variability between courses at the university level. The IQR for community colleges is 4 novels (14 – 10 = 4), while the university IQR of 6 (15 – 9 = 6) is one-and-a-half times as wide.

**Outliers:** One course at a university is an outlier; 25 books are assigned in that course. 25 books is far away from the bulk of novels assigned in university courses. The TI-83/84+ calculator can mark an outlier on a boxplot with dots.

**Conclusions:** The university professors claim that their courses are more demanding because they assign more novels. However, that data does not bear this claim out. 25% of university courses assign more novels than any of the community college courses (the right “whisker,” or the top 25% of the courses, for universities is beyond the entire boxplot for community colleges). But just as dramatically, 25% of the university classes assign fewer books than any of the community colleges (the left “whisker,” or lowest 25%, for universities is below the entire boxplot for community colleges). Furthermore, the median number of novels assigned at the two universities is the same—12 books. Community colleges are more consistent from course-to-course in the number of novels they assign (IQR is 4) than are the universities (IQR is 6).
Example 2

A rabbit breeder kept track of the number of offspring from five does (female rabbits) this year. The does had: 243, 215, 184, 280, and 148 kits (baby rabbits) respectively. Show how to calculate the mean and standard deviation of the number of kits per doe without using the statistical functions of a calculator.

The mean is \( \frac{243+215+184+280+148}{5} = 214 \) kits.

The standard deviation is the square root of the average of the distances to the mean, after the distances have been made positive by squaring. To find the standard deviation, first find the distance each doe is from the mean:

\[
243 - 214 = 29, 
215 - 214 = 1, 
184 - 214 = -30, 
280 - 214 = 66, 
148 - 214 = -66
\]

Find each of the distances squared:

\[
29^2 = 841, 
1^2 = 1, 
(-30)^2 = 900, 
66^2 = 4356, 
(-66)^2 = 4356
\]

The mean distance-squared is:

\[
\frac{841+1+900+4356+4356}{5} = 2090.8
\]

The square root is 45.725. Since the precision of the original measurements was an integer, the final result should also be an integer. The mean number of kits per doe is 214 with a standard deviation of 46 kits.

Problems

1. Different types of toads tend to lay different numbers of eggs. The following data was collected from two different species. Compare the number of eggs laid by American toads to the number laid by Fowler toads. Is it appropriate to summarize the distributions by using mean and standard deviation? Use a bin width of 250 eggs.

American toads: 9100, 8700, 10300, 9500, 7800, 8900, 9200, 9300, 8800, 9400, 8000, 9000, 8400, 9700, 10000, 8600, 8900, 9900, 9300

Checksum 181,200.

Fowler Toads: 9500, 9100, 9400, 8800, 9000, 8400, 9200, 9000, 9100, 8600, 9200, 8700, 9800, 9300, 8800, 9200, 9300, 9000, 9100

Checksum 181,600.

2. Without using the statistical functions on your calculator, find the standard deviation of the number of eggs laid by each of the first five American toads.

3. Do low birth weight babies start crawling at a later age than babies born at an average weight? A psychologist collected the following data for the age at which children started crawling:

Low weight babies: 10, 12, 11, 11, 7, 13, 10, 12, 11, 13, 10, 11, 15, 11, 14, 10 months

Checksum 181.

Average weight babies: 7, 6, 13, 9, 8, 7, 5, 7, 9, 8, 10, 8, 11, 7, 7, 10, 6, 8, 7, 6, 12, 8, 7 months

Checksum 186.
4. Compare the amount of time the flavor lasted for people chewing brand “10” chewing gum to the amount of time the flavor lasted in “Strident” chewing gum. See the graph at right. Estimate the mean for each type of gum.

Answers

1. Mean and standard deviation are appropriate statistics because both distributions are fairly symmetric with no outliers.

Both types of toads lay a mean of between 9000 and 9100 eggs. Both distributions are single-peaked and symmetric with no apparent outliers. However there is much greater variability in the number of eggs that American toads lay. The standard deviation for American toads is about 637 while the standard deviation for Fowler toads is only half as much, about 316 eggs.

$$2 \sqrt{\frac{20^2 + (-380)^2 + 1220^2 + 420^2 + (-1280)^2}{5}} \approx 830 \text{ eggs}$$
3. The median and IQR will be used to compare statistics since mean and standard deviation are not appropriate—both distributions are skewed and one has an outlier.

The median age at which low-weight babies start crawling is 11 months, while the median age for average-weight babies is 8 months.

Both distributions are single-peaked and skewed. The low-weight babies appear to have an outlier at 7 months, although the calculator does not identify it as a true outlier. The average-weight babies have an outlier at 13 months.

The variability in the crawling age is roughly the same for low-weight babies (IQR is 2.5 months) as for average-weight babies (IQR is 2 months).

Low-weight babies have their development delayed by about 3 months. About 75% of low-weight babies have not started crawling at an age when 75% of average-weight babies are already crawling.

4. The median for both types of gum was about 18 minutes of flavor time. The times for “10” were skewed, while the times for Strident were symmetric. The lower half of the distributions for both gums was the same. But there was much more variability in the upper half of people chewing “10” than in the upper half of Strident. Indeed, more than 25% of “10” chewers reported flavor lasting longer than any of the Strident chewers. Neither gum had outliers in flavor time.

There was more variability in flavor time for “10”—the IQR was about 9 minutes \((25 - 16 = 9)\). The IQR of 4 minutes \(20 - 16 = 4\) for Strident was less than half that of “10”. That variability is an advantage. If you chew “10,” you will probably be no worse off than chewing Strident, and you could have much longer flavor.

