SOLVING ONE- VARIABLE INEQUALITIES  9.1.1 and 9.1.2

To solve an inequality in one variable, first change it to an equation (a mathematical sentence with an “=” sign) and then solve. Place the solution, called a “boundary point”, on a number line. This point separates the number line into two regions. The boundary point is included in the solution for situations that involve ≥ or ≤, and excluded from situations that involve strictly > or <. On the number line boundary points that are included in the solutions are shown with a solid filled-in circle and excluded solutions are shown with an open circle. Next, choose a number from within each region separated by the boundary point, and check if the number is true or false in the original inequality. If it is true, then every number in that region is a solution to the inequality. If it is false, then no number in that region is a solution to the inequality.

For additional information, see the Math Notes boxes in Lessons 9.1.1 and 9.1.3.

Example 1
Solve: $3x - (x + 2) \geq 0$
Change to an equation and solve.
Place the solution (boundary point) on the number line. Because $x = 1$ is also a solution to the inequality (≥), we use a filled-in dot.
Test a number on each side of the boundary point in the original inequality. Highlight the region containing numbers that make the inequality true.
The solution is $x \geq 1$.

Example 2
Solve: $-x + 6 > x + 2$
Change to an equation and solve.
Place the solution (boundary point) on the number line. Because $x = 2$ is not a solution to the original inequality (>), we use an open dot.
Test a number on each side of the boundary point in the original inequality. Highlight the region containing numbers that make the inequality true.
The solution is $x < 2$. 
Problems

Solve each inequality.

1. \(4x - 1 \geq 7\)
2. \(2(x - 5) \leq 8\)
3. \(3 - 2x < x + 6\)
4. \(\frac{1}{2}x > 5\)
5. \(3(x + 4) > 12\)
6. \(2x - 7 \leq 5 - 4x\)
7. \(3x + 2 < 11\)
8. \(4(x - 6) \geq 20\)
9. \(\frac{1}{4}x < 2\)
10. \(12 - 3x > 2x + 1\)
11. \(\frac{x - 5}{7} \leq -3\)
12. \(3(5 - x) \geq 7x - 1\)
13. \(3y - (2y + 2) \leq 7\)
14. \(\frac{m + 2}{5} < \frac{2m}{3}\)
15. \(\frac{m - 2}{3} \geq \frac{2m + 1}{7}\)

Answers

1. \(x \geq 2\)
2. \(x \leq 9\)
3. \(x > -1\)
4. \(x > 10\)
5. \(x > 0\)
6. \(x \leq 2\)
7. \(x < 3\)
8. \(x \geq 11\)
9. \(x < 8\)
10. \(x < \frac{11}{3}\)
11. \(x \leq -16\)
12. \(x \leq 1.6\)
13. \(y \leq 9\)
14. \(m > \frac{6}{7}\)
15. \(m \geq 17\)
To **solve an equation with absolute value**, first break the problem into two equations since the quantity inside the absolute value can be positive or negative. Then solve each part separately. For additional information, see the Math Notes box in Lesson 9.2.2.

There are several methods for solving absolute value and other inequalities, but one method that works for all kinds of inequalities is to change the inequality to an equation, solve it, then place the solution(s) on a number line. The solution(s), called “boundary point(s),” divide the number line into regions. Check any point within each region in the original inequality. If that point makes the original inequality true, then all the points in that region are solutions. If that point makes the original inequality false, then none of the points in that region are solutions. The boundary points are included in (≥ or ≤) or excluded from (> or <) the solution depending on the inequality sign.

Solving an absolute value inequality is very similar to solving a linear inequality in one variable, except that there are often three solution regions on the number line instead of just two. For more on solving linear inequalities see the section “Solving One-Variable Inequalities (Lessons 9.1.1 and 9.1.2)” of this *Parent Guide with Extra Practice*.

**Example 1**

Solve \( |2x + 3| = 11 \)

“Looking inside” the absolute value can solve this problem. (See “Multiple Methods for Solving Equations” (3.3.1 – 3.3.3) in this *Parent Guide with Extra Practice*.)

Since \( |11| = |-11| = 11 \), \( 2x + 3 = 11 \) or \( 2x + 3 = -11 \).

Solving both equations yields two answers: \( 2x + 3 = 11 \) or \( 2x + 3 = -11 \)

\[
\begin{align*}
2x & = 8 \\
x & = 4
\end{align*}
\]

\[
\begin{align*}
2x & = -14 \\
x & = -7
\end{align*}
\]

**Example 2**

Solve \( |2x - 3| = 7 \)

Separate into two equations. \( 2x - 3 = 7 \) or \( 2x - 3 = -7 \)

Add 3. \( 2x = 10 \) or \( 2x = -4 \)

Divide by 2. \( x = 5 \) or \( x = -2 \)

*Remember that you can always check your solutions in the original equation to make sure they are correct!*
Example 3

Solve: \( |x - 3| \leq 5 \)

Change to an equation and solve.

\[ |x - 3| = 5 \]

\[ x - 3 = 5 \text{ or } x - 3 = -5 \]

\[ x = 8 \text{ or } x = -2 \]

(The boundary points)

Choose \( x = -3, \ x = 0, \) and \( x = 9 \) to test in the original inequality. \( x = -3 \) is false, \( x = 0 \) is true, and \( x = 9 \) is false.

