Two triangles are congruent if there is a sequence of rigid transformations (reflections, rotations, and translations) that carries one triangle onto the other. In these lessons, students find shortcuts that enable them to prove triangles congruent in the least amount of steps, by developing five triangle congruence conditions.

The triangle congruence conditions are SSS ≅, ASA ≅, AAS ≅, SAS ≅, and HL ≅, illustrated below. Note: “S” stands for “side” and “A” stands for “angle.” HL ≅ is only used with right triangles. The “H” stands for “hypotenuse” and the “L” stands for leg. The pattern appears to be “SSA” but this arrangement is NOT one of our conjectures, since it is only true for right triangles.

See the Math Notes boxes in Lessons 7.1.4 and 7.1.7. For additional examples and more practice, see the Checkpoint 10 Materials.
Example 1

Use the triangle congruence conditions to decide whether or not each pair of triangles must be congruent. Base each decision on the markings, not on appearances. Justify each answer.

a. The triangles are congruent by SAS ≅.

b. The triangles are congruent by SSS ≅.

c. The triangles are congruent by AAS ≅.

d. The triangles are not necessarily congruent. The first triangle displays an ASA arrangement, while the second triangle displays an AAS arrangement. The triangles could still be congruent, but based on the markings, we cannot conclude that they definitely are congruent.

e. The triangles are right triangles and congruent by HL ≅.

f. The triangles are congruent by ASA ≅.
Example 2

Using the information given in the diagrams below, decide if any triangles are congruent. If you claim the triangles are congruent, create a flowchart justifying your answer.

a. 

b. 

In part (a), \( \triangle ABD \cong \triangle CBD \) by the SAS \( \cong \) conjecture. Note: If you only see “SA,” observe that \( BD \) is congruent to itself.

In part (b), the triangles are not necessarily congruent; they could be congruent, but since we only have information about angles, we cannot conclude anything else. Note: They are similar by AA ~.

Problems

Briefly explain if each of the following pairs of triangles are congruent or not. If so, state the triangle congruence condition that supports your conclusion.

1. 

2. 

3. 

4. 

5. 

6.
Chapter 7

7. 

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24. 

25.
In each diagram below, are any triangles congruent? If so, prove it.

26.  

27.  

28.  

29.  

30.  

31.  

Complete a proof for each problem below.

32. Given: $\overline{TR}$ and $\overline{MN}$ bisect each other.  
Prove: $\triangle NTP \cong \triangle MRP$

33. Given: $\overline{CD}$ bisects $\angle ACB$; $\angle 1 \equiv \angle 2$.  
Prove: $\triangle CDA \cong \triangle CDB$

34. Given: $\overline{AB} \parallel \overline{CD}$, $\angle B \equiv \angle D$, $\overline{AB} \equiv \overline{CD}$  
Prove: $\triangle ABE \cong \triangle CDE$

35. Given: $\overline{PG} \equiv \overline{SG}$, $\overline{TP} \equiv \overline{TS}$  
Prove: $\triangle TPG \cong \triangle TSG$

36. Given: $\overline{OE} \perp \overline{MP}$, $\overline{OE}$ bisects $\angle MOP$  
Prove: $\triangle MOE \cong \triangle POE$

37. Given: $\overline{AD} \parallel \overline{BC}$, $\overline{DC} \parallel \overline{BA}$  
Prove: $\triangle ADB \cong \triangle CBD$
38. Given: $\overline{AC}$ bisects $\overline{DE}$, $\angle A \equiv \angle C$
Prove: $\triangle ADB \cong \triangle CEB$

![Diagram](image1)

39. Given: $\overline{PQ} \perp \overline{RS}$, $\angle R \equiv \angle S$
Prove: $\triangle PQR \cong \triangle PQS$

![Diagram](image2)

40. Given: $\angle S \equiv \angle R$, $\overline{PQ}$ bisects $\angle SQR$
Prove: $\triangle SPQ \cong \triangle RPQ$

![Diagram](image3)

41. Given: $\overline{TU} \equiv \overline{GY}$, $\overline{KY} \parallel \overline{HU}$, $\overline{KT} \perp \overline{TG}$, $\overline{HG} \perp \overline{TG}$
Prove: $\angle K \equiv \angle H$

![Diagram](image4)

42. Given: $\overline{MQ} \parallel \overline{WL}$, $\overline{MQ} \equiv \overline{WL}$
Prove: $\overline{ML} \parallel \overline{WQ}$

![Diagram](image5)

Consider the diagram at right.