The distribution is symmetric, so the mean for Strident is about the same as the median—about 18 minutes. But the mean for “10” is greater than 18 minutes due to the skew in the shape—maybe 22 minutes or so.
In Lesson 10.2.1, students transform functions by adding “k” to a given function \( f(x) \). If the original function is \( f(x) \), then the transformed function \( f(x) + k \) is a vertical shift of the original function by \( k \) units. If \( k \) is positive, then the function is shifted upward. If \( k \) is negative, then the function is shifted downward.

Example 1

\( f(x) = 2x \)  \hspace{1cm} \text{Graph} \ f(x) + 3.

Start by making a table for \( f(x) \).

Add a column to the table for \( f(x) + 3 \). Note that \( f(x) \) can be thought of as \( y \), so when completing the table for \( f(x) + 3 \), think of it as \( y + 3 \).

Graph \( f(x) \) and \( f(x) + 3 \) on the same set of axes.

In the graph at right, \( f(x) \) is graphed with a dashed line and \( f(x) + 3 \) is graphed with a solid line.

Example 2

\( f(x) = \left(\frac{1}{2}\right)^x \)  \hspace{1cm} \text{Graph} \ f(x) - 4.

Make a table similar to the one made in the previous example.

Graph \( f(x) \) and \( f(x) - 4 \) on the same set of axes.

In the graph at right, \( f(x) \) is graphed with a dashed curve and \( f(x) - 4 \) is graphed with a solid curve.
Problems

Graph the original function and the transformed function on the same set of axes.

1. \( f(x) = -2x \)  
2. \( f(x) = \frac{3}{4}x \)  
3. \( f(x) = 2^x \)  
4. \( f(x) = \left(\frac{2}{3}\right)^x \)  

\( f(x) + 3 \)  
\( f(x) - 1 \)  
\( f(x) + 1 \)  
\( f(x) - 5 \)

Explain how each transformation is different from \( f(x) \).

5. \( f(x) - 8 \)  
6. \( f(x) - 50 \)  
7. \( f(x) + 31 \)  
8. \( f(x) + 36 \)

Given the graph of \( f(x) \) at right, graph each transformation.

9. \( f(x) - 4 \)  
10. \( f(x) + 2 \)

Answers

In answer graphs for problems 1 through 4, the original function is graphed using a dotted line or curve and the transformed function is graphed using a solid line or curve.

5. Shifted down 8 units.  
6. Shifted down 50 units.  
7. Shifted up 31 units.  
8. Shifted up 36 units.

9 and 10.
ARITHMETIC OPERATIONS WITH FUNCTIONS 10.2.2

In Lesson 10.2.2, students combine functions using addition and subtraction. This is done in the context of different situations.

**Example 1**

Thomas has $750 saved and earns $228 each day at work, all of which goes into his bank account. Thomas spends $133 each week.

Write one equation $B(x)$ for the amount of money Thomas has in his bank account after $x$ weeks. Write another equation $S(x)$ for the amount of money Thomas has spent after $x$ weeks. Then combine the equations to create a function for the amount of money Thomas has left after $x$ weeks.

Solution: $B(x) = 228x + 750$  
$S(x) = 133x$

Since Thomas is saving money in the bank, but then spending it, we need to subtract the amount he spends from the amount he has in the bank.

$B(x) – S(x) = (228x + 750) – (133x)$  or  $B(x) – S(x) = 95x + 750$

**Example 2**

Javier and Earnest are raking leaves for a neighbor with a large yard and lots of trees. Javier can rake 2.5 bags of leaves per hour and at noon has raked 7 bags of leaves. Earnest can rake 3 bags of leaves per hour, but he arrived late this morning and only has 2 bags of leaves raked at noon.

Write one equation $J(x)$ for the number of bags of leaves Javier has raked $x$ hours after noon. Write another equation $E(x)$ for the number of bags of leaves Earnest has raked $x$ hours after noon. Then combine the equations to create a function for the total number of bags of leaves they have raked together $x$ hours after noon.

Solution: $J(x) = 2.5x + 7$  
$E(x) = 3x + 2$

Since Javier and Earnest are working together, we need to add their equations to write a function for the total number of bags of leaves rakes.

$J(x) + E(x) = (2.5x + 7) + (3x + 2)$  or  $J(x) + E(x) = 5.5x + 9$
Problems

1. Rex owns a business. It currently costs him $2000 per month to operate his business, but the operational costs are increasing by $10 each month. Rex’s business currently makes a revenue of $3000 per month and the revenue is increasing by $35 each month.

Write one equation for the amount it costs Rex to run his business, another equation for the revenue that his business earns, and a combined equation to model the profit Rex’s business makes after \( m \) months.

2. Randy owns two plots of land on which he grows and sells trees. In Plot A there are 982 trees and Randy sells 5 of these trees each day. In Plot B there are 35 trees and Randy plants 4 trees per day.

If \( d \) = the number of days that have passed, write one equation for the number of trees in Plot A, another equation for the number of trees in Plot B, and a combined equation for the total number of trees on Randy’s two plots.

