The graph of a quadratic function, a parabola, is a symmetrical curve. Its highest or lowest point is called the vertex. The graph of a parabola can be created by using an equation in the form \( y = ax^2 + bx + c \). In previous lessons students graphed parabolas by substituting values for \( x \) and then evaluating \( y \). This can be a tedious process. Another method for graphing a parabola is to determine the \( x \)-intercepts first, and then solve for the vertex and/or the \( y \)-intercept. To determine the \( x \)-intercepts, substitute 0 for \( y \) and solve the quadratic equation, \( 0 = ax^2 + bx + c \). One method for solving a quadratic equation is to factor and use the Zero Product Property. This method uses two ideas:

1. When a product is equal to zero, then at least one of the factors must be zero.
2. Some quadratic expressions can be factored into the product of two simple binomials.

For additional information see the Math Notes box in Lesson 5.1.4.

**Example 1**

Determine the \( x \)-intercepts of the parabola \( y = x^2 + 6x + 8 \). Then find the \( y \)-intercept and sketch the parabola.

First, substitute \( y = 0 \):

\[
0 = x^2 + 6x + 8
\]

Then factor the quadratic expression:

\[
(x + 4)(x + 2) = 0
\]

Set each factor equal to 0.

\[
(x + 4) = 0 \quad \text{or} \quad (x + 2) = 0
\]

Solve each equation for \( x \).

\[
x = -4 \quad \text{or} \quad x = -2
\]

Find the \( y \)-intercept by substituting \( x = 0 \) and evaluating.

\[
y = (0)^2 + 6(0) + 8 = 8
\]

The graph is a parabola with \( x \)-intercepts at \((-4, 0) \) and \((-2, 0) \) and a \( y \)-intercept at \((0, 8)\).
Example 2

Use the x-intercepts and the vertex to sketch the graph of \(-2x^2 + 7x + 15 = y\).

Substitute \(y = 0\).

\[-2x^2 + 7x + 15 = 0\]

Factor the quadratic expression.

\[-(2x - 3)(x - 5) = 0\]

Set each factor equal to 0.

\[-2x - 3 = 0\] or \[(x - 5) = 0\]

Solve each equation for \(x\).

\[-2x = 3\] or \(x = 5\)

\[x = -\frac{3}{2}\] or \(x = 5\)

To find the vertex, average the x-intercepts and use that x-value to calculate the corresponding y-value. The average of the x-intercepts is \(\frac{5 + (-1.5)}{2} = 1.75\). Substituting this into the original equation yields:

\[-2(1.75)^2 + 7(1.75) + 15 = 21.125\]

Thus the vertex is located at \((1.75, 21.125)\). Use the vertex and the x-intercepts at \((-\frac{3}{2}, 0)\) and \((5, 0)\) to sketch the graph of the parabola.

Example 3

Sketch a graph of \(9x^2 - 6x + 1 = y\).

Substitute \(y = 0\).

\[9x^2 - 6x + 1 = 0\]

Factor the quadratic expression.

\[(3x - 1)(3x - 1) = 0\]

Solve each equation for \(x\). Notice that the factors are the same so there will be only one solution.

\[(3x - 1) = 0\]

\[3x = 1\] \[x = \frac{1}{3}\]

Substitute \(x = 0\) into the original equation to determine that the y-intercept is \((0, 1)\). The graph is a parabola that has a single x-intercept at \((\frac{1}{3}, 0)\) and y-intercept at \((0, 1)\).
Problems

Use the Zero Product Property to determine the x-intercepts for the graph of each quadratic function.

1. \( y = x^2 + x - 20 \)
2. \( y = -x^2 - 7x - 12 \)
3. \( y = 3x^2 - 7x - 6 \)
4. \( y = 3x^2 + 11x + 10 \)
5. \( y = 6x^2 + 5x - 4 \)
6. \( y = -x^2 + 2x + 8 \)
10. \( y = 6x^2 - x - 15 \)
11. \( y = 4x^2 + 12x + 9 \)
12. \( y = 2x^2 + 8x + 6 \)

Determine the x- and y-intercepts of the graph of each of the following quadratic functions and then sketch the graph.