The solution is all numbers greater than or equal to \(-2\) and less than or equal to \(8\), written as \(-2 \leq x \leq 8\).

Example 4

Solve: \( 3|2y + 1| + 1 > 28 \)

Change to an equation and solve.

\[ 3|2y + 1| + 1 = 28 \]

Get the absolute value by itself by subtracting 1 and then dividing by 3 on both sides. Then solve the absolute value equation.

\[ |2y + 1| = 9 \]

\[ 2y + 1 = 9 \text{ or } 2y + 1 = -9 \]

\[ y = 4 \text{ or } y = -5 \]

(The boundary points)

Choose \( y = -6, \ y = 0, \) and \( y = 5 \) to test in the original inequality. \( y = -6 \) is true, \( y = 0 \) is false, and \( y = 5 \) is true.

The solution is all numbers less than \(-5\) or greater than \(4\), written as \(y < -5\) or \(y > 4\).

Problems

Solve each absolute value equation.

1. \( |x - 2| = 5 \)
2. \( |3x + 2| = 11 \)
3. \( |5 - x| = 9 \)
4. \( |3 - 2x| = 7 \)
5. \( |2x + 3| = -7 \)
6. \( |4x + 1| = 10 \)

Solve each absolute value inequality.

7. \( |x + 4| \geq 7 \)
8. \( |x| - 5 \leq 8 \)
9. \( |x - 5| \leq 8 \)
10. \( |4r - 2| > 8 \)
11. \( |3x| \leq 12 \)
12. \( |1 - 3x| \leq 13 \)
13. \( |2x - 3| > 15 \)
14. \( |5x| > -15 \)
15. \( -2|x - 3| + 6 < -4 \)
16. \( |4 - d| \leq 7 \)
17. \( |x - 4| \leq 0 \)
18. \( |2x + 1| - 2 < -3 \)
Answers

1. $x = 7$ or $-3$
2. $x = 3$ or $-\frac{13}{3}$
3. $x = -4$ or $14$
4. $x = -2$ or $5$
5. no solution
6. $x = \frac{9}{4}$ or $-\frac{11}{4}$
7. $x \geq 3$ or $x \leq -11$
8. $-13 \leq x \leq 13$
9. $-3 \leq x \leq 13$
10. $r < -\frac{3}{2}$ or $r > \frac{5}{2}$
11. $-4 \leq x \leq 4$
12. $-4 < x < \frac{14}{3}$
13. $x < -6$ or $x > 9$
14. all real numbers
15. $x > 8$ or $x < -2$
16. $-3 \leq d \leq 11$
17. $x = 4$
18. no solution
To graph the solutions to an inequality in two variables, first graph the corresponding equation. This graph is the boundary line (or curve), since all points that make the inequality true lie on one side or the other of the line. Before graphing the equation, decide whether the line or curve is part of the solution or not, that is, whether it is solid or dashed. If the inequality symbol is either \( \leq \) or \( \geq \), then the points on the boundary line are solutions to the inequality and the line must be solid. If the symbol is either < or >, then the points on the boundary line are not solutions to the inequality and the line is dashed.

Next, decide which side of the boundary line must be shaded to show the part of the graph that represents all \((x, y)\) coordinate pairs that make the inequality true. To do this, choose a point not on the boundary line. Substitute the \(x\)- and \(y\)-values of this point into the \textit{original} inequality. If the inequality is true for the test point, then shade the region on the side of the boundary line that contains the test point. If the inequality is false for the test point, then shade the opposite region.

The shaded portion represents all of the \((x, y)\) coordinate pairs that are solutions to the original inequality.

Caution: If you need to rewrite the inequality in order to graph it, such as rewriting it in slope-intercept form, always use the \textit{original} inequality to test a point, not the rearranged form.

For additional information, see the Math Notes box in Lesson 9.3.1.

**Example 1**

Graph the solutions to the inequality \( y > 3x - 2 \).

First, graph the line \( y = 3x - 2 \), but draw it dashed since > means the boundary line is not part of the solution. For example, the point \((0, -2)\) is on the boundary line, but it is not a solution to the inequality because \(-2 \not> 3(0) - 2\) or \(-2 \not> -2\).

Next, test a point that is not on the boundary line. For this example, use the point \((-2, 4)\).

\[ 4 > 3(-2) - 2, \text{ so } 4 > -8 \] which is a true statement.