For problems 3 through 8, write an equation for the combined functions given the functions below.

\[
\begin{align*}
f(x) &= -4x - 7 \\
g(x) &= 3^x + 6 \\
h(x) &= -7x + 6
\end{align*}
\]

3. \( g(x) - h(x) \)  
4. \( f(x) + g(x) \)  
5. \( g(x) + h(x) \)
6. \( h(x) - g(x) \)  
7. \( f(x) - g(x) \)  
8. \( f(x) + h(x) \)

Answers

1. \( C(m) = 10m + 2000 \)  
   \( R(m) = 35m + 3000 \)  
   \( P(m) = R(m) - C(m) = (35m + 3000) - (10m + 2000) = 25m + 1000 \)
2. \( A(d) = -5d + 982 \)  
   \( B(d) = 4d + 35 \)  
   \( T(d) = A(d) + B(d) = (-5d + 982) + (4d + 35) = -1d + 1017 \)
3. \( g(x) - h(x) = (3^x + 6) - (-7x + 6) = 3^x + 7x \)
4. \( f(x) + g(x) = (-4x - 7) + (3^x + 6) = 3^x - 4x - 1 \)
5. \( g(x) + h(x) = (3^x + 6) + (-7x + 6) = 3^x - 7x + 12 \)
6. \( h(x) - g(x) = (-7x + 6) - (3^x + 6) = -7x - 3^x \)
7. \( f(x) - g(x) = (-4x - 7) - (3^x + 6) = -3^x - 4x - 13 \)
8. \( f(x) + h(x) = (-4x - 7) + (-7x + 6) = -11x - 1 \)
The first section of Chapter 11 introduces constructions. Historically, before there were uniform measuring devices, straightedges and compasses were the only means to draw shapes. A straightedge is not a ruler in that it has no measurement markings on it. Despite the fact that students do not have access to rulers or protractors in this part of the chapter, they are still able to draw shapes, create certain specific angle measurements, bisect angles and segments, and create congruent figures.

See the Math Notes box in Lesson 11.1.3.

Example 1

Using only a straightedge and compass, construct the perpendicular bisector of $AB$ at right. Then bisect one of the right angles.

The perpendicular bisector of $AB$ is a line that is perpendicular to $AB$ and also goes through the midpoint of $AB$. Although we could find this line by folding point $A$ onto point $B$, we want a way to find it with just the straightedge and compass. Also, with no markings on the straightedge, we cannot measure to find the midpoint.

Because of the nature of a compass, circles are the basis for constructions. For this construction, we draw two congruent circles, one with its center at point $A$, the other with its center at point $B$. The radii of these circles must be large enough so that the circles intersect at two points. Drawing a line through the two intersection points of the circles gives $l$, the perpendicular bisector of $AB$. 
To bisect a right angle, we begin by drawing a circle with a center at point $M$. There are no restrictions on the length of the radius, but we need to see the points of intersection, $P$ and $Q$. Also, we are only concerned with the arc of the circle that is within the interior of the angle we are bisecting, $PQ$. Next we use points $P$ and $Q$ as the centers of two congruent circles that intersect in the interior of $\angle PMQ$. This gives us point $X$ which, when connected to point $M$, bisects the right angle.

Note: With this construction, we have created two $45^\circ$ angles. From this we also get a $135^\circ$ angle, $\angle XMA$. Another bisection (of $\angle XMQ$) would give us a $22.5^\circ$ angle.

Example 2

Construct $\triangle MUD$ so that $\triangle MUD$ is congruent to $\triangle ABC$ by SAS $\cong$(side-angle-side congruence).

To construct a congruent triangle, we will need to use two constructions: copying a segment and copying an angle. In this example we want to construct the triangle with the SAS $\cong$, so we will copy a side, then an angle, and then the adjacent side. It does not matter which side we start with as long as we do the remainder of the parts in a SAS order. Here we will start by copying $BC$ with a compass. First draw a ray like the one at right ($UD$). Next, put the compass point on point $B$ and open the compass so that it reaches to point $C$. Keeping that measurement, mark off a congruent segment on the ray ($UD$). Next copy $\angle BCA$ so that its vertex $C$ is at point $D$ on the ray and one of its sides is $UD$. Then copy $CA$ to create $DM$. Finally, connect point $U$ to point $M$ and $\triangle MUD \cong \triangle ABC$. 
Problems

1. Construct a triangle congruent to $\triangle XYZ$ using SSS $\cong$.

2. Construct a rhombus with sides congruent to $AB$.

3. Construct a regular hexagon with sides congruent to $PQ$.

4. Use constructions to find the centroid of $\triangle WKD$. The centroid is the intersection of the medians of the triangle.

5. Construct the perpendicular bisectors of each side of $\triangle TES$. Do they all meet in one point?
**Answers**

1. Draw a ray and copy one side of $\triangle XYZ$ on it—for example, $\overline{XZ}$. To copy a second side ($\overline{XY}$), put one endpoint at $X$, and swing an arc above $\overline{XZ}$. Finally, copy the third side, place the compass point at $Z$, and swing an arc above $\overline{XZ}$ so that it intersects the arc from $X$. Connect points $X$ and $Z$ to the point of intersection of the arcs and label it $Y$.

2. Copy $\overline{AB}$ on a ray. Draw another ray from $A$ above the ray and mark a point $C$ on it at the length of $\overline{AB}$. Swing arcs of length $\overline{AB}$ from $B$ and $C$ and label their intersection $D$.

3. Construct a circle of radius $\overline{PQ}$. Mark a point on the circle, then make consecutive arcs around the circle using length $\overline{PQ}$. Connect the six points to form the hexagon. Alternately, construct an equilateral triangle using $\overline{PQ}$, then make five copies of the triangle to complete the hexagon.

4. Find the midpoint of each side of $\triangle WKD$ by following the instructions in Example 1. The intersection of the perpendicular bisector and the side will be the midpoint. Then draw a segment from each vertex to the midpoint of the opposite side. The point where the medians intersect is the centroid.

5. Yes, this is the circumcenter of the triangle.
Work problems are solved using the concept that if a job can be completed in \( x \) units of time, then the rate (or fraction of the job completed per unit of time) is \( \frac{1}{x} \). For example, if Diann can complete a job in 5 hours, then her rate is \( \frac{1}{5} \) of the job per hour.

Mixture problems are solved using the concept that the product (value or %)(quantity) must be consistent throughout the equation.

Example 1 (Work)

John can completely wash and dry the dishes in 20 minutes. His brother can do it in 30 minutes. How long will it take them working together?