13. \( y = x^2 + 4x + 3 \)
14. \( y = x^2 + 5x - 6 \)
15. \( y = 2x^2 - 7x - 4 \)
16. \( y = -3x^2 - 10x + 8 \)
17. \( y = 16x^2 - 25 \)

Answers

1. \((-5, 0)\) and \((4, 0)\)
2. \((-4, 0)\) and \((-3, 0)\)
3. \((-\frac{2}{3}, 0)\) and \((3, 0)\)
4. \((-\frac{3}{2}, 0)\) and \((-2, 0)\)
5. \((-\frac{4}{3}, 0)\) and \((\frac{1}{2}, 0)\)
6. \((4, 0)\) and \((-2, 0)\)
10. \((-\frac{3}{2}, 0)\) and \((\frac{5}{3}, 0)\)
11. \((-\frac{3}{2}, 0)\)
12. \((-1, 0)\) and \((-3, 0)\)

13. x-intercepts: \((-1, 0), (-3, 0)\)
   y-intercept: \(0, 3\)
14. x-intercepts: \((-6, 0), (1, 0)\)
   y-intercept: \(0, -6\)

15. x-intercepts: \((-0.5, 0), (4, 0)\)
   y-intercept: \(0, -4\)
16. x-intercepts: \((-4, 0), \left(\frac{2}{3}, 0\right)\)
   y-intercept: \(0, 8\)

17. x-intercepts: \(-\frac{5}{4}, 0\), \(\frac{5}{4}, 0\)
   y-intercept: \(0, -25\)
SOLVING BY COMPLETING THE SQUARE  5.2.1 – 5.2.3

A quadratic equation in the form \((ax - p)^2 = q\) is in **perfect square form**. There are two solutions to a perfect square equation like \(x^2 = 25\), because \((5)^2 = 25\) and \((-5)^2 = 25\). However, students often forget to consider both solutions, because the radical sign only refers to the principal square root, or the positive square root. Thus, the fact that \(\sqrt{x^2} = |x|\) is used in the solution process.

**Example 1**

\[
(x - 3)^2 = 25
\]
\[
\sqrt{(x - 3)^2} = \sqrt{25}
\]
\[
|x - 3| = 5
\]
\[
x - 3 = 5 \text{ or } x - 3 = -5
\]
\[
x = 8 \text{ or } -2
\]

**Example 2**

\[
(x - 3)^2 = 5
\]
\[
\sqrt{(x - 3)^2} = \sqrt{5}
\]
\[
|x - 3| = \sqrt{5}
\]
\[
x - 3 = \sqrt{5} \text{ or } x - 3 = -\sqrt{5}
\]
\[
x = 3 \pm \sqrt{5}
\]

The process of rewriting a quadratic equation in standard form, \(ax^2 + bx + c = 0\), into perfect square form is called **completing the square**. This process is illustrated in the following examples.

See the Math Notes box in Lesson 5.2.4 for more information.

**Example 1**  Solve \(x^2 + 4x + 4 = 25\).

The left side is already a perfect square trinomial, as demonstrated by the algebra tiles.

Factor the left side to write it as a perfect square.

Then solve the perfect square equation.

Simplify.

\[
(x + 2)(x + 2) = 25
\]
\[
(x + 2)^2 = 25
\]
\[
\sqrt{(x + 2)^2} = \sqrt{25}
\]
\[
|x + 2| = 5
\]
\[
x + 2 = 5 \text{ or } x + 2 = -5
\]
\[
x = 3 \text{ or } x = -7
\]
Example 2  Solve \( x^2 + 8x + 10 = 0 \).

First, rewrite the equation so that the constant term is on the other side of the equal sign.

Make a partial algebra tile square for the left side of the equation.  Then determine how many unit tiles are needed to complete the square.

In this example, 16 unit tiles are needed, so add 16 unit tiles to complete the square.

To keep the equation balanced, add 16 to both sides.

Factor the left side to write it as a perfect square, and solve.

\[
\begin{align*}
(x + 4)^2 &= 6 \\
|x + 4| &= \sqrt{6} \\
x + 4 &= \sqrt{6} \text{ or } x + 4 &= -\sqrt{6} \\
x &= -4 \pm \sqrt{6} \\
x &= -1.55 \text{ or } x = -6.45
\end{align*}
\]

Example 3  Solve \( x^2 + 5x + 2 = 0 \).

Rewrite the equation with the constant term on the other side.

We need to make \( x^2 + 5x \) into a perfect square.

Imagine building a square for \( x^2 + 5x \) using algebra tiles. The five \( x \)-tiles must be equally divided, so the sides of the square will be \( x + 2.5 \) and \( x + 2.5 \).

There should be \( (2.5)^2 = 6.25 \) tiles in the upper right corner to complete the square. Add 6.25, or \( \frac{25}{4} \), to each side of the equation.

