Applications of geometry in everyday settings often involve the measures of angles. In this chapter students investigate and prove angle pair relationships and begin a Theorem Graphic Organizer. The graphic organizer helps students record important relationships that they have proven. Students investigate vertical angles (which are congruent), and angle pairs formed by parallel lines cut by a transversal, such corresponding angles, alternate interior angles, and same-side interior angles.

See the Math Notes box in Lesson 1.3.4 for more information about angle pair relationships.

At the end of this section, students use technology to explore the Triangle Inequality, which determines the restrictions on the possible lengths of the third side of a triangle given the lengths of its other two sides. They also observe that in a given triangle, the largest angle is opposite the longest side, and the smallest angle is opposite the shortest side.

Example 1

In each figure below, determine the measures of angles \( a \), \( b \), and/or \( c \). Justify your answers.

a. \[
\begin{align*}
\angle a &\quad \angle b \quad \angle c \\
72^\circ &\quad 90^\circ &\quad 22^\circ
\end{align*}
\]

b. \[
\begin{align*}
\angle b &\quad \angle c \quad \angle a \\
22^\circ &\quad \text{straight angle} &\quad 92^\circ
\end{align*}
\]

c. \[
\begin{align*}
\angle a &\quad \angle b \quad \angle c \\
92^\circ &\quad 92^\circ &\quad 50^\circ
\end{align*}
\]

d. \[
\begin{align*}
\angle a &\quad \angle b \quad \angle c \\
97^\circ &\quad 90^\circ &\quad 90^\circ
\end{align*}
\]

Part (a): The small box at angle \( b \) tells us that angle \( b \) is a right angle, so \( m\angle b = 90^\circ \). The angle labeled \( c \) is a straight angle (it forms a straight line) so \( m\angle c = 180^\circ \). To calculate \( m\angle a \), realize that \( \angle a \) and the \( 72^\circ \) angle are complementary (they sum to \( 90^\circ \)). Therefore, \( m\angle a + 72^\circ = 90^\circ \) which tells us that \( m\angle a = 18^\circ \).
Part (b): First, \( m\angle a \) and the 22° angle are supplementary because they form a straight angle (line), so the sum of their measures is 180°. Subtracting from 180° we see that \( m\angle a = 158° \). Vertical angles are formed when two lines intersect. They are a pair of angles that are opposite (across from) each other where the lines cross. Their angle measures are always equal. Since the 22° angle and \( \angle b \) are a pair of vertical angles, \( m\angle b = 22° \). Similarly, \( \angle a \) and \( \angle c \) are vertical angles, so \( m\angle a = m\angle c = 158° \).

Part (c): This diagram shows two parallel lines (as indicated by the double arrows on the lines) that are intersected by a transversal. \( \angle a \) and the 92° angle are called alternate interior angles, and since the lines are parallel these angles have equal measures. Therefore, \( m\angle a = 92° \).

There are several ways to calculate the remaining angles. One way is to realize that \( \angle a \) and \( \angle b \) are supplementary. Another uses the fact that \( \angle b \) and the 92° angle are same-side interior angles, which makes them supplementary because the lines are parallel. Either way gives the same result: \( m\angle b = 180° - 92° = 88° \). There is also more than one way to calculate \( m\angle c \). \( \angle c \) and \( \angle b \) are supplementary because they form a straight angle. Alternately, \( \angle c \) and the 92° angle are corresponding angles, which have equal measures because the lines are parallel. A third way is to see that \( \angle a \) and \( \angle c \) are vertical angles. With any of these approaches, \( m\angle c = 92° \).

Part (d): The sum of the measures of the three angles of a triangle is always 180°. Therefore \( m\angle a + 50° + 97° = 180° \), so \( m\angle a = 33° \).

Example 2

Can a triangle have sides lengths of:

a. 3, 4, 5? b. 8, 2, 12?