43. Is $\triangle BCD \cong \triangle EDC$? Prove it!

![Diagram](image6)

44. Is $\overline{AB} \equiv \overline{DC}$? Prove it!

![Diagram](image7)

45. Is $\overline{AB} \equiv \overline{ED}$? Prove it!
Answers

1. \( \triangle ABC \cong \triangle DEF \) by ASA \( \cong \).
2. \( \triangle GIH \cong \triangle LJK \) by SAS \( \cong \).
3. \( \triangle PNM \cong \triangle PNO \) by SSS \( \cong \).
4. \( \overline{QS} \cong \overline{QS} \), so \( \triangle QRS \cong \triangle QTS \) by HL \( \cong \).
5. Not necessarily congruent.
6. \( \triangle ABC \cong \triangle DFE \) by ASA \( \cong \) or AAS \( \cong \).
7. \( \overline{GI} \cong \overline{GI} \), so \( \triangle GHI \cong \triangle IJG \) by SSS \( \cong \).
8. Alternate interior angles = used twice, so \( \triangle KLN \cong \triangle NMK \) by ASA \( \cong \).
10. Vertical angles and/or alternate interior angles \( \cong \), so \( \triangle TUX \cong \triangle VWX \) by ASA \( \cong \).
11. No, the length of each hypotenuse is different.
12. Pythagorean Theorem, so \( \triangle EGH \cong \triangle IHG \) by SSS \( \cong \).
13. Sum of angles of triangle = 180\(^\circ\), but since the equal angles do not correspond, the triangles are not congruent.
14. \( AF + FC = FC + CD \), so \( \triangle ABC \cong \triangle DEF \) by SSS \( \cong \).
15. \( \overline{XZ} \cong \overline{XZ} \), so \( \triangle WXZ \cong \triangle YXZ \) by AAS \( \cong \).
16. \( \triangle ABC \cong \triangle EDC \) by AAS \( \cong \).
17. \( \triangle PQS \cong \trianglePRS \) by AAS \( \cong \), with \( \overline{PS} \cong \overline{PS} \) (segment congruent to itself).
18. \( \triangle VXW \cong \triangle ZXY \) by ASA \( \cong \), with \( \angle VXW \cong \angle ZXY \) because vertical angles are \( \cong \).
20. \( \triangle KLB \cong \triangle EBL \) by HL \( \cong \), with \( \overline{BL} \cong \overline{BL} \) (segment congruent to itself).
21. Vertical angles \( \cong \) at \( O \), so \( \triangle POQ \cong \triangle ROS \) by SAS \( \cong \).
22. Not necessarily congruent.
23. \( \triangle TEA \cong \triangle SAE \) by SSS \( \cong \), with \( \overline{EA} \cong \overline{EA} \) (segment congruent to itself).
25. \( \triangle KRS \cong \triangle ISR \) by HL \( \cong \).
26. \( \angle BAD \equiv \angle BCD \)
   \( \overline{BD} \equiv \overline{BD} \)
   \( \angle BDC \equiv \angle BDA \)
   \( \Delta ABD \equiv \Delta CBD \)
   Given
   Side congruent to itself
   Right \( \angle \)s are \( \equiv \)
   AAS \( \equiv \)

27. \( \angle B \equiv \angle E \)
   \( \overline{BC} \equiv \overline{CE} \)
   \( \angle BCA \equiv \angle BCD \)
   \( \Delta ABC \equiv \Delta DCE \)
   Given
   Vertical \( \angle \)s are \( \equiv \)

28. \( \overline{AC} \equiv \overline{CD} \)
   \( \overline{BC} \equiv \overline{BC} \)
   \( \angle BCD \equiv \angle BCA \)
   \( \Delta ABC \equiv \Delta DBC \)
   Given
   Side congruent to itself
   Right \( \angle \)s are \( \equiv \)
   SAS \( \equiv \)

29. \( \overline{AD} \equiv \overline{BC} \)
   \( \overline{CA} \equiv \overline{CA} \)
   \( \overline{BA} \equiv \overline{CD} \)
   \( \Delta ABC \equiv \Delta CDA \)
   Given
   Side congruent to itself
   SSS \( \equiv \)

30. Not necessarily. Counterexample:

31. \( \overline{BC} \equiv \overline{EF} \)
   \( \overline{AC} \equiv \overline{FD} \)
   \( \Delta ABC \equiv \Delta DEF \)
   Given
   Given
   HL \( \equiv \)

32. \( \overline{NP} \equiv \overline{MP} \) and \( \overline{TP} \equiv \overline{RP} \) by definition of bisector. \( \angle NPT \equiv \angle MPR \) because vertical angles are equal. So, \( \Delta NTP \equiv \Delta MRP \) by SAS \( \equiv \).

33. \( \angle ACD \equiv \angle BCD \) by definition of angle bisector. \( \overline{CD} \equiv \overline{CD} \) (segment congruent to itself) so \( \Delta CDA \equiv \Delta CDB \) by ASA \( \equiv \).