Factor the left side to write it as a perfect square, and solve.

\[
\begin{align*}
(x + \frac{5}{2})^2 &= -\frac{8}{4} + \frac{25}{4} \\
\sqrt{(x + \frac{5}{2})^2} &= \sqrt{\frac{17}{4}} \\
|x + \frac{5}{2}| &= \frac{\sqrt{17}}{2} \\
x &= \frac{-5 \pm \sqrt{17}}{2} \approx -0.44, -4.56
\end{align*}
\]
Problems
Solve each quadratic equation by completing the square.

1. \((x + 2)^2 = 3\)
2. \(y^2 - 6y + 9 = 25\)
3. \(x^2 + 2x - 3 = 0\)
4. \(x^2 + 8x = 5\)
5. \(x^2 + 6x + 2 = 0\)
6. \(x^2 + 10x - 75 = 0\)
7. \(x^2 - 6x + 2 = 0\)
8. \(y^2 + 5y = 14\)
9. \(x^2 + 6x + 1 = -10\)
10. \(x^2 - 2x - 3 = 0\)
11. \(x^2 + 4x = 3\)
12. \(x^2 + 3x = 3\)
13. \(x^2 - 3x - 13.75 = 0\)
14. \(x^2 - x - 3 = 0\)
15. \(x^2 - 10x + 12 = 6\)

Answers
1. \(x = -2 \pm \sqrt{3}\)
2. \(x = 8\) or \(-2\)
3. \(x = 1\) or \(-3\)
4. \(x = -4 \pm \sqrt{21}\)
5. \(x = -3 \pm \sqrt{7}\)
6. \(x = 5\) or \(-15\)
7. \(x = 3 \pm \sqrt{7}\)
8. \(x = -7\) or \(2\)
9. no real solution
10. \(x = -1\) or \(3\)
11. \(x = -2 \pm \sqrt{7}\)
12. \(x = \frac{-3 \pm \sqrt{21}}{2}\)
13. \(x = -2.5\) or \(5.5\)
14. \(x = \frac{1 \pm \sqrt{13}}{2}\)
15. \(x = 5 \pm \sqrt{19}\)
When a quadratic equation is not factorable, another method is needed to solve for \( x \). You can always complete the square to solve a quadratic equation, but that can be challenging when the coefficients are large. The Quadratic Formula can be used to solve any quadratic equation, no matter how complicated.

The solution(s) to any quadratic equation \( ax^2 + bx + c = 0 \) are:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

The ± symbol is read as “plus or minus.” It is shorthand notation that tells you to calculate the formula twice, once with + and again with – to get both \( x \)-values.

To use the formula, the quadratic equation must be written in standard form: \( ax^2 + bx + c = 0 \). This is necessary to correctly identify the values of \( a \), \( b \), and \( c \). Once the equation is in standard form and equal to 0, \( a \) is the coefficient of the \( x^2 \)-term, \( b \) is the coefficient of the \( x \)-term and \( c \) is the constant term.

For additional information, see the Math Notes box in Lesson 5.2.5.

**Example 1**

Solve \( 2x^2 - 5x - 3 = 0 \).

Identify \( a \), \( b \), and \( c \). Watch your signs. \( a = 2 \), \( b = -5 \), \( c = -3 \)

Write the Quadratic Formula.

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

Substitute \( a \), \( b \), and \( c \) into the formula and do the initial calculations.

\[
x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-3)}}{2(2)}
\]

\[
x = \frac{5 \pm \sqrt{25 - (-24)}}{4}
\]

\[
x = \frac{5 \pm \sqrt{49}}{4}
\]

Simplify the \( \sqrt{\cdot} \).

\[
x = \frac{5 \pm 7}{4}
\]

Calculate both values of \( x \).

\[
x = \frac{5 + 7}{4} = \frac{12}{4} = 3 \quad \text{or} \quad x = \frac{5 - 7}{4} = \frac{-2}{4} = -\frac{1}{2}
\]

The solutions are \( x = 3 \) or \( x = -\frac{1}{2} \).
Example 2

Solve \(3x^2 + 5x + 1 = 0\).

Identify \(a\), \(b\), and \(c\).

\(a = 3, b = 5, c = 1\)

Write the Quadratic Formula.

\(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\)

Substitute \(a\), \(b\), and \(c\) into the formula and do the initial calculations.

\(x = \frac{-5 \pm \sqrt{25-12}}{6}\)

Simplify the \(\sqrt{}\).