Solution: John works at a rate of \( \frac{1}{20} \) of the job per minute. His brother works at a rate of \( \frac{1}{30} \) of the job per minute. Since they are working together we add the two rates together to get the combined rate.
Let \( t \) = the time (minutes) to complete the task and then using dimensional analysis:

\[
\left( \frac{1 \text{ job}}{20 \text{ minutes}} + \frac{1 \text{ job}}{30 \text{ minutes}} \right) (t \text{ minutes}) = 1 \text{ job}
\]

Solving the equation:

\[
\left( \frac{5 \text{ job}}{60 \text{ minutes}} \right) (t \text{ minutes}) = 1 \text{ job}
\]

\[
\frac{5}{60} t = 1
\]

\[
t = \frac{60}{5} = 12 \text{ minutes}
\]

Example 2 (Work)

With two inflowing pipes open, a water tank can be filled in 5 hours. If the larger pipe can fill the tank alone in 7 hours, how long would the smaller pipe take to fill the tank?

Solution: In this problem, we know the rate of the large pipe \( \left( \frac{1 \text{ job}}{7 \text{ hours}} \right) \), but we do not know the rate of the small pipe. We do know the time. Using the same set-up as Example 1, let \( t = \) time (hours) and the equation is:

\[
\left( \frac{1 \text{ job}}{7 \text{ hours}} + \frac{1 \text{ job}}{t \text{ hours}} \right) (5 \text{ hours}) = 1 \text{ job}
\]

Solving the equation:

\[
\frac{5}{7} + \frac{5}{t} = 1
\]

\[
7t \left( \frac{5}{7} + \frac{5}{t} \right) = 1(7t)
\]

\[
5t + 35 = 7t
\]

\[
35 = 2t
\]

\[
17.5 \text{ hours} = t
\]
Example 3 (Mixture)

Alicia has 10 liters of an 80% acid solution. How many liters of water should she add to form a 30% acid solution?

Solution: Draw a diagram to understand the problem.

If the dark gray represents the acid, the amount of acid (in liters) from the two containers on the right must equal the amount of acid in the one container on the left. Using this fact and the concept that \((\text{value or percentage}) \times \text{(quantity)} = \text{amount of acid}\), write and solve the following equation:

\[
(0.80)(10) + (0)(x) = (0.30)(10 + x)
\]

\[
8 = 3 + 0.3x
\]

\[
x = \frac{5}{0.3} = \frac{50}{3} \text{ or } 16 \frac{2}{3} \text{ L}
\]

Example 4 (Mixture)

A store has candy worth $0.90 a pound and candy worth $1.20 a pound. If the owners want 60 pounds of candy worth $1.00 a pound, how many pounds of each candy should they use?

Solution: Draw a diagram to understand the problem.

In this problem there are two unknowns, but we can still use the idea of Example 3 to solve this problem. Since there are two unknowns, we will need to write two equations (a system of equations). The cost of the two “bags” of candy on the right must equal the cost of the combined amount on the left. Using this fact and the concept that \((\text{value or price}) \times \text{(quantity)} = \text{cost}\), write and solve the following system of equations:

\[
x + y = 60 \quad \text{rewrite as } y = 60 - x \quad \Rightarrow \quad 0.9x + 1.2(60 - x) = 60
\]

\[
0.90x + 1.20y = 1.00(60) \quad \text{then use substitution} \quad 0.9x + 72 - 1.2x = 60
\]

\[
-0.3x = -12
\]

\[
x = 40 \text{ lbs}
\]

Therefore \(y = 20 \text{ lbs}\)

Note: This problem could have been solved with one variable. If the first bag has \(x\) lbs of candy and the final bag has 60 lbs of candy, then the second bag would have \((60 - x)\) lbs of candy. In this case, start with the equation on the top right.
Problems

1. Susan can paint her living room in 2 hours. Her friend Jaime estimates it would take him 3 hours to paint the same room. If they work together, how long will it take them to paint Susan’s living room?

2. Professor Minh can complete a set of experiments in 4 hours. Her assistant can do it in 6 hours. How long will it take them to complete the experiments working together?

3. With one hose a swimming pool can be filled in 12 hours. Another hose can fill it in 16 hours. How long will it take to fill the pool using both hoses?

4. Together, two machines can harvest a tomato crop in 6 hours. The larger machine can do it alone in 10 hours. How long does it take the smaller machine to harvest the crop working alone?

5. Steven can look up 20 words in a dictionary in an hour. His teammate Mary Lou can look up 30 words per hour. Working together, how long will it take them to look up 100 words?

6. A water tank is filled by one pump in 6 hours and is emptied by another pump in 12 hours. If both pumps are operating, how long will it take to fill the tank?

7. Two crews can service the space shuttle in 12 days. The faster crew can service the shuttle in 20 days alone. How long would the slower crew need to service the shuttle working alone?

8. Janelle and her assistant Ryan can carpet a house in 8 hours. If Janelle could complete the job alone in 12 hours, how long would it take Ryan to carpet the house working alone?

9. Able can harvest a strawberry crop in 4 days. Barney can do it in 5 days. Charlie would take 6 days. If they all work together, how long will it take them to complete the harvest?

10. How much coffee costing $6 a pound should be mixed with 3 pounds of coffee costing $4 a pound to create a mixture costing $4.75 a pound?

11. Sam’s favorite recipe for fruit punch requires 12% apple juice. How much pure apple juice should he add to 2 gallons of punch that already contains 8% apple juice to meet his standards?

