The first section of Chapter 11 introduces constructions. Historically, before there were uniform measuring devices, straightedges and compasses were the only means to draw shapes. A straightedge is not a ruler in that it has no measurement markings on it. Despite the fact that students do not have access to rulers or protractors in this part of the chapter, they are still able to draw shapes, create certain specific angle measurements, bisect angles and segments, and create congruent figures.

See the Math Notes box in Lesson 11.1.3.

Example 1

Using only a straightedge and compass, construct the perpendicular bisector of $\overline{AB}$ at right. Then bisect one of the right angles.

The perpendicular bisector of $\overline{AB}$ is a line that is perpendicular to $\overline{AB}$ and also goes through the midpoint of $\overline{AB}$.

Although we could find this line by folding point $A$ onto point $B$, we want a way to find it with just the straightedge and compass. Also, with no markings on the straightedge, we cannot measure to find the midpoint.

Because of the nature of a compass, circles are the basis for constructions. For this construction, we draw two congruent circles, one with its center at point $A$, the other with its center at point $B$. The radii of these circles must be large enough so that the circles intersect at two points. Drawing a line through the two intersection points of the circles gives $l$, the perpendicular bisector of $\overline{AB}$.
To bisect a right angle, we begin by drawing a circle with a center at point $M$. There are no restrictions on the length of the radius, but we need to see the points of intersection, $P$ and $Q$. Also, we are only concerned with the arc of the circle that is within the interior of the angle we are bisecting, $PQ$. Next we use points $P$ and $Q$ as the centers of two congruent circles that intersect in the interior of $\angle PMQ$. This gives us point $X$ which, when connected to point $M$, bisects the right angle.

Note: With this construction, we have created two $45^\circ$ angles. From this we also get a $135^\circ$ angle, $\angle XMA$. Another bisection (of $\angle XMQ$) would give us a $22.5^\circ$ angle.

Example 2

Construct $\triangle MUD$ so that $\triangle MUD$ is congruent to $\triangle ABC$ by SAS $\cong$ (side-angle-side congruence).

To construct a congruent triangle, we will need to use two constructions: copying a segment and copying an angle. In this example we want to construct the triangle with the SAS $\cong$, so we will copy a side, then an angle, and then the adjacent side. It does not matter which side we start with as long as we do the remainder of the parts in a SAS order. Here we will start by copying $BC$ with a compass. First draw a ray like the one at right ($UD$). Next, put the compass point on point $B$ and open the compass so that it reaches to point $C$. Keeping that measurement, mark off a congruent segment on the ray ($UD$). Next copy $\angle BCA$ so that its vertex $C$ is at point $D$ on the ray and one of its sides is $UD$. Then copy $CA$ to create $DM$. Finally, connect point $U$ to point $M$ and $\triangle MUD \cong \triangle ABC$. 
Problems

1. Construct a triangle congruent to $\triangle XYZ$ using SSS $\cong$.

2. Construct a rhombus with sides congruent to $\overline{AB}$.

3. Construct a regular hexagon with sides congruent to $\overline{PQ}$.

4. Use constructions to find the centroid of $\triangle WKD$.
   The centroid is the intersection of the medians of the triangle.

5. Construct the perpendicular bisectors of each side of $\triangle TES$. Do they all meet in one point?
Answers

1. Draw a ray and copy one side of $\triangle XYZ$ on it—for example, $XZ$. To copy a second side ($XY$), put one endpoint at $X$, and swing an arc above $XZ$. Finally, copy the third side, place the compass point at $Z$, and swing an arc above $XZ$ so that it intersects the arc from $X$. Connect points $X$ and $Z$ to the point of intersection of the arcs and label it $Y$.

2. Copy $AB$ on a ray. Draw another ray from $A$ above the ray and mark a point $C$ on it at the length of $AB$. Swing arcs of length $AB$ from $B$ and $C$ and label their intersection $D$.

3. Construct a circle of radius $PQ$. Mark a point on the circle, then make consecutive arcs around the circle using length $PQ$. Connect the six points to form the hexagon. Alternately, construct an equilateral triangle using $PQ$, then make five copies of the triangle to complete the hexagon.

4. Find the midpoint of each side of $\triangle WKD$ by following the instructions in Example 1. The intersection of the perpendicular bisector and the side will be the midpoint. Then draw a segment from each vertex to the midpoint of the opposite side. The point where the medians intersect is the centroid.

5. Yes, this is the circumcenter of the triangle.
WORK AND MIXTURE PROBLEMS

**Work problems** are solved using the concept that if a job can be completed in $x$ units of time, then the rate (or fraction of the job completed per unit of time) is $\frac{1}{x}$. For example, if Diann can complete a job in 5 hours, then her rate is $\frac{1}{5}$ of the job per hour.