At first, students might think that the lengths of the sides of a triangle can be any three lengths, but that is not so. The Triangle Inequality for three side lengths to form a triangle, the length of any side must be less than the sum of the lengths of the other two sides. For a triangle with the side lengths given in part (a) to exist, all of these inequalities must be true:

\[
5 < 3 + 4 \quad 3 < 4 + 5 \quad 4 < 5 + 3.
\]

Since each inequality is true, a triangle with sides of lengths 3, 4, and 5 is possible.

In part (b) we need to check if:

\[
12 < 8 + 2 \quad 8 < 2 + 12 \quad 2 < 12 + 8.
\]

In this case, only two of the three inequalities are true, namely, the last two. The first inequality is not true so a triangle with side lengths of 8, 2, and 12 is not possible.
Example 3

Longest Side, Largest Angle Conjecture

a. List the angles in order from smallest to largest.

b. List the sides in order from shortest to longest.

In triangles, the longest side is opposite the largest angle and vice versa, while the smallest side is opposite the smallest angle and vice versa.

Part (a): \( \angle b \) is the largest angle since it is opposite the side marked 25 units. \( \angle c \) is the smallest angle as it is opposite the 7 unit side. \( \angle c, \angle a, \angle b \)

Part (b): Side \( c \) is the longest side (it is opposite the 105º angle) and side \( b \) is the shortest side (it is opposite the 25º angle). \( b, a, c \)

Problems

Use the geometric properties and theorems you have learned to solve for \( x \) in each diagram.

1. 2. 3. 4. 5. 6. 7. 8. 9. 10. 11. 12. 13. 14.
Use what you know about angle measures to determine the values of $x$, $y$, or $z$.

15. \[
\begin{align*}
x + 5^\circ & \quad 4x \\
6x - 4^\circ & \quad 4x - 6^\circ
\end{align*}
\]

16. \[
\begin{align*}
x + 13^\circ & \quad 2x + 7^\circ \\
5x & \quad B
\end{align*}
\]

17. \[
\begin{align*}
y & \\
6x - 4^\circ & \quad 4x - 6^\circ
\end{align*}
\]

18. \[
\begin{align*}
x - 7^\circ & \quad 3x - 3^\circ \\
\end{align*}
\]

19. \[
\begin{align*}
z & \\
28^\circ & \quad 100^\circ
\end{align*}
\]

20. \[
\begin{align*}
z & \\
30^\circ & \quad 20^\circ
\end{align*}
\]

21. In the triangle in problem 16, list the side lengths in order from shortest to longest.

Can the triangle have side lengths of:

22. 1, 2, 3?
23. 7, 8, 9?
24. 4.5, 2.5, 6?
25. 9.5, 1.25, 11.75?
Answers

1. \( x = 45^\circ \)  
2. \( x = 40^\circ \)  
3. \( x = 12.5^\circ \)  
4. \( x = 15^\circ \)  
5. \( x = 20^\circ \)  
6. \( x = 3^\circ \)  
7. \( x = 7^\circ \)  
8. \( x = 7^\circ \)  
9. \( x = 81^\circ \)  
10. \( x = 9^\circ \)  
11. \( x = 15.6^\circ \)  
12. \( x = 2^\circ \)  
13. \( x = 65^\circ \)  
14. \( x = 7 \frac{1}{6}^\circ \)  
15. \( x = 35^\circ \)  
16. \( x = 20^\circ \)  
17. \( x = 19^\circ, y = 110^\circ \)  
18. \( x = 25^\circ, y = 90^\circ \)  
19. \( x = 28^\circ, y = 52^\circ, z = 80^\circ \)  
20. \( x = 150^\circ, y = 160^\circ, z = 130^\circ \)  
21. \( \angle A = 33^\circ, \angle B = 100^\circ, \angle C = 47^\circ \) so \( \overline{CB}, \overline{AB}, \overline{AC} \).  
22. no  
23. yes  
24. yes  
25. no