Since the inequality is true for this test point, shade the region containing the point \((-2, 4)\). All of the coordinate pairs that are solutions lie in the shaded region.
Example 2

Graph the solutions to the inequality \( y \leq 2^x - 6 \).

First, graph the exponential function \( y = 2^x - 6 \) and draw it as a solid curve, since \( \leq \) means that the points on the boundary curve are solutions to the inequality. For example, the point \((0, -5)\) is on the boundary curve. It is a solution to the inequality because 
\[-5 \leq 2^0 - 6 \text{ or } -5 \leq 1 - 6.\]

Next, test a point not on the boundary curve. For this example use the point \((2, 2)\).
\[2 \leq 2^2 - 6, \text{ so } 2 \leq -2 \text{ which is a false statement.}\]

Since the inequality is false for this test point, shade below the region that does not contain this point. All of the coordinate pairs that are solutions lie in the shaded region.

Problems

Graph the solutions to each of the following inequalities on a separate set of axes.

1. \( y \leq 3x + 1 \)  
2. \( y \geq -2x + 3 \)  
3. \( y > 4x - 2 \)  
4. \( y < -3x - 5 \)  
5. \( y \leq 3 \)  
6. \( x > 1 \)  
7. \( y > \frac{2}{3}x + 8 \)  
8. \( y < -\frac{3}{5}x - 7 \)  
9. \( 3x + 2y \geq 7 \)  
10. \( -4x + 2y < 3 \)  
11. \( y \leq 2^x \)  
12. \( y > 2^x - 3 \)  
13. \( y \geq \left(\frac{1}{2}\right)^x - 2 \)  
14. \( y < 4 \left(\frac{1}{2}\right)^x \)  
15. \( y \leq -(2)^x \)

Answers

1.  
2.  
3.
To graph the solutions to a system of inequalities, follow the same procedure outlined in the previous section but do it twice—once for each inequality. The solution to the system of inequalities is the overlap of the shading from the individual inequalities. When two boundary lines are graphed, there are often four regions. The region containing the coordinate pairs that make both of the inequalities true is the solution region.

Example 1

Graph the solutions to the system: 
\[ y \leq \frac{1}{2}x + 2 \]
\[ y > -\frac{2}{3}x + 1 \]

Graph the lines \( y = \frac{1}{2}x + 2 \) and \( y = -\frac{2}{3}x + 1 \).
The first is solid and the second is dashed.

Test a point in the first inequality. For this example, \((-4, 5)\).
\[ 5 \leq \frac{1}{2}(-4) + 2 \quad \text{or} \quad 5 \leq 0 \]
This inequality is false, so shade the region of the first boundary line that does not contain the point \((-4, 5)\).

Test a point in the second inequality. For this example \((0, 0)\).
\[ 0 > -\frac{2}{3}(0) + 1 \quad \text{or} \quad 0 > 1 \]
This inequality is false, so shade the region of the second boundary line that does not contain the point \((0, 0)\).

The coordinate pair solutions are represented by the overlap of the two shaded regions shown by the darkest shading in the graph at right.
Example 2

Graph the solutions to the system: \( y \leq -x + 5 \)
\[ y \geq 2^x - 1 \]

Graph the line \( y = -x + 5 \) and the curve \( y = 2^x - 1 \) with a solid line and curve.

Test the point \((0, -4)\) in the first inequality.
\[
0 \leq -(−4)+5 \quad \text{or} \quad 0 \leq 1
\]
This inequality is true, so shade the region containing the point \((0, -4)\).

Test the point \((0, 3)\) in the second inequality.
\[
3 \geq 2^0 - 1 \quad \text{or} \quad 3 \geq 0
\]
This inequality is true, so shade the region containing the point \((0, 3)\).

The coordinate pair solutions are represented by the overlap of the two shaded regions shown by the darkest shading in the graph at right.

Problems

Graph the solutions to each of the following pairs of inequalities on the same set of axes.

1. \( y > 3x - 4 \)
   \( y \leq -2x + 5 \)
2. \( y \geq -3x - 6 \)
   \( y > 4x - 4 \)
3. \( y < -\frac{3}{2}x + 4 \)
   \( y \leq \frac{1}{3}x + 3 \)
4. \( y < -\frac{3}{4}x - 1 \)
   \( y \geq \frac{4}{3}x + 1 \)
5. \( y < 3 \)
   \( y > \frac{1}{2}x + 2 \)
6. \( x \leq 3 \)
   \( y < \frac{3}{4}x - 4 \)
7. \( y \leq 2x + 1 \)
   \( y \geq 2^x - 4 \)
8. \( y < -x + 5 \)
   \( y \geq 2^x + 1 \)
9. \( y < 2^x + 3 \)
   \( y \geq (\frac{1}{2})^x \)
Answers

1. 

2. 

3. 

4. 

5. 

6. 

7. 

8. 

9.