There are two special right triangles that occur often in mathematics: the 30°-60°-90° triangle and the 45°-45°-90° triangle. By AA~, all 30°-60°-90° triangles are similar, and all 45°-45°-90° triangles are similar. Consequently, for each type of triangle, the side lengths are proportional. The sides of these triangles follow these patterns. (To understand and generate these patterns, it can be helpful to think of a 30°-60°-90° triangle as half of an equilateral triangle, and of a 45°-45°-90° triangle as half of a square.)

Another useful tool when working with right triangles is to recognize when side lengths of right triangles are Pythagorean Triples. The lengths 3, 4, and 5 form a Pythagorean Triple because the three side lengths are integers and satisfy the Pythagorean Theorem. The sides of all triangles similar to the 3 : 4 : 5 triangle will have lengths in the same ratio that form Pythagorean Triples (6 : 8 : 10, 9 : 12 : 15, etc.). Another common Pythagorean Triple is 5 : 12 : 13.

For additional information see the Math Notes box in Lesson 6.1.3.

**Example 1**

Use the 30°-60°-90° and 45°-45°-90° triangle patterns to determine the lengths of the unlabeled sides of each triangle below.

a.  

b.  

c.  

d.  

e.  

f.
Part (a): This is a $30^\circ$-$60^\circ$-$90^\circ$ triangle. The pattern shows that the hypotenuse is twice the length of the short leg. Since the short leg has a length of 6, the hypotenuse has a length of 12. The long leg is the length of the short leg times $\sqrt{3}$, so the long leg has a length of $6\sqrt{3}$. (Note: You can also use the Pythagorean Theorem to solve for the last side once you know the lengths of two sides.)

Part (b): This is also a $30^\circ$-$60^\circ$-$90^\circ$ triangle, but this time the length of the hypotenuse is known. Following the pattern, this means the length of the short leg is half the hypotenuse: 7. As before, multiply the length of the short leg by $\sqrt{3}$ to get the length of the long leg: $7\sqrt{3}$.

Part (c): This is a $45^\circ$-$45^\circ$-$90^\circ$ triangle. (You can verify that the missing angle measure is $45^\circ$ by remembering that the sum of the angles of a triangle is $180^\circ$.) The legs of a $45^\circ$-$45^\circ$-$90^\circ$ triangle are equal in length (it is isosceles) so the length of the missing leg is also 5. To calculate the length of the hypotenuse, multiply the length of a leg by $\sqrt{2}$. Therefore the hypotenuse has length $5\sqrt{2}$.

Part (d): This is another $30^\circ$-$60^\circ$-$90^\circ$ triangle. This time the length of the long leg is given. To calculate the length of the short leg, divide the length of the longer leg by $\sqrt{3}$. Therefore, the length of the short leg is 8. To calculate the length of the hypotenuse, double the length of the short leg, so the hypotenuse is 16.

Part (e): This is a $45^\circ$-$45^\circ$-$90^\circ$ triangle with the length of the hypotenuse is given. To determine the lengths of the legs (which are equal in length), divide the length of the hypotenuse by $\sqrt{2}$. Therefore, each leg has length 6.

Part (f): This is another $45^\circ$-$45^\circ$-$90^\circ$ triangle with the length of the hypotenuse given. In the part (e), when given the length of the hypotenuse, we divided by $\sqrt{2}$ to calculate the length of the legs. Do the same thing here.

$$\frac{10}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$$

Note: Multiplying by $\frac{1}{\sqrt{2}}$ is called rationalizing the denominator. It is a technique for removing the radical from the denominator. The Math Notes box in Lesson 6.1.1 describes this process.
Example 2

Use what you know about Pythagorean Triples and similar triangles to fill in the missing lengths of sides below.

There are a few common Pythagorean Triples that students should recognize; 3:4:5, 5:12:13, 8:15:17, and 7:24:25 are the most common. If you forget about a particular triple or do not recognize one, you can always determine the unknown side by using the Pythagorean Theorem if two of the sides are given.

Part (a): This triangle is similar to the 3 : 4 : 5 triangle. The length of the hypotenuse is 500.

Part (b): Each leg has a length that is a multiple of 4. Knowing this, you can rewrite the lengths as 48 = 4(12), and 20 = (4)(5). This is to the 5 : 12 : 13 triangle. The length of the hypotenuse is 4(13) = 52.

Part (c): Do not let the decimal bother you. In fact, since we are working with Pythagorean Triples and their multiples, double both sides to create a similar triangle. This eliminates the decimal and makes a similar triangle with a leg length of 24 and a hypotenuse of 25. This makes it easier to recognize the triple as 7 : 24 : 25. Since the sides of this triangle are half of the 7 : 24 : 25 triangle, the length of the leg corresponding to 7 is 3.5.
Problems
Identify the special triangle relationships. Then solve for $x$, $y$, or both.