**Mixture problems** are solved using the concept that the product (value or %)(quantity) must be consistent throughout the equation.

Example 1 (Work)

John can completely wash and dry the dishes in 20 minutes. His brother can do it in 30 minutes. How long will it take them working together?

Solution: John works at a rate of $\frac{1}{20}$ of the job per minute. His brother works at a rate of $\frac{1}{30}$ of the job per minute. Since they are working together we add the two rates together to get the combined rate.

Let $t$ = the time (minutes) to complete the task and then using dimensional analysis:

$$\left(\frac{1 \text{ job}}{20 \text{ minutes}} + \frac{1 \text{ job}}{30 \text{ minutes}}\right)(t \text{ minutes}) = 1 \text{ job}$$

Solving the equation:

$$\left(\frac{5 \text{ job}}{60 \text{ minutes}}\right)(t \text{ minutes}) = 1 \text{ job}$$

$$\frac{5}{60} t = 1$$

$$t = \frac{60}{5} = 12 \text{ minutes}$$

Example 2 (Work)

With two inflowing pipes open, a water tank can be filled in 5 hours. If the larger pipe can fill the tank alone in 7 hours, how long would the smaller pipe take to fill the tank?

Solution: In this problem, we know the rate of the large pipe $\left(\frac{1 \text{ job}}{7 \text{ hours}}\right)$, but we do not know the rate of the small pipe. We do know the time. Using the same set-up as Example 1, let $t$ = time (hours) and the equation is:

$$\left(\frac{1 \text{ job}}{7 \text{ hours}} + \frac{1 \text{ job}}{t \text{ hours}}\right)(5 \text{ hours}) = 1 \text{ job}$$

Solving the equation:

$$\frac{5}{7} + \frac{5}{t} = 1$$

$$7t\left(\frac{5}{7} + \frac{5}{t}\right) = 1(7t)$$

$$5t + 35 = 7t$$

$$35 = 2t$$

$$17.5 \text{ hours} = t$$
**Example 3 (Mixture)**

Alicia has 10 liters of an 80% acid solution. How many liters of water should she add to form a 30% acid solution?

Solution: Draw a diagram to understand the problem.

If the dark gray represents the acid, the amount of acid (in liters) from the two containers on the right must equal the amount of acid in the one container on the left. Using this fact and the concept that \((\text{value or percentage}) \times (\text{quantity}) = \text{amount of acid}\), write and solve the following equation:

\[
(0.80)(10) + (0)(x) = (0.30)(10 + x)
\]

\[
8 = 3 + 0.3x
\]

\[
5 = 0.3x
\]

\[
x = \frac{5}{0.3} = \frac{50}{3} \text{ or } 16 \frac{2}{3} \text{ L}
\]

**Example 4 (Mixture)**

A store has candy worth $0.90 a pound and candy worth $1.20 a pound. If the owners want 60 pounds of candy worth $1.00 a pound, how many pounds of each candy should they use?

Solution: Draw a diagram to understand the problem.

In this problem there are two unknowns, but we can still use the idea of Example 3 to solve this problem. Since there are two unknowns, we will need to write two equations (a system of equations). The cost of the two “bags” of candy on the right must equal the cost of the combined amount on the left. Using this fact and the concept that \((\text{value or price}) \times (\text{quantity}) = \text{cost}\), write and solve the following system of equations:

\[
x + y = 60 \quad \text{rewrite as } y = 60 - x \quad \Rightarrow \quad 0.9x + 1.2(60 - x) = 60
\]

\[
0.90x + 1.20y = 1.00(60) \quad \text{then use substitution} \quad 0.9x + 72 - 1.2x = 60
\]

\[
-0.3x = -12
\]

\[
x = 40 \text{ lbs}
\]

Therefore \(y = 20 \text{ lbs}\)

Note: This problem could have been solved with one variable. If the first bag has \(x\) lbs of candy and the final bag has 60 lbs of candy, then the second bag would have \((60 - x)\) lbs of candy. In this case, start with the equation on the top right.
Problems

1. Susan can paint her living room in 2 hours. Her friend Jaime estimates it would take him 3 hours to paint the same room. If they work together, how long will it take them to paint Susan’s living room?

2. Professor Minh can complete a set of experiments in 4 hours. Her assistant can do it in 6 hours. How long will it take them to complete the experiments working together?

3. With one hose a swimming pool can be filled in 12 hours. Another hose can fill it in 16 hours. How long will it take to fill the pool using both hoses?

