Two triangles are congruent if there is a sequence of rigid transformations that carries one onto the other. Two triangles are also congruent if all three pairs of corresponding angles and all three pairs of corresponding sides are congruent. In these lessons, students review ways to prove triangles congruent in fewer steps, by using five triangle congruence theorems. They are SSS \cong, ASA \cong, AAS \cong, SAS \cong, and HL \cong, illustrated below.

Note: “S” stands for “side” and “A” stands for “angle.” HL \cong is only used with right triangles. The “H” stands for “hypotenuse” and the “L” stands for leg. The pattern for HL in the diagram appears to be “SSA,” but this arrangement is NOT one of the theorems, because SSA is only valid for right triangles.

See the Math Notes box in Lesson 2.1.1.
Example 1

Determine whether or not each pair of triangles must be congruent. Base each decision on the markings, not on appearances. Justify each answer with a triangle congruence theorem.

a. In part (a), the triangles are congruent by the SAS ≅ theorem. The triangles are also congruent in part (b), this time by the SSS ≅ theorem. In part (c), the triangles are congruent by the AAS ≅ theorem. Part (d) shows a pair of triangles that are not necessarily congruent. The first triangle displays an ASA arrangement, while the second triangle displays an AAS arrangement. The triangles could still be congruent, but based on the given markings, we cannot conclude that they definitely are congruent. The triangles in part (e) are right triangles and the markings fit the HL ≅ theorem. Lastly, in part (f), the triangles are congruent by the ASA ≅ theorem.
**Example 2**

Using the information given in the diagram at right, decide if the triangles are congruent. If you claim the triangles are congruent, create a flowchart justifying your answer.

\[ \triangle ABD \cong \triangle CBD \] by the SAS \( \cong \) theorem. Note: If you only see “SA,” observe that \( BD \) is congruent to itself. The **Reflexive Property** justifies stating that something is equal or congruent to itself.

**Problems**

Decide if each of the following pairs of triangles must be congruent. If so, state the triangle congruence theorem that supports your conclusion.

1. 

2. 

3. 

4. 

5. 

6. 

7. 

8. 

9. 

[Diagram of triangles with markings and labels]
Use the triangle congruence theorems to decide whether or not each pair of triangles must be congruent. Base your decision on the markings, not on appearances. Justify your answer.
In each diagram below, are any triangles congruent? If so, prove it.

26. [Diagram A]
27. [Diagram B]
28. [Diagram C]

29. [Diagram D]
30. [Diagram E]
31. [Diagram F]

**Answers**

1. \( \triangle ABC \cong \triangle DEF \) by ASA
2. \( \triangle GHI \cong \triangle LJK \) by SAS
3. \( \triangle PNM \cong \triangle PNO \) by SSS
4. \( \overline{QS} \cong \overline{QS} \), so \( \triangle QRS \cong \triangle QTS \) by HL
5. The triangles are not necessarily congruent.
6. \( \triangle ABC \cong \triangle DFE \) by ASA or AAS
7. \( \overline{GI} \cong \overline{GI} \), so \( \triangle GHI \cong \triangle JIG \) by SSS
8. \( \parallel \) lines \( \rightarrow \) alternate interior angles \( \cong \) can be used twice, so \( \triangle KLN \cong \triangle NMK \) by ASA
9. Vertical angles \( \cong \), so \( \triangle POQ \cong \triangle ROS \) by SAS
10. Vertical angles and/or \( \parallel \) lines \( \rightarrow \) alternate interior angles \( \cong \), so \( \triangle TUX \cong \triangle VWX \) by ASA or AAS
11. No, the lengths of the hypotenuses of the triangles are different.
12. By the Pythagorean Theorem, \( EG = IH \) and \( EH = IG \), so \( \triangle EGH \cong \triangle IHG \) by SSS
13. Sum of angles of triangle = 180°, but since the congruent angles do not correspond, the triangles are not congruent.
14. \( AF + FC = FC + CD \), so \( \triangle ABC \cong \triangle DEF \) by SSS
15. \( \overline{XZ} \cong \overline{XZ} \), so \( \triangle WXZ \cong \triangle YXZ \) by AAS
16. \( \triangle ABC \cong \triangle EDC \) by AAS
17. \( \triangle PQS \cong \triangle PRS \) by AAS or HL \( \equiv \) with \( \overline{PS} \equiv \overline{PS} \) by the Reflexive Property.
18. \( \triangle VWX \cong \triangle ZXY \) by ASA \(\cong\), with \( \angle VXW \cong \angle ZXY \) because vertical angles are \(\cong\).