34. \( \angle A \equiv \angle C \) since alternate interior angles of parallel lines congruent so \( \Delta ABF \equiv \Delta CDE \) by ASA \( \equiv \).

35. \( \overline{TG} \equiv \overline{TG} \) (segment congruent to itself) so \( \Delta TPG \equiv \Delta TSG \) by SSS \( \equiv \).
36. $\angle MEO \cong \angle PEO$ because perpendicular lines form $\cong$ right angles $\angle MOE \cong \angle POE$ by angle bisector and $\overline{OE} \cong \overline{OE}$ (segment congruent to itself). So, $\triangle MOE \cong \triangle POE$ by ASA $\cong$.

37. $\angle CDB \cong \angle ABD$ and $\angle ADB \cong \angle CBD$ since parallel lines give congruent alternate interior angles. $\overline{DB} \cong \overline{DB}$ (segment congruent to itself) so $\triangle ADB \cong \triangle CBD$ by ASA $\cong$.

38. $\overline{DB} \cong \overline{EB}$ by definition of bisector. $\angle DBA \cong \angle EBC$ since vertical angles are congruent. So $\triangle ADB \cong \triangle CEB$ by AAS $\cong$.

39. $\angle RQP \cong \angle SQP$ since perpendicular lines form congruent right angles. $\overline{PQ} \cong \overline{PQ}$ (segment is congruent to itself) so $\triangle PQR \cong \triangle PQS$ by AAS $\cong$.

40. $\angle SQP \cong \angle RQP$ by angle bisector and $\overline{PQ} \cong \overline{PQ}$ (segment congruent to itself), so $\triangle SPQ \cong \triangle RPQ$ by AAS $\cong$.

41. $\angle KYT \cong \angle HUG$ because parallel lines form congruent alternate exterior angles. $\angle T + \angle U = \angle Y + \angle G$ so $\angle TY \cong \angle GU$ by subtraction. $\angle T \cong \angle G$ since perpendicular lines form congruent right angles. So $\triangle KTY \cong \triangle HGU$ by ASA. Therefore, $\angle K \cong \angle H$ since congruent triangles have congruent parts.

42. $\angle MQL \cong \angle WLQ$ since parallel lines form congruent alternate interior angles. $\overline{QL} \cong \overline{QL}$ (segment congruent to itself) so $\triangle MQL \cong \triangle WLQ$ by SAS so $\angle WQL \cong \angle MLQ$ since congruent triangles have congruent parts. So $\overline{ML} \parallel \overline{WQ}$ since congruent alternate interior angles are formed by parallel lines.

43. $\triangle ABCD \cong \triangle EDC$ by SAS $\cong$.

44. Not necessarily congruent.

45. Not necessarily congruent.
COORDINATE GEOMETRY 7.2.1 – 7.2.3

Now that students are familiar with many of the properties of various triangles, quadrilaterals, and special quadrilaterals, they can apply their algebra skills and knowledge of the coordinate grid to study coordinate geometry. In this section, polygons are plotted on coordinate axes. Using familiar ideas, such as the Pythagorean Theorem and slope, students can prove whether or not quadrilaterals have special properties.

See the Math Notes boxes in Lessons 7.2.2 and 7.2.3.

Example 1

On a set of coordinate axes, plot the points $A(-3, -1), B(1, -4), C(5, -1)$, and $D(1, 2)$ and connect them in the order given. Is this quadrilateral a rhombus? Justify your answer.

To show that this quadrilateral is a rhombus, we must show that all four sides are the same length (definition of a rhombus). When we want to determine the length of a segment on the coordinate graph, we use the Pythagorean Theorem. To begin, plot the points on a graph.

Although the shape appears to be a parallelogram, and possibly a rhombus, we cannot base our decision on appearances. To use the Pythagorean Theorem, we outline a slope triangle, creating a right triangle with $DC$ as the hypotenuse. The lengths of the legs of this right triangle are 3 and 4 units.

Using the Pythagorean Theorem,

\[3^2 + 4^2 = (DC)^2\]
\[9 + 16 = (DC)^2\]
\[25 = (DC)^2\]
\[DC = 5\]

Similarly, we can draw slope triangles for the other three sides of the quadrilateral and use the Pythagorean Theorem again. In each case, we find the lengths are all 5 units. Therefore, since all four sides have the same length, the polygon is a rhombus.
Example 2

On a set of coordinate axes, plot the points \(A(-4, 1), B(1, 3), C(8, -1),\) and \(D(4, -3)\), and connect them in the order given. Is this quadrilateral a parallelogram? Justify your answer.

When we plot the points, the quadrilateral appears to be a parallelogram, but we cannot base our decision on appearances. To prove it is a parallelogram, we must show that the opposite sides are parallel. On a graph, we show that lines are parallel by showing that they have the same slope.