4. Together, two machines can harvest a tomato crop in 6 hours. The larger machine can do it alone in 10 hours. How long does it take the smaller machine to harvest the crop working alone?

5. Steven can look up 20 words in a dictionary in an hour. His teammate Mary Lou can look up 30 words per hour. Working together, how long will it take them to look up 100 words?

6. A water tank is filled by one pump in 6 hours and is emptied by another pump in 12 hours. If both pumps are operating, how long will it take to fill the tank?

7. Two crews can service the space shuttle in 12 days. The faster crew can service the shuttle in 20 days alone. How long would the slower crew need to service the shuttle working alone?

8. Janelle and her assistant Ryan can carpet a house in 8 hours. If Janelle could complete the job alone in 12 hours, how long would it take Ryan to carpet the house working alone?

9. Able can harvest a strawberry crop in 4 days. Barney can do it in 5 days. Charlie would take 6 days. If they all work together, how long will it take them to complete the harvest?

10. How much coffee costing $6 a pound should be mixed with 3 pounds of coffee costing $4 a pound to create a mixture costing $4.75 a pound?

11. Sam’s favorite recipe for fruit punch requires 12% apple juice. How much pure apple juice should he add to 2 gallons of punch that already contains 8% apple juice to meet his standards?

12. Jane has 60 liters of 70% acid solution. How many liters of water must be added to form a solution that is 40% acid?

13. How many pounds of nuts worth $1.05 a pound must be mixed with nuts worth $0.85 a pound to get a mixture of 200 pounds of nuts worth $0.90 a pound?
14. A coffee shop mixes Kona coffee worth $8 per pound with Brazilian coffee worth $5 per pound. If 30 pounds of the mixture is to be sold for $7 per pound, how many pounds of each coffee should be used?

15. How much tea costing $8 per pound should be mixed with 2 pounds of tea costing $5 per pound to get a mixture costing $6 per pound?

16. How many liters of water must evaporate from 50 liters of an 8% salt solution to make a 25% salt solution?

17. How many gallons of pure lemon juice should be mixed with 4 gallons of 25% lemon juice to achieve a mixture which contains 40% lemon juice?

18. Brian has 20 ounces of a 15% alcohol solution. How many ounces of a 50% alcohol solution must he add to make a 25% alcohol solution?

Answers

1. \( \left( \frac{1}{2} + \frac{1}{3} \right) t = 1 \) \hspace{1cm} 2. \( \left( \frac{1}{4} + \frac{1}{6} \right) t = 1 \) \hspace{1cm} 3. \( \left( \frac{1}{12} + \frac{1}{16} \right) t = 1 \)
   
   1.2 hours \hspace{1cm} 2.4 hours \hspace{1cm} \frac{6}{7} \approx 6.86 \text{ hours}

4. \( \left( \frac{1}{10} + \frac{1}{x} \right) (6) = 1 \) \hspace{1cm} 5. \( (20 + 30) t = 100 \) \hspace{1cm} 6. \( \left( \frac{1}{6} - \frac{1}{12} \right) t = 1 \)
   
   15 hours \hspace{1cm} 2 hours \hspace{1cm} 12 hours

7. \( \left( \frac{1}{20} + \frac{1}{x} \right) (12) = 1 \) \hspace{1cm} 8. \( \left( \frac{1}{12} + \frac{1}{x} \right) (8) = 1 \) \hspace{1cm} 9. \( \left( \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \right) t = 1 \)
   
   30 days \hspace{1cm} 24 hours \hspace{1cm} \frac{60}{37} \approx 1.62 \text{ days}

10. \( 6x + 4(3) = 4.75(x + 3) \) \hspace{1cm} 11. \( 0.12(2) + 1x = 0.08(2 + x) \)
    
    1.8 pounds \hspace{1cm} \frac{1}{11} \text{ gallon}

12. \( 0.7(60) + 0x = 0.4(60 + x) \) \hspace{1cm} 13. \( 1.05x + 0.85y = 0.90(200) \)

    45 liters \hspace{1cm} x + y = 200

14. \( 8k + 5b = 7(30) \) \hspace{1cm} 15. \( 8x + 5(2) = 6(x + 2) \)

    \( k + b = 30 \) \hspace{1cm} 1 pound

    20 Kona, 10 Brazilian

16. \( 0.08(50) - 0x = 0.25(50 - x) \) \hspace{1cm} 17. \( 1x + 0.25(4) = 0.40(x + 4) \)

    34 liters \hspace{1cm} 1 gallon

18. \( 0.15(20) + 0.50x = 0.25(20 + x) \)

    8 ounces
Students have been solving equations throughout this course. Now they focus on what a solution means, and learn to solve equations and systems of equations graphically.