19. \( \triangle TEA \cong \triangle SAE \) by SSS \(\cong\), with \( \overline{EA} \cong \overline{EA} \) by the Reflexive Property.

20. \( \triangle KLB \cong \triangle EBL \) by HL \(\cong\), with \( \overline{BL} \cong \overline{BL} \) by the Reflexive Property.


26. Yes

27. Yes

28. Yes

29. Yes

30. Not necessarily. Counterexample:

31. Yes
CONVERSES 2.1.3 and 2.1.4

A conditional statement is a sentence in “If …, then …” form. “If all of the sides are equal in length, then the triangle is equilateral” is an example of a conditional statement. Conditional statements can be abbreviated by creating an arrow diagram. When the clause after the “if” in a conditional statement (called the hypothesis) changes places with the clause after the “then” (called the conclusion), the new statement is called the converse of the original. If the conditional statement is true, the converse is not necessarily true, and vice versa.

In Lesson 2.1.4 students use what they have learned about angle measures to create proofs by contradiction. For additional information see the Math Notes box in Lesson 2.2.1.

Example 1

Read each conditional statement below. Write it as an arrow diagram and state whether or not it is true. Then write the converse of the statement and state whether or not the converse is true.

a. If a triangle is equilateral, then it is equiangular.

b. If \( x = 4 \), then \( x^2 = 16 \).

c. If \( ABCD \) is a square, then \( ABCD \) is a parallelogram.

Part (a): \( \Delta \) is equilateral \( \Rightarrow \) \( \Delta \) is equiangular

The converse is: If a triangle is equiangular, then it is equilateral. This statement and the original conditional statement are both true.

In part (b), the conditional statement is true and the arrow diagram is: \( x = 4 \) \( \Rightarrow \) \( x^2 = 16 \).

The converse of this statement, “If \( x^2 = 16 \), then \( x = 4 \),” is not necessarily true because \( x \) could equal \(-4\).

In part (c), the arrow diagram is: \( ABCD \) is a square \( \Rightarrow \) \( ABCD \) is a parallelogram.

This statement is true, but the converse, “If \( ABCD \) is a parallelogram, then \( ABCD \) is a square,” is not necessarily true. Any parallelogram that does not have four equal sides or four right angles is not a square.
Problems

Rewrite each conditional statement below as an arrow diagram and state whether or not it is true. Then write the converse of the statement and state whether or not the converse is true.

1. If an angle is a straight angle, then the angle measures 180°.
   Converse: If an angle measures 180°, then it is a straight angle. True.

2. If a triangle is a right triangle, then the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.
   Converse: If the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse, then the triangle is a right triangle. True.

3. If the measures of two angles of one triangle are equal to the measures of two angles of another triangle, then the measures of the third angles are also equal.

4. If one angle of a quadrilateral is a right angle, then the quadrilateral is a rectangle.

5. If two angles of a triangle have equal measures, then the two sides of the triangle opposite those angles have equal length.

Use the method of proof by contradiction to justify each of your conclusions to problems 6 and 7 below.

6. Nik scored 40 points lower than Tess on their last math test. The scores could range from 0 to 100 points. Could Tess have scored a 30 on this test? Justify using a proof by contradiction.

7. Jolie claims that a triangle can have two right angles. Prove her wrong! Justify your answer with a proof by contradiction.

Answers

1. Conditional: True

   ![Arrow Diagram]

   Converse: If an angle measures 180°, then it is a straight angle. True.

2. Conditional: True

   ![Arrow Diagram]

   Converse: If the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse, then the triangle is a right triangle. True.
3. Conditional: True

\[ \begin{array}{c}
\text{Original Triangle} & \to & \text{Transformed Triangle} \\
\end{array} \]

Converse: If the measures of one pair of corresponding angles of two triangles are equal, then the measures of the two other pairs of corresponding angles are also equal. False.