We can use slope triangles to find the slope of each side.

\[
\text{Slope of } \overline{BC} = \frac{-4}{7} = -\frac{4}{7} \quad \text{Slope of } \overline{AD} = \frac{-4}{8} = -\frac{1}{2}
\]

\[
\text{Slope of } \overline{BA} = \frac{2}{5} \quad \text{Slope of } \overline{DC} = \frac{2}{4} = \frac{1}{2}
\]

Although the values for the slopes of the opposite sides are close, they are not equal. Therefore this quadrilateral is not a parallelogram.

Example 3

On a set of coordinate axes, plot the points \(A(-1, 5), B(6, 1), C(-3, -1)\). Determine the midpoints of \(\overline{AC}\) and \(\overline{AB}\). Then connect the midpoints to draw a midsegment of the triangle. Show that the midsegment is parallel to, and half the length of \(\overline{BC}\).

To determine the midpoints of \(\overline{AC}\) and \(\overline{AB}\), draw slope triangles.

The midpoint of a segment is found by adding half of the change in \(x\left(\frac{1}{2} \Delta x\right)\) and half the change in \(y\left(\frac{1}{2} \Delta y\right)\) to the coordinates of the leftmost endpoint.

For \(\overline{AC}\), \(\Delta x = 2\) and \(\Delta y = 6\), so \(\frac{1}{2} \Delta x = 1\) and \(\frac{1}{2} \Delta y = 3\).
Point \(C(-3, -1)\) is the leftmost point, so the midpoint of \(\overline{AC}\) is \((-3 + \frac{1}{2} \Delta x, -1 + \frac{1}{2} \Delta y)\) or \((-3 + 1, -1 + 3) = (-2, 2)\).

Repeat this process to determine that the midpoint of \(\overline{AB}\):

\(\Delta x = 7\) and \(\Delta y = -4\), so \(\frac{1}{2} \Delta x = 3 \frac{1}{2}\) and \(\frac{1}{2} \Delta y = -2\).
Point \(A\) is the leftmost point, so the midpoint of \(\overline{AB}\) is \((-1 + 3 \frac{1}{2}, 5 + (-2)) = \left(2 \frac{1}{2}, 3\right)\).

To show that the midsegment is parallel to \(\overline{BC}\), we need to show that they have the same slope. The slope of the midsegment is \(\frac{1}{4} = \frac{2}{9}\). The slope of \(\overline{BC}\) is also \(\frac{2}{9}\).

Use your slope triangles and the Pythagorean Theorem to determine the length of the midsegment and the length of \(\overline{BC}\).

Midsegment: \(4.5^2 + 1^2 = c^2\) \(21.25 = c^2\) \(c \approx 4.61\)

\(\overline{BC}: 9^2 + 2^2 = c^2\) \(85 = c^2\) \(c \approx 9.22\) \(4.61\) is half of \(9.22\)
Problems

1. If $ABCD$ is a rectangle, and $A(1, 2), B(5, 2),$ and $C(5, 5)$, what is the coordinate of $D$?

2. If $P(2, 1)$ and $Q(6, 1)$ are the endpoints of the base of an isosceles right triangle, what is the $x$-coordinate of the third vertex?

3. The three points $S(-1, -1), A(1, 4),$ and $M(2, -1)$ are vertices of a parallelogram. What are the coordinates of three possible points for the other vertex?

4. Graph the following lines on the same set of axes. These lines enclose a shape. What is the name of that shape? Justify your answer.

   \[
   y = \frac{3}{2} x + 7 \\
   y = 0.6x \\
   y = -\frac{10}{6} x - 1 \\
   y = -\frac{5}{3} x + 9
   \]

5. If $W(-4, -5), X(1, 0), Y(-1, 2),$ and $Z(-6, -3)$, what shape is $WXYZ$? Justify your answer.

6. If $\overline{DT}$ has endpoints $D(2, 2)$ and $T(6, 4)$, what is the equation of the line containing the perpendicular bisector of $\overline{DT}$?

Answers

1. $(1, 5)$

2. $(4, 4)$

3. $(4, 4), (0, -6),$ or $(-2, 4)$

4. Since the slopes of opposites side are equal, this is a parallelogram. Additionally, since the slopes of intersecting lines are negative reciprocals of each other, they are perpendicular. This means the angles are all right angles, so the figure is a rectangle.

5. The slopes are: $WX = 1$, $XY = -1$, $YZ = 1$, and $ZW = -1$. This shows that $WXYZ$ is a rectangle.

6. $y = -2x + 11$

   The midpoint of $\overline{DT}$ is $(4, 3)$. The slope of $\overline{DT}$ is $\frac{1}{2}$, so the perpendicular slope is $-2$.

   Start with $3 = -2(4) + b$ and solve for $b$. 