\(x = \frac{-5 \pm \sqrt{13}}{6}\)

The solutions are \(x = \frac{-5 + \sqrt{13}}{6} \approx -0.23\) or \(x = \frac{-5 - \sqrt{13}}{6} \approx -1.43\).

Example 3

Solve \(25x^2 - 20x + 4 = 0\).

Identify \(a\), \(b\), and \(c\).

\(a = 25, b = -20, c = 4\)

Write the Quadratic Formula.

\(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\)

Substitute \(a\), \(b\), and \(c\) into the formula and do the initial calculations.

\(x = \frac{20 \pm \sqrt{400-400}}{50}\)

Simplify the \(\sqrt{}\).

\(x = \frac{20 \pm \sqrt{0}}{50}\)

This quadratic equation has only one solution: \(x = \frac{2}{5}\).
**Example 4**

Solve \( x^2 + 4x = -10 \).

Rewrite the equation in standard form. \( x^2 + 4x + 10 = 0 \)

Identify \( a, b, \) and \( c \). \( a = 1, \ b = 4, \ c = 10 \)

Write the Quadratic Formula. \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)

Substitute \( a, b, \) and \( c \) into the formula and do the initial calculations.

\[ x = \frac{-4 \pm \sqrt{16 - 40}}{2} \]

Simplify the \( \sqrt{ } \).

\[ x = \frac{-4 \pm \sqrt{-24}}{2} \]

There are no real numbers that can be squared to give \(-24\); therefore this quadratic equation has no real solutions.

**Problems**

Solve each equation by using the Quadratic Formula.

1. \( x^2 - x - 2 = 0 \)  
2. \( x^2 - x - 3 = 0 \)  
3. \( -3x^2 + 2x + 1 = 0 \)
4. \(-2 - 2x^2 = 4x \)  
5. \( 7x = 10 - 2x^2 \)  
6. \(-6x^2 - x + 6 = 0 \)
7. \( 6 - 4x + 3x^2 = 8 \)  
8. \( 4x^2 + x - 1 = 0 \)  
9. \( x^2 - 5x + 3 = 0 \)
10. \( 0 = 10x^2 - 2x + 3 \)  
11. \( x(-3x + 5) = 7x - 10 \)  
12. \( (5x + 5)(x - 5) = 7x \)

Identify the error in each of the following solutions. Then write a correct solution to the problem.

13. **Solve:** \( 3x^2 + 6x + 1 = 0 \)

\[ a = 3, \ b = 6, \ c = 1 \]

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ x = \frac{-6 \pm \sqrt{36 - 12}}{6} \]

\[ x = \frac{-6 \pm \sqrt{24}}{6} \]

\[ x = 1 \pm \frac{\sqrt{24}}{6} \]

14. **Solve:** \( -2x^2 + 7x + 5 = 0 \)

\[ a = -2, \ b = 7, \ c = 5 \]

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ x = \frac{-7 \pm \sqrt{49 - 40}}{2(-2)} \]

\[ x = \frac{-7 \pm \sqrt{9}}{4} \]

\[ x = \frac{-7 \pm 3}{4} \]

\[ x = \frac{-4}{4} = 1 \text{ or } x = \frac{-10}{4} = 2.5 \]
Answers

1. $x = 2$ or $-1$

2. $x = \frac{1 + \sqrt{13}}{2}$
   
   $\approx 2.30$ or $-1.30$

3. $x = -\frac{1}{3}$ or $1$

4. $x = -1$

5. $x = \frac{-7 \pm \sqrt{129}}{4}$

   $\approx 1.09$ or $-4.59$

   $\approx -1.09$ or $0.92$

6. $x = \frac{1 \pm \sqrt{145}}{-12}$

7. $x = \frac{4 \pm \sqrt{40}}{6} = \frac{2 \pm \sqrt{10}}{3}$

   $\approx 1.72$ or $-0.39$

8. $x = \frac{-1 \pm \sqrt{17}}{8}$

   $\approx 0.39$ or $-0.64$

   $\approx 4.30$ or $0.70$

9. $x = \frac{5 \pm \sqrt{13}}{2}$

10. no real solution

11. $x = \frac{2 \pm \sqrt{124}}{-6} = \frac{4 \pm \sqrt{31}}{-3}$

    $\approx -2.19$ or $1.52$

    $\approx 6.21$ or $-0.81$

12. $x = \frac{27 \pm \sqrt{1229}}{10}$

13. The formula starts with “$-b$”; the negative sign was left off. $x = -1 \pm \frac{\sqrt{24}}{6}$

14. Under the radical, “$-4ac$” should equal $+40$. $x = \frac{-7 \pm \sqrt{89}}{-4}$
Complex numbers arise when trying to solve some equations such as $x^2 + 1 = 0$, which has no real solution. The equation does, however, have a complex solution.