4. Conditional: False

\[ \begin{array}{c}
A & C \\
B & D \\
\end{array} \]

Converse: If a quadrilateral is a rectangle, then one angle is a right angle. True, in fact, all four angles are right angles.

5. Conditional: True

\[ \begin{array}{c}
\text{Original Triangle} & \to & \text{Transformed Triangle} \\
\end{array} \]

Converse: If two sides of a triangle are equal in length, then the two angles opposite those sides are equal in measure. True.

6. Assume that Tess scored 30 points. Then Nik’s score was 30 – 40 = –10, which is impossible. So Tess cannot have a score of 30 points.

7. Assume that a triangle has two right angles. Using the Triangle Sum Theorem, the measure of the third angle must be zero. However, this is impossible, so a triangle cannot have two right angles. OR: If a triangle has two 90° angles, the two sides that intersect with the side between them would be parallel and never meet to complete the triangle, as shown in the figure.
In this section students focus on comparing geometric shapes. They begin by dilating shapes: enlarging them as one might on a copy machine. When students compare the original and enlarged shapes closely, they discover that the shape of the figure remains exactly the same (this means the angle measures of the enlarged figure are equal to those of the original figure), but the size changes (the lengths of the sides increase). Although the size changes, the lengths of the corresponding sides all have a constant ratio, known as the scale factor, or ratio of similarity.

See the Math Notes boxes in Lessons 2.2.2 and 2.3.1 for more information about dilations and similar figures.

Example 1

Enlarge the figure at right from the origin by a factor of 3.

Students used rubber bands to create a dilation (enlargement) of several shapes. We can do this using a grid and slope triangles. Create a right triangle so that the segment from the origin to point A (2, 4), is the hypotenuse, one leg lies on the positive x-axis, and the other leg connects point A to the endpoint of the first leg at (2, 0). This triangle is called a slope triangle since it represents the slope of the hypotenuse from (0, 0) to vertex A. Add two more slope triangles exactly like this one along the ray from (0, 0) through point A as shown in the figure at right. Using three triangles creates an enlargement by a factor of 3 and gives us the new point A’ at (6, 12). Repeat this process for the other two vertices, forming a new slope triangle for each vertex.

This will give us new points B’ at (12, –6) and C’ at (–12, –12). Connecting points A’, B’, and C’, we form a new triangle that is an enlargement of the original triangle by a factor of 3, as shown at left. Notice that the sides of the dilated triangle are parallel to the sides of the original triangle.
Example 2

The two quadrilaterals at right are similar. What parts are equal? Can you determine the lengths of any of the unlabeled sides?

Since the quadrilaterals are similar, we know that all the corresponding angles have the same measure. This means that \(m \angle A = m \angle A', m \angle B = m \angle B', m \angle C = m \angle C',\) and \(m \angle D = m \angle D'.\) In addition, the corresponding sides are proportional, which means the ratio of corresponding sides is a constant. To determine the ratio, we need to know the lengths of one pair of corresponding sides. From the diagram we see that \(\overline{AD}\) corresponds to \(\overline{A'D'}\). Since these sides correspond, we can write \(\frac{AD}{A'D'} = \frac{4}{6}\).

Therefore, the ratio of similarity is \(\frac{4}{6}\), or \(\frac{2}{3}\). We can use this value to calculate the length of another side if we know the length of its corresponding side.

\[
\begin{align*}
\frac{AB}{A'B'} &= \frac{4}{6} & \frac{BC}{B'C'} &= \frac{4}{6} & \frac{CD}{C'D'} &= \frac{4}{6} \\
\frac{AB}{6} &= \frac{4}{6} & \frac{8}{B'C'} &= \frac{4}{6} & \frac{CD}{12} &= \frac{4}{6} \\
AB &= 4 & 4B'C' &= 48 & 6CD &= 48 \\
& & B'C' &= 12 & CD &= 8
\end{align*}
\]

Example 3

The pair of shapes at right is similar (\(ABCDEF \sim UVWXYZ\)). Label the second figure correctly to reflect the similarity statement. Assume the second figure is drawn to scale.