1. 
\[ \begin{array}{c}
\text{60°}
\end{array} \]

2. 
\[ \begin{array}{c}
8\sqrt{2}
\end{array} \]

3. 
\[ \begin{array}{c}
12
\end{array} \]

4. 
\[ \begin{array}{c}
1000
\end{array} \]

5. 
\[ \begin{array}{c}
45°
\end{array} \]

6. 
\[ \begin{array}{c}
22.5\sqrt{3}
\end{array} \]

7. 
\[ \begin{array}{c}
30°
\end{array} \]

8. 
\[ \begin{array}{c}
60°
\end{array} \]

9. 
\[ \begin{array}{c}
30
\end{array} \]

10. 
\[ \begin{array}{c}
50
\end{array} \]

Answers
1. \( x = 8\sqrt{3} \), \( y = 8 \)
2. \( x = y = 8 \)
3. \( x = 13 \)
4. \( y = 800 \)
5. \( x = 6, y = 6\sqrt{2} \)
6. \( x = y = \frac{12}{\sqrt{2}} = 6\sqrt{2} \)
7. \( x = 11\sqrt{3}, y = 22 \)
8. \( x = 45, y = 22.5 \)
9. \( x = 34 \)
10. \( x = 48 \)
A fractional exponent is equivalent to an expression with roots or radicals.

For \( x \neq 0 \) and \( n \neq 0 \),

\[
\frac{x^m}{n} = \left(\frac{x^m}{n}\right)^{1/n} = \sqrt[n]{x^m} \quad \text{or} \quad x^{m/n} = (x^{1/n})^m = (\sqrt[n]{x})^m.
\]

Fractional exponents can also be used to solve equations containing exponents. For additional information, see the Math Notes box in Lesson 6.2.1.

### Example 1

Rewrite each expression in radical form and simplify if possible.

a. \(16^{5/4}\)

b. \((-8)^{2/3}\)

**Solution:**

a. \(16^{5/4}\) or \(16^{5/4}\)

\[
= (16^{1/4})^5 = (16^{5/4})^{1/4}
\]

\[
= (\sqrt[4]{16})^5 = (1,048,576)^{1/4}
\]

\[
= (2)^5 = \sqrt[4]{1,048,576}
\]

\[
= 32 = 32
\]

b. \((-8)^{2/3}\) or \((-8)^{2/3}\)

\[
= (-8)^{1/3} \quad \text{or} \quad (-8)^{1/3}
\]

\[
= (\sqrt[3]{-8})^2 = (64)^{1/3}
\]

\[
= (-2)^2 = \sqrt[3]{64}
\]

\[
= 4 = 4
\]

### Example 2

Simplify each expression. Answer should contain no parentheses and no negative exponents.

a. \((144x^{-12})^{1/2}\)

b. \(\left(\frac{8x^7y^3}{x}\right)^{-1/3}\)

**Using the Laws of Exponents:**

a. \((144x^{-12})^{1/2} = \left(\frac{144}{x^{12}}\right)^{1/2}\)

\[
= \sqrt{\frac{144}{x^{12}}} = \frac{12}{x^6}
\]

b. \(\left(\frac{8x^7y^3}{x}\right)^{-1/3} = \left(\frac{x^7y^3}{8x^6y^3}\right)^{1/3}\)

\[
= \left(\frac{1}{8x^6y^3}\right)^{1/3}
\]

\[
= \sqrt[3]{\frac{1}{8x^6y^3}} = \frac{1}{2x^2y}
\]
Problems

Rewrite each expression as at least three different equivalent expressions and then simplify.

1. $(64)^{2/3}$
2. $16^{-1/2}$
3. $(-27)^{1/3}$

Simplify the following expressions. Your final expressions should contain no negative exponents and no parentheses.

4. $\left(\frac{3}{5x}\right)^{-2}$
5. $(36^{1/2}x^4)(16x^3)$
6. $\left(\frac{x^7y^3}{x}\right)^{1/3}$

7. $(16a^8b^{12})^{3/4}$
8. $\frac{144^{1/2}x^{-3}}{(16^{3/4}x^7)^0}$
9. $\frac{a^{2/3}b^{-3/4}c^{7/8}}{a^{-1/3}b^{1/4}c^{1/8}}$

Answers

Example answers are given for problems 1–3; other answers are possible.

1. $\sqrt[3]{64^2}$, $(64^2)^{1/3}$, $(\sqrt[3]{64})^2$, 16
2. $\frac{1}{(16)^{1/2}} \cdot \sqrt[4]{16}$, $\frac{1}{\sqrt[8]{16}}$, $\frac{1}{4}$
3. $\left((-3)^3\right)^{1/3}$, $\sqrt[3]{-1} \cdot \sqrt[3]{27}$, $\sqrt[3]{-27}$, $-3$
4. $\frac{25x^2}{9}$
5. $96x^7$
6. $x^2y$
7. $8a^6b^9$
8. $\frac{12}{x^3}$
9. $\frac{ac^{3/4}}{b}$