12. Jane has 60 liters of 70% acid solution. How many liters of water must be added to form a solution that is 40% acid?

13. How many pound of nuts worth $1.05 a pound must be mixed with nuts worth $0.85 a pound to get a mixture of 200 pounds of nuts worth $0.90 a pound?
14. A coffee shop mixes Kona coffee worth $8 per pound with Brazilian coffee worth $5 per pound. If 30 pounds of the mixture is to be sold for $7 per pound, how many pounds of each coffee should be used?

15. How much tea costing $8 per pound should be mixed with 2 pounds of tea costing $5 per pound to get a mixture costing $6 per pound?

16. How many liters of water must evaporate from 50 liters of an 8% salt solution to make a 25% salt solution?

17. How many gallons of pure lemon juice should be mixed with 4 gallons of 25% lemon juice to achieve a mixture which contains 40% lemon juice?

18. Brian has 20 ounces of a 15% alcohol solution. How many ounces of a 50% alcohol solution must he add to make a 25% alcohol solution?

**Answers**

1. \( \left( \frac{1}{2} + \frac{1}{3} \right) t = 1 \)  
   1.2 hours

2. \( \left( \frac{1}{4} + \frac{1}{6} \right) t = 1 \)  
   2.4 hours

3. \( \left( \frac{1}{12} + \frac{1}{16} \right) t = 1 \)  
   6.67 hours

4. \( \left( \frac{1}{10} + \frac{1}{x} \right)(6) = 1 \)  
   15 hours

5. \( (20 + 30) t = 100 \)  
   2 hours

6. \( \left( \frac{1}{6} - \frac{1}{12} \right) t = 1 \)  
   12 hours

7. \( \left( \frac{1}{20} + \frac{1}{x} \right)(12) = 1 \)  
   30 days

8. \( \left( \frac{1}{12} + \frac{1}{x} \right)(8) = 1 \)  
   24 hours

9. \( \left( \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \right) t = 1 \)  
   60/37 ≈ 1.62 days

10. \$6x + $4(3) = $4.75(x + 3) \)  
    1.8 pounds

11. \( 0.12(2) + 1x = 0.08(2 + x) \)  
    \( \frac{1}{11} \) gallon

12. \( 0.7(60) + 0x = 0.4(60 + x) \)  
    45 liters

13. \( 1.05x + 0.85y = 0.90(200) \)  
    \( x + y = 200 \)  
    50 pounds

14. \$8k + $5b = $7(30) \)  
    \( k + b = 30 \)  
    20 Kona, 10 Brazilian

15. \$8x + $5(2) = $6(x + 2) \)  
    1 pound

16. \( 0.08(50) - 0x = 0.25(50 - x) \)  
    34 liters

17. \( 1x + 0.25(4) = 0.40(x + 4) \)  
    1 gallon

18. \( 0.15(20) + 0.50x = 0.25(20 + x) \)  
    8 ounces
Students have been solving equations throughout this course. Now they focus on what a solution means, and learn to solve equations and systems of equations graphically. Note: A graphing calculator is needed for this lesson. Instructions for using a graphing calculator can be found at www.cpm.org.

Example 1

The graph of \( y = \left( \frac{1}{2} \right)^x - 4 \) is shown at right. Use the graph to solve each of the following equations. Explain how the graph can be used to solve the equations.

a. \( \left( \frac{1}{2} \right)^x - 4 = y \)

b. \( \left( \frac{1}{2} \right)^x - 4 = -3 \)

c. \( \left( \frac{1}{2} \right)^x - 4 = 4 \)

Solution (a): Because this is one equation with two variables, there are infinitely many solutions. The solutions are all of the \((x, y)\) coordinate pairs that lie on the graph of the function.

Solution (b): This is the same equation in part (a), but now \( y = -3 \). To determine the value of \( x \) that will solve the equation, draw a horizontal line at \( y = -3 \), as shown in the graph at right. The intersection point of the two graphs is \((0, -3)\), so the solution to the given equation is \( x = 0 \). This can be verified by checking the solution in the original equation:

\[
\left( \frac{1}{2} \right)^0 - 4 = -3 \\
1 - 4 = -3 \\
-3 = -3
\]

Solution (c): This is the same equation in part (a), but now \( y = 4 \). To determine the value of \( x \) that will solve the equation, draw a horizontal line at \( y = 4 \), as shown in the graph at right. The intersection point of the two graphs is \((-3, 4)\), so the solution to the given equation is \( x = -3 \). This can be verified by checking the solution in the original equation:

\[
\left( \frac{1}{2} \right)^{-3} - 4 = 4 \\
2^3 - 4 = 4 \\
8 - 4 = 4 \\
4 = 4
\]
Example 2

Solve the equation $2^x - 3 = 6x$ by graphing.

Remember that a solution is a value that makes the equation true. In our original equation, this would mean that both sides of the equation would be equal for certain values of $x$. Using the graphs, the solution is the $x$-value that has the same $y$-value for both graphs, or the point(s) at which the graphs intersect.

To solve this equation, graph $y = 2^x - 3$ and $y = 6x$ as shown at right above. Notice that the screen shows the graphs only intersecting at one point, so zoom out to check for other possible points of intersections. This is shown in the graph at right. Determining these points will help solve the equation.

Zoom in to be able to see the intersection point you are determining more clearly. Then use the “calculate” feature of the graphing calculator to calculate the point of intersection.

One point of intersection is $(-0.3711, -2.226)$. The original equation only has one variable ($x$), so the solution to the equation $2^x - 3 = 6x$ is $x \approx -0.3711$.

The second point of intersection is $(5.060, 30.360)$. The solution to the equation is $x \approx 5.0600$.