Since the polygons are similar, this means that their corresponding angles have equal measure. When we write a similarity statement, we write the letters so that the corresponding angles match up. By the similarity statement, we must have \(m \angle A = m \angle U, m \angle B = m \angle V, m \angle C = m \angle W, m \angle D = m \angle X, m \angle E = m \angle Y,\) and \(m \angle F = m \angle Z.\)

The smaller figure is labeled at right. If it is difficult to tell which original angle corresponds to its enlargement or reduction, try rotating the figures so that they have the same orientation.
Problems

1. Copy the figure below onto graph paper and then enlarge it by a factor of 2.

2. Create a figure similar to the one below with a scale factor of 0.5.

For each pair of similar figures below, determine the ratio of similarity for large:small. Diagrams are drawn roughly to scale; you can assume that sides that look longer are longer.

3.

4.

5.

6.

7.

8.

For each pair of similar figures, state the ratio of similarity. Then use it to calculate the value of $x$. Diagrams are drawn roughly to scale; you can assume that sides that look longer are longer.

9.

10.

11.

12.

13.

14.
15. The shadow of a statue is 20 feet long, while the shadow of a student is 4 ft long. If the student is 6 ft tall, how tall is the statue?

Each pair of figures below is similar. Use what you know about similarity to solve for $x$. Diagrams are drawn roughly to scale; you can assume that sides that look longer are longer.

16.

17.

18.

19.

Solve for the missing lengths in the pairs of similar figures below.

20. $\triangle ABC \sim \triangle PQR$

21. $\triangle JKL \sim \triangle WXY$

22. $\triangle STU \sim \triangle MNP$

23. $\triangle DAV \sim \triangle ISW$
24. $ABCDE \sim FGHIJ$

25. $\triangle ABC \sim \triangle DBE$

Answers

1. 

2. 

3. $\frac{4}{3}$

4. $\frac{5}{1}$

5. $\frac{2}{1}$

6. $\frac{24}{7}$

7. $\frac{6}{1}$

8. $\frac{15}{8}$

9. $\frac{7}{8}; \ x = 32$

10. $\frac{2}{1}; \ x = 72$

11. $\frac{1}{3}; \ x = 15$

12. $\frac{5}{6}; \ x = 15$

13. $\frac{4}{5}; \ x = 20$

14. $\frac{3}{2}; \ x = 16.5$

15. 30 ft

16. $x = 12$

17. $x = 9$

18. $x = 0.8$

19. $x = \frac{40}{3} \approx 13.33$

20. $x = 7.5$

21. $x = 1.25$

22. $x = 16$

23. $x \approx 3.7$

24. $x = 13.5$

25. $x = 12$
CONDITIONS FOR TRIANGLE SIMILARITY

When two figures are related by a series of transformations (including dilations), they are similar. Another way to check for similarity is to measure all the angles and sides of two figures. In this section students develop conditions to shorten the process. These are the **AA Triangle Similarity Condition** (AA ~), the **SAS Triangle Similarity Condition** (SAS ~), and the **SSS Triangle Similarity Condition** (SSS ~). The first condition states that if two pairs of corresponding angles are congruent, then the triangles are similar. The second condition states that if two pairs of corresponding side lengths have the same ratio, and their included angles are congruent, then the triangles are similar. The third condition states that if all three pairs of corresponding side lengths have the same ratio, then the triangles are similar. Additionally, students find that if similar figures have a ratio of similarity of 1, then the shapes are congruent, that is, they have the same size and shape. Students use flowcharts in this section to help organize their information and make logical conclusions about similar triangles. Now students are able to use similar triangles to determine side lengths, perimeters, heights, and other measurements.

See the Math Notes box in Lesson 2.3.2 for more information about similar triangles.

Example 1

Based on the given information, is each pair of triangles similar? If they are similar, write the similarity statement. Justify your answer completely.

- a.
- b.
- c.
- d.
- e.
- f.
We will use the three similarity conditions to test whether or not the triangles are similar.