Note: A graphing calculator is needed for this lesson. Instructions for using a graphing calculator can be found at www.cpm.org.

**Example 1**

The graph of $y = \left(\frac{1}{2}\right)^x - 4$ is shown at right. Use the graph to solve each of the following equations. Explain how the graph can be used to solve the equations.

a. $\left(\frac{1}{2}\right)^x - 4 = y$

b. $\left(\frac{1}{2}\right)^x - 4 = -3$

c. $\left(\frac{1}{2}\right)^x - 4 = 4$

**Solution (a):** Because this is one equation with two variables, there are infinitely many solutions. The solutions are all of the $(x, y)$ coordinate pairs that lie on the graph of the function.

**Solution (b):** This is the same equation in part (a), but now $y = -3$. To determine the value of $x$ that will solve the equation, draw a horizontal line at $y = -3$, as shown in the graph at right. The intersection point of the two graphs is $(0, -3)$, so the solution to the given equation is $x = 0$.

This can be verified by checking the solution in the original equation:

$\left(\frac{1}{2}\right)^0 - 4 = -3$

$1 - 4 = -3$

$-3 = -3$

**Solution (c):** This is the same equation in part (a), but now $y = 4$. To determine the value of $x$ that will solve the equation, draw a horizontal line at $y = 4$, as shown in the graph at right. The intersection point of the two graphs is $(-3, 4)$, so the solution to the given equation is $x = -3$.

This can be verified by checking the solution in the original equation:

$\left(\frac{1}{2}\right)^{-3} - 4 = 4$

$2^3 - 4 = 4$

$8 - 4 = 4$

$4 = 4$
Example 2

Solve the equation \(2^x - 3 = 6x\) by graphing.

Remember that a solution is a value that makes the equation true. In our original equation, this would mean that both sides of the equation would be equal for certain values of \(x\). Using the graphs, the solution is the \(x\)-value that has the same \(y\)-value for both graphs, or the point(s) at which the graphs intersect.

To solve this equation, graph \(y = 2^x - 3\) and \(y = 6x\) as shown at right above. Notice that the screen shows the graphs only intersecting at one point, so zoom out to check for other possible points of intersections. This is shown in the graph at right. Determining these points will help solve the equation.

Zoom in to be able to see the intersection point you are determining more clearly. Then use the “calculate” feature of the graphing calculator to calculate the point of intersection.

One point of intersection is \((-0.3711, -2.226)\). The original equation only has one variable (\(x\)), so the solution to the equation \(2^x - 3 = 6x\) is \(x \approx -0.3711\).

The second point of intersection is \((5.060, 30.360)\). The solution to the equation is \(x \approx 5.0600\).

These solutions can be verified by using a calculator to check the solutions in the original equation:

\[
\begin{align*}
2^{-0.3711} - 3 & \approx 6(-0.3711) \quad \text{and} \quad 2^{5.060} - 3 \approx 6(5.060) \\
0.7732 - 3 & \approx -2.2266 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\end{align*}
\]

Note that the solutions and checks are approximate as students do not yet have an algebraic method for solving this type of equation to be able to get an exact answer.
Problems

The graph of \( y = \frac{1}{3} x^3 - 5 \) is shown at right. Use the graph to estimate the solutions to the equations in problems 1 through 4. Note: The \( y \)-axis is scaled by 5s.

1. \( \frac{1}{27} x^3 - 5 = 0 \)
2. \( \frac{1}{27} x^3 - 5 = 5 \)
3. \( \frac{1}{27} x^3 - 5 = -10 \)
4. \( \frac{1}{27} x^3 - 5 = -20 \)

Use a graphing calculator to solve each of the equations below.

5. \( 2^x = 3x + 8 \)
6. \( 5x = 3^x - 7 \)
7. \( 2x + 8 = 3^x + 6 \)
8. \( \left( \frac{1}{6} \right)^x + 2 = -\frac{3}{8} x \)
9. \( -2^x + 2 = 3x \)
10. \( -\left( \frac{9}{4} \right)^x = -9x + 6 \)

Answers

1. \( x = 3 \)
2. \( x \approx 3.5 \)
3. \( x = -3 \)
4. \( x \approx -4.2 \)
5. \( x \approx -2.612 \) or 4.406
6. \( x \approx -1.355 \) or 2.763
7. \( x \approx -0.790 \) or 1.445
8. no solution
9. \( x \approx 0.266 \)
10. \( x \approx 0.897 \) or 4.300