These solutions can be verified by using a calculator to check the solutions in the original equation:

$2^{-0.3711} - 3 \approx 6(-0.3711)$ and $2^{5.060} - 3 \approx 6(5.060)$

$0.7732 - 3 \approx -2.2266$ and $33.3359 - 3 \approx 30.36$

$-2.2268 \approx -2.2266$ and $30.3359 \approx 30.36$

Note that the solutions and checks are approximate as students do not yet have an algebraic method for solving this type of equation to be able to get an exact answer.
Problems

The graph of \( y = \frac{1}{3}x^3 - 5 \) is shown at right. Use the graph to estimate the solutions to the equations in problems 1 through 4. Note: The y-axis is scaled by 5s.

1. \( \frac{1}{27}x^3 - 5 = 0 \)
2. \( \frac{1}{27}x^3 - 5 = 5 \)
3. \( \frac{1}{27}x^3 - 5 = -10 \)
4. \( \frac{1}{27}x^3 - 5 = -20 \)

Use a graphing calculator to solve each of the equations below.

5. \( 2^x = 3x + 8 \)
6. \( 5x = 3^x - 7 \)
7. \( 2x + 8 = 3^x + 6 \)
8. \( \left( \frac{1}{6} \right)^x + 2 = -\frac{3}{8}x \)
9. \( -2^x + 2 = 3x \)
10. \( -\left( \frac{9}{4} \right)^x = -9x + 6 \)

Answers

1. \( x = 3 \)
2. \( x \approx 3.5 \)
3. \( x = -3 \)
4. \( x \approx -4.2 \)
5. \( x \approx -2.612 \) or 4.406
6. \( x \approx -1.355 \) or 2.763
7. \( x \approx -0.790 \) or 1.445
8. no solution
9. \( x \approx 0.266 \)
10. \( x \approx 0.897 \) or 4.300
SIMPLIFYING ALGEBRAIC EXPRESSIONS  A.1.1 and A.1.2

Algebra tiles provide students with the opportunity to “see” abstract algebraic expressions and equations with two variables. Regular use of algebra tiles will help students access abstract concepts through the use of concrete physical representations.

In the figures at right, the dimensions of each tile are shown along its sides, and the area is shown on the tile itself. Algebra tiles are named by their areas. For example, the $x^2$-tile is in the upper left corner; it has an area of $x^2$.

In algebraic expressions, combining terms that have the same area to write a simpler expression is called combining like terms.

For additional information, refer to the Math Notes box in Lesson A.1.1.

The Lesson A.1.1B Resource Page (available at cpm.org or in the eBook) provides algebra tiles for home use. An algebra tiles eTool is also available at cpm.org or in the eBook.

Example 1

Write a simplified algebraic expression for the tile collection below.

Solution:

$$xy + x^2 + x + x + y^2 + 1 + x + y^2$$

or

$$x^2 + 2y^2 + xy + 3x + 1$$

Example 2

The expression $x^2 + xy + y^2 + 3xy + y^2 + 7$ can be rewritten as $x^2 + xy + 3xy + y^2 + y^2 + 7$ and simplified to $x^2 + 4xy + 2y^2 + 7$ by combining like terms.

Example 3

$$3x^2 - 4x + 3 + -x^2 + 3x - 7$$

$$= 3x^2 - x^2 - 4x + 3x + 3 - 7$$

$$= 2x^2 - x - 4$$
An Expression Mat is a physical representation of an algebraic expression. The upper half of an expression mat is the positive (addition) region and the lower half is the negative (subtraction) region. Positive algebra tiles are shaded and negative tiles are blank. (The illustration to the right reminds you that shaded tiles are positive.) A matching pair of tiles with one tile shaded and the other tile blank represents two opposites—with a value of 0. We refer to them as “zero pairs.” (The Lesson A.1.1A Resource Page has an Expression Mat.)

On an Expression Mat, tiles may be removed or moved in one of two “legal” ways:

1. Flip tiles and move them from the negative region to the positive region. That is, change subtraction to adding the opposite.
2. Remove an equal number of opposite tiles (one shaded and one not shaded) that are within the same region. These pairs of opposite tiles have a value of zero.
3. Group tiles that are alike together. That is, combine like terms.

Example 4

Simplify $3x + 2 - (2x - 3)$.

Create Expression Mat:

Flip tiles in subtraction region to addition region:

Remove zero pairs:

Therefore, $3x + 2 - (2x - 3)$ simplifies to $x + 5$. 
Example 5

Simplify $1 - (2y - 3) + y - 2$. Create Expression Mat:

Flip tiles in subtraction region to addition region:

Remove zero pairs:

Therefore, $1 - (2y - 3) + y - 2$ simplifies to $2 - y$.

Problems

Simplify each expression by combining like terms. Use algebra tiles if needed.

1. $2x^2 + x + 3 + 4x^2 + 3x + 5$
2. $y^2 + 2y + x^2 + 3y^2 + x^2$
3. $x^2 - 3x + 2 + x^2 + 4x - 7$
4. $y^2 + 2y - 3 - 4y^2 - 2y + 3$
5. $4xy + 3x + 2y - 7 + 6xy + 2x + 7$
6. $x^2 - y^2 + 2x + 3y + x^2 + y^2 + 3y$
7. $(4x^2 + 4x - 1) + (x^2 - x + 7)$
8. $(y^2 + 3xy + x^2) + (2y^2 + 4xy - x^2)$
Write the algebraic expression that corresponds to each Expression Mat, then simplify.

9.  

10.  

11.  

12.  

13.  

14.  

Use algebra tiles and an Expression Mat to simplify each expression.