In part (a), we have the lengths of the three sides, so it makes sense to check whether the SSS ~ holds true. Write the ratios of the corresponding side lengths and compare them to see if they are the same, as shown at right. Each ratio reduces to 3, so they are equal. Therefore, \( \triangle TES \sim \triangle AWK \) by SSS ~.

The measurements given in part (b) suggest we look at SAS ~. \( \angle A \) and \( \angle R \) are the included angles. Since they are both right angles, they have equal measures. Now we need to check that the corresponding sides lengths have the same ratio, as shown at right.

Although the triangles display the SAS ~ pattern and the included angles have equal measures, the triangles are not similar because the corresponding side lengths do not have the same ratio.

In part (c), we are given the measures of two angles of each triangle, but not corresponding angles. \( m\angle K = 55^\circ = m\angle N \) which is one pair of corresponding angles. For AA ~, we need two pairs of congruent angles. If we use the fact that the measures of the three angles of a triangle add up to 180°, we can calculate the measures of \( \angle O \) and \( \angle E \) as shown at right. Now we see that all pairs of corresponding angles have equal measures, so \( \triangle POK \sim \triangle EMN \) by AA ~.

Part (d) shows the SAS ~ pattern and we can see that the included angles have equal measures, \( m\angle G = m\angle H \). We also need the ratio of the corresponding side lengths to be equal. Since the two fractions are equal (the second reduces to the first), the corresponding side lengths have the same ratio. Therefore, \( \triangle YUG \sim \triangle IOH \) by SAS ~.

In part (e), we see that the included angles have equal measures, \( m\angle B = m\angle N \).
Since \( \frac{45}{15} = \frac{9}{3} = \frac{3}{1} \), the corresponding sides are proportional. Therefore, \( \triangle BOX \sim \triangle NTE \) by SAS ~.

In part (f), we only have one pair of angles that are congruent (the right angles), but those angles are not between the sides with known lengths. However, we can calculate the lengths of the third sides using the Pythagorean Theorem.

\[
\begin{align*}
8^2 + (IL)^2 &= 10^2 \\
64 + (IL)^2 &= 100 \\
(IL)^2 &= 36 \\
IL &= 6
\end{align*}
\]

\[
\begin{align*}
12^2 + (AB)^2 &= 20^2 \\
144 + (AB)^2 &= 400 \\
(AB)^2 &= 256 \\
AB &= 16
\end{align*}
\]

Now that we know all three sides, we can check to see if the triangles are similar by SSS ~. Since the ratios of the corresponding sides are the same, \( \triangle ELI \sim \triangle BZA \) by SSS ~.
Example 2

Using the information given in the diagram at right, decide if the triangles are similar or congruent. If you claim the triangles are similar or congruent, create a flowchart justifying your answer.

\( \triangle WXV \sim \triangle ZYV \) by the AA \( \sim \) theorem. The triangles are not necessarily congruent; they could be congruent, but since we only have information about angles, we cannot conclude anything else.

\[ \angle XVW \equiv \angle YVZ \]

Vertical angles \( \equiv \)

\[ \angle W \equiv \angle Z \]

\( \parallel \) lines \( \Rightarrow \) alt. int. angles \( \equiv \)

There is more than one way to justify this conclusion. There is another pair of alternate interior angles (\( \angle X \) and \( \angle Y \)) that are congruent that we could have used rather than the vertical angles, or we could have used that pair along with the vertical angles.

Example 3

In the diagram at right, \( \overline{AY} \parallel \overline{HP} \). Decide whether or not there are any similar triangles in the figure. Justify your answer with a flowchart.

Can you determine the length of \( \overline{AY} \)? Justify your answer.

Recalling information we studied in earlier chapters, the parallel lines give us congruent angles. In this figure, we have two pairs of corresponding angles that are congruent: \( \angle PHR \equiv \angle YAR \) and \( \angle HPR \equiv \angle AYR \). Because two pairs of corresponding angles are congruent, we know the triangles are similar: \( \triangle PHR \sim \triangle YAR \) by AA \( \sim \). Since the triangles are similar, the lengths of corresponding sides are proportional (i.e., have the same ratio). This means we can write the solution at right.