The imaginary number $i$ is defined to be $\sqrt{-1}$, so $i^2 = -1$. When $i$ is multiplied by a real number, the result is another imaginary number, such as $2i, 3i,$ and $i\sqrt{2}$. When an imaginary number is added to a real number, the result is called a complex number. Complex numbers are written in the form $a + bi$, where $a$ and $b$ are real numbers.

For additional information see the Historical Note and Math Notes box in Lesson 5.2.6.

Example 1

Use the definition of $i$ to simplify each of the following expressions.

a. $3 + \sqrt{-16}$  
b. $(3 + 4i) + (-2 - 6i)$

c. $(4i)(-5i)$  
d. $(8 - 3i)(8 + 3i)$

When simplifying, remember that $i = \sqrt{-1}$ and $i^2 = -1$.

Part (a): $3 + \sqrt{-16} = 3 + \sqrt{16}\sqrt{-1} = 3 + 4i$. This is the simplest form; the real and imaginary parts of the complex number cannot be combined.

Part (b): Combine like terms: $(3 + 4i) + (-2 - 6i) = 1 - 2i$.

Part (c): Use the Commutative Property to rearrange the expression: $(4i)(-5i) = (4)(-5)(i)(i) = -20i^2 = -20(-1) = 20$.

Part (d): Use the Distributive Property or an area model to compute this product.

\[
(8 - 3i)(8 + 3i) = 8(8) + 8(3i) - 3i(8) - 3i(3i) \\
= 64 + 24i - 24i - 9i^2 \\
= 64 - 9(1) \\
= 64 - 9 = 55
\]
Example 2

Solve the equation below using the Quadratic Formula. Explain what the solution tells you about the graph of the related function.

\[2x^2 - 20x + 53 = 0\]

In this example, \(a = 2\), \(b = -20\), and \(c = 53\).

Solution: \[x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(2)(53)}}{2(2)}\]

\[= \frac{20 \pm \sqrt{400 - 424}}{4}\]
\[= \frac{20 \pm \sqrt{-24}}{4}\]
\[= \frac{20 \pm 2\sqrt{6}i}{4}\]
\[= \frac{10 \pm \sqrt{6}}{2}i\]

Because the equation \(0 = 2x^2 - 20x + 53\) has complex solutions, this means that the graph of the related function \(y = 2x^2 - 20x + 53\) does not cross the \(x\)-axis. Verify this with your graphing tool.

Problems

Simplify the following expressions.

1. \((6 + 4i) - (2 - i)\)
2. \(8i - \sqrt{-16}\)
3. \((-3)(4i)(7i)\)

4. \((5 - 7i)(-2 + 3i)\)
5. \((3 + 2i)(3 - 2i)\)
6. \((6 - 5i)(6 + 5i)\)

7. \(\sqrt{-49}\)
8. \((8i)^2\)
9. \((i - 3)^2\)

10. \((3 + 4i) + (7 - 2i)\)
11. \((5i)(2i)^2\)
12. \((4 + 9i)(1 - i)\)

Solve the following quadratic equations.

13. \(0 = 3x^2 + 5x + 4\)
14. \(x^2 + 2x = -5\)
15. \(8 = -x^2 - x\)

16. \(6x^2 + 5x + 3 = 0\)
17. \(-4x = x^2 + 4\)
18. \(2x^2 + 2x + 5 = 0\)
Answers

1. $4 + 5i$
2. $4i$
3. $84$
4. $11 + 29i$
5. $13$
6. $61$
7. $7i$
8. $-64$
9. $8 - 6i$
10. $10 + 2i$
11. $-20i$
12. $13 + 5i$

13. $x = \frac{-5 \pm \sqrt{5^2 - 4(3)(4)}}{2(3)} = \frac{-5 \pm \sqrt{23}}{6}$

14. $x = -1 \pm 2i$
15. $x = \frac{-1 \pm i\sqrt{31}}{2} = \frac{1}{2} \pm \frac{\sqrt{31}}{2}i$
16. $x = \frac{-5 \pm i\sqrt{47}}{12} = \frac{5}{12} \pm \frac{\sqrt{47}}{12}i$

17. $x = -2$
18. $x = \frac{-1 \pm 3i}{2} = -\frac{1}{2} \pm \frac{3}{2}i$