15.  \[ 3 + 5x - 4 - 7x \]

16.  \[ -x - 4x - 7 \]

17.  \[ -(x + 3) \]

18.  \[ 4x - (x + 2) \]

19.  \[ 5x - (-3x + 2) \]

20.  \[ x - 5 - (2 - x) \]

21.  \[ 1 - 2y - 2y \]

22.  \[ -3x + 5 + 5x - 1 \]

23.  \[ 3 - (y + 5) \]

24.  \[ -(x + y) + 4x + 2y \]

25.  \[ 3x - 7 - (3x - 7) \]

26.  \[ -(x + 2y + 3) - 3x + y \]

27.  \[ (7x^2 - 6x - 9) - (9x^2 + 3x - 4) \]

28.  \[ (3x^2 - 8x - 4) - (5x^2 + x + 1) \]
Answers

1. \(6x^2 + 4x + 8\)
2. \(4y^2 + 2y + 2x^2\)
3. \(2x^2 + x - 5\)
4. \(-3y^2\)
5. \(10xy + 5x + 2y\)
6. \(2x^2 + 2x + 6y\)
7. \(5x^2 + 3x + 6\)
8. \(3y^2 + 7xy\)
9. \(3 + (-2) - 4 - (-3) = 0\)
10. \(3x + 1 - x - (-1) = 2x + 2\)
11. \(5 - (-2y) - (3) \text{ or } 5 - (-2y + 3) = 2y + 2\)
12. \(-4x - x - (-2) = -5x + 2\)
13. \(-(2y) - 1 \text{ or } -(2y + 1) = 2y - 1\)
14. \(3 + (-2y) - (-y) - (-2) = -y + 5\)
15. \(-2x - 1\)
16. \(-5x - 7\)
17. \(x - 3\)
18. \(3x - 2\)
19. \(8x - 2\)
20. \(2x - 7\)
21. \(-4y + 1\)
22. \(2x + 4\)
23. \(-y - 2\)
24. \(3x + y\)
25. \(0\)
26. \(-4x - y - 3\)
27. \(-2x^2 - 9x - 5\)
28. \(-2x^2 - 9x - 5\)
Expressions on two side-by-side Expression Mats can be compared to determine which expression is greater.

To compare two expressions, represent each expression using algebra tiles on its own Expression Mat. Simplify the expression on each Expression Mat by moving or removing tiles using “legal” moves:

- “Flip” tiles (change them from negative to positive and vice versa) and move them from the negative region to the positive region. That is, change subtraction to adding the opposite.
- Remove an equal number of opposite tiles (one shaded and one not shaded) that are within the same region. That is, remove the zero pairs.
- Group tiles that are alike within the same region together. That is, combine like terms.

Continue to make “legal” moves, in any order, until the expressions cannot be simplified any more.

- Remove tiles that are the same from both Expression Mats if necessary.

Compare the expression on the left with the one on the right to determine which expression is greater. If there are variable tiles remaining after simplifying, you do not have enough information to tell which side is greater—depending on what number the variable tile represents, either expression could be larger than the other.

**Example 1**

The Expression Comparison Mat at right represents the expressions $-2x + (-3) + 1 - (-x + 3)$ and $2 + (-3) - (x - 2)$. Use legal moves to simplify and determine which side is greater.

**Solution:**

Flip tiles and move them from the negative region to the positive region.

*Solution continues on next page →*
Solution continued from previous page.

Remove an equal number of opposite tiles (one shaded and one not shaded) that are within the same region. Also remove the same tiles from both Expression Mats.

−5 < 1; The right side is greater.

Students are also asked to record their steps. Different teachers have different expectations, but here are two possible ways to record the steps. The steps may also be done in a different order.

Recording the steps symbolically:

Left Expression

\[
\begin{align*}
-2x + 1 - 3 - (-x + 3) & \quad \text{flip} \\
-2x + 1 - 3 + x - 3 & \quad = \\
-x - 5 & = \frac{-(x)}{-5}
\end{align*}
\]

Right Expression

\[
\begin{align*}
2 - 3 - (x - 2) & \quad \text{flip} \\
2 - 3 - x + 2 & = \\
1 - x & = \frac{-(x)}{1}
\end{align*}
\]

Recording the steps with justifications:

<table>
<thead>
<tr>
<th>Left Expression</th>
<th>Right Expression</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2x + 1 - 3 - (-x + 3))</td>
<td>(2 - 3 - (x - 2))</td>
<td>Starting expressions</td>
</tr>
<tr>
<td>(-2x + 1 - 3 + x - 3)</td>
<td>(2 - 3 - (x - 2))</td>
<td>Flip (-x + 3) from “-” to “+”</td>
</tr>
<tr>
<td>(-x - 5)</td>
<td>(2 - 3 - (x - 2))</td>
<td>Combine like terms</td>
</tr>
<tr>
<td>(-x - 5)</td>
<td>(2 - 3 - x + 2)</td>
<td>Flip (x - 2) from “-” to “+”</td>
</tr>
<tr>
<td>(-5)</td>
<td>(2 - 3 + 2)</td>
<td>Remove (-x) from both sides</td>
</tr>
<tr>
<td>(-5)</td>
<td>(1)</td>
<td>Combine like terms</td>
</tr>
</tbody>
</table>
Example 2

Create the expressions \(x + 1 - (-1 - 2x)\) and \(3 + x - 1 - (x - 4)\) and then use legal moves to simplify and determine which side is greater.

Since we do not know the value of \(x\), it is not possible to determine the greater side.

Recording the steps with justifications:

<table>
<thead>
<tr>
<th>Left Expression</th>
<th>Right Expression</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x + 1 - (-1 - 2x))</td>
<td>(3 + x - 1 - (x - 4))</td>
<td>Starting expressions</td>
</tr>
<tr>
<td>(x + 1 + 1 + 2x)</td>
<td>(3 + x - 1 - x + 4)</td>
<td>Flip from “-” to “+”</td>
</tr>
<tr>
<td>(3x + 2)</td>
<td>(6)</td>
<td>Combine like terms</td>
</tr>
<tr>
<td>(3x)</td>
<td>(4)</td>
<td>Remove 2 from both sides</td>
</tr>
</tbody>
</table>
Problems

Write a set of expressions for problem. Use legal moves to simplify and determine which expression greater. Carefully record your steps.