We can justify this result with a flowchart as well. The flowchart at right organizes and states what is written above.
Problems

Each pair of figures below is similar. Write a correct similarity statement and solve for \(x\) for each pair of figures. Diagrams are drawn roughly to scale; you can assume that sides that look longer are longer.

1. 

2. 

3. 

4. 

Determine if each pair of triangles is similar. If they are similar, justify your answer.

5. 

6. 

7. 

8. 

9. 

10.
Decide if each pair of triangles is similar. If they are similar, write a correct similarity statement and justify your answer.

17. \( \triangle O \) and \( \triangle C \):

18. \( \triangle O \) and \( \triangle W \):

19. \( \triangle A \) and \( \triangle E \):

20. \( \triangle A \) and \( \triangle M \):
Using the information given in each diagram below, decide if any triangles are congruent, similar but not congruent, or not similar. If you claim the triangles are congruent or similar, create a flowchart justifying your answer.

21. 

22. 

23. 

24. 

25. In the figure at right $\overline{AB} \parallel \overline{DE}$. Is $\triangle ABC$ similar to $\triangle EDC$? Use a flowchart to organize and justify your answer.

26. Standing 4 feet from a mirror resting on the flat ground, Palmer, whose eye height is 5 feet, 9 inches, can see the reflection of the top of a tree. He measures the mirror to be 24 feet from the tree. How tall is the tree? Draw a picture to help solve the problem.
Answers

1. \( ABCDEF \sim UZYXWV, x = 3.75 \)
2. \( RECT \sim NGLA, x = 8 \)
3. \( \Delta MNS \sim \Delta RCH, x = 72 \)
4. \( LACEY \sim ITHOM, x = 16.5 \)
5. \( AA \sim \)
6. \( SSS \sim \)
7. \( AA \sim \)
8. \( SAS \sim \)
9. not \( \sim \)
10. not \( \sim \)
11. \( SAS \sim \) or \( SSS \sim \)
12. not \( \sim \)
13. \( AA \sim \)
14. \( SSS \sim \)
15. \( AA \sim \)
16. \( AA \sim \)
17. \( \Delta BOX \sim \Delta NCA \) by \( AA \sim \)
18. The triangles are not similar because the sides are not proportional.
\[
\frac{12}{15} = \frac{18}{22.5} = 0.8, \quad \frac{10}{13} \approx 0.76
\]
19. \( \Delta ALI \sim \Delta MES \) by \( SAS \sim \)
20. The triangles are not similar. On \( \Delta SAM \), the 60° is included between the two given sides, but on \( \Delta UEL \) the angle is not included.

21. \( \Delta DAV \sim \Delta ISV \) by \( SAS \sim \)

22. \( \Delta LUN \) and \( \Delta HTC \) are not necessarily similar based on the markings.

23. \( \Delta SAP \sim \Delta SJE \) by \( AA \sim \)
24. \( \triangle KRS \equiv \triangle ISR \) by HL \( \equiv \)

\( \triangle KRS \) and \( \triangle ISR \) are right triangles.

\( KR \equiv IS \)

Given

\( RS \equiv RS \)

Ref. Prop.

\( \triangle KRS \equiv \triangle ISR \)

HL \( \equiv \)

25. \( AB \parallel DE \)

Given

\( \angle ACB \equiv \angle ECD \)

Vertical angles \( \equiv \)

\( \angle ABC \equiv \angle EDC \)

\( \parallel \) lines \( \Rightarrow \) alt. int. angles \( \equiv \)

\( \triangle ABC \sim \triangle ECD \)

AA \( \sim \)

Note: There is more than one way to solve this problem. Corresponding angles could have been used twice rather than mentioning vertical angles.

26. A sketch of the situation and how it translates into a diagram with triangles is shown at right. \( \triangle PFM \sim \triangle TRM \) by AA \( \sim \).

The proportion is:

\[
\frac{x}{5.75} = \frac{24}{4}
\]

\[
4x = 138
\]

\[
x = 34.5
\]

Therefore, the tree is 34.5 feet tall.