1. Which is greater? 2.

3. Which is greater?

4. Which is greater?

In problems 5 through 10, record your steps as you use legal moves to simplify each expression and determine which expression has the greater value.

5. Which is greater: $6 - (2x - 4) - 3$ or $-x - (1 + x) + 4$?

6. Which is greater: $3x - (2 - x) + 1$ or $-5 + 4x + 3$?

7. Which is greater: $-1 + 6x - 2 + 4y - 2x$ or $y + 5x - (-2 + x) + 3y - 2$?

8. Which is greater: $x^2 - 2x + 6 - (-3x)$ or $-(3 - x^2) + 5 + 2x$?

9. Which is greater: $x + 2 - (2 - 2x)$ or $4 + x - 2 - (x - 4)$?

10. Which is greater: $2x + 4 - x - (-2) + x^2$ or $3 + x^2 + 4x - (-3 + 3x)$?

Answers (Expressions and explanations will vary.)

1. $-4 > -7$; left side is greater

2. $-5 < -1$; right side is greater

3. $x > 1$; not enough information

4. $4 > 3$; left side is greater

5. $7 > 3$; left side is greater

6. $-1 > -2$; left side is greater

7. $-3 < 0$; right side is greater

8. $4 > x$; not enough information

9. $3x > 6$; not enough information

10. $0 = 0$; both sides equal
An Equation Mat can be used to represent the process of solving an equation. An Equation Mat is created by putting two Expression Mats side by side—one for each side of the equal sign.

When the process of solving an equation ends with different numbers on each side of the equal sign (for example, $2 = 4$), there is no solution to the problem. When the result is the same expression or number on each side of the equation (for example, $x + 2 = x + 2$) it means that there are infinitely many solutions, or all real numbers are solutions.

See the Math Notes box in Lesson A.1.7 for a list of all the legal moves and their corresponding algebraic language. Also see the Math Notes box in Lesson A.1.8 for solving a linear equation and checking the solution.

For additional examples and practice, see the Checkpoint 1 materials.

Example 1

Solve $x + 2 - (-2x) = x + 5 - (x - 3)$.

First, build the equation on an Equation Mat.

Second, flip the tiles in the subtraction region to the addition region (change subtraction to adding the opposite).

Continue to simplify using legal moves. For example, remove zero pairs.

$3x + 2 = 8$
Example continued from previous page.

Isolate \(x\)-terms on one side and non-\(x\)-terms on the other by placing or removing matching tiles from both sides of the Equation Mat. Remove zero pairs again if needed.

![Equation Mat with tiles]

Finally, arrange tiles into equal-sized groups on both sides. Since both sides of the equation are equal, determine the value of \(x\). In this case, the tiles can be arranged into three groups, resulting in \(x = 2\).

**Example 2**

Solve \(3x + 3x - 1 = 4x + 9\)

\[
3x + 3x + (-1) = 4x + 9 \\
6x + (-1) = 4x + 9 \\
6x = 4x + 10 \\
2x = 10 \\
x = 5
\]

Flip all tiles from subtraction region to addition region.

Combine like terms.

Add 1 to each side, remove zero pairs.

Remove \(4x\) from each side.

Arrange into two groups.

**Example 3**

Solve \(-2x + 1 - (-3x + 3) = -4 + (-x - 2)\)

\[
-2x + 1 + 3x + (-3) = -4 + (-x) + (-2) \\
x + (-2) = (-x) + (-6) \\
x = (-x) + (-4) \\
2x = -4 \\
x = -2
\]

Flip all tiles from subtraction region to addition region.

Combine like terms.

Add 2 to each side, remove zero pairs.

Add \(x\) to both sides, remove zero pairs.

Arrange into two groups.
Problems
Solve each equation.

1. \(2x - 3 = -x + 3\)  
2. \(1 + 3x - x = x - 4 + 2x\)
3. \(4 - 3x = 2x - 6\)  
4. \(3 + 3x - (x - 2) = 3x + 4\)
5. \(-(x + 3) = 2x - 6\)  
6. \(-4 + 3x - 1 = 2x + 1 + 2x\)
7. \(-x + 3 = 10\)  
8. \(5x - 3 + 2x = x + 7 + 6x\)
9. \(4y - 8 - 2y = 4\)  
10. \(9 - (1 - 3y) = 4 + y - (3 - y)\)
11. \(2x - 7 = -x - 1\)  
12. \(-2 - 3x = x - 2 - 4x\)
13. \(-3x + 7 = x - 1\)  
14. \(1 + 2x - 4 = -3 - (-x)\)
15. \(2x - 1 - 1 = x - 3 - (-5 + x)\)  
16. \(-4x - 3 = x - 1 - 5x\)
17. \(10 = x + 6 + 2x\)  
18. \(-(x - 2) = x - 5 - 3x\)
19. \(6 - x - 3 = 4x - 8\)  
20. \(0.5x - (-x + 3) = x - 5\)

Answers
1. \(x = 2\)  
2. \(x = 5\)  
3. \(x = 2\)  
4. \(x = 1\)  
5. \(x = 1\)
6. \(x = -6\)  
7. \(x = -7\)  
8. no solution  
9. \(y = 6\)  
10. \(y = -7\)
11. \(x = 2\)  
12. all real numbers  
13. \(x = 2\)  
14. \(x = 0\)  
15. \(x = 2\)
16. no solution  
17. \(x = 1 \frac{1}{3}\)  
18. \(x = -7\)  
19. \(x = 2 \frac{1}{5}\)  
20. \(x = -4\)