Complex numbers arise when trying to solve some equations such as \( x^2 + 1 = 0 \), which has no real solution. The equation does, however, have a complex solution.

The imaginary number \( i \) is defined to be \( \sqrt{-1} \), so \( i^2 = -1 \). When \( i \) is multiplied by a real number, the result is another imaginary number, such as \( 2i, 3i, \) and \( i\sqrt{2} \). When an imaginary number is added to a real number, the result is called a complex number. Complex numbers are written in the form \( a + bi \), where \( a \) and \( b \) are real numbers.

For additional information see the Historical Note and Math Notes box in Lesson 5.2.6.

Example 1

Use the definition of \( i \) to simplify each of the following expressions.

a. \( 3 + \sqrt{-16} \)

b. \( (3 + 4i) + (-2 - 6i) \)

c. \( (4i)(-5i) \)

d. \( (8 - 3i)(8 + 3i) \)

When simplifying, remember that \( i = \sqrt{-1} \) and \( i^2 = -1 \).

Part (a): \( 3 + \sqrt{-16} = 3 + \sqrt{16}\sqrt{-1} = 3 + 4i \). This is the simplest form; the real and imaginary parts of the complex number cannot be combined.

Part (b): Combine like terms: \( (3 + 4i) + (-2 - 6i) = 1 - 2i \).

Part (c): Use the Commutative Property to rearrange the expression:
\[
(4i)(-5i) = (4)(-5)(i)(i) = -20i^2 = -20(-1) = 20.
\]

Part (d): Use the Distributive Property or an area model to compute this product.

\[
(8 - 3i)(8 + 3i) = 8(8) + 8(3i) - 3i(8) - 3i(3i)
= 64 + 24i - 24i - 9i^2
= 73
\]
Example 2

Solve the equation below using the Quadratic Formula. Explain what the solution tells you about the graph of the related function.

\[2x^2 - 20x + 53 = 0\]

In this example, \(a = 2\), \(b = -20\), and \(c = 53\).

Solution: 

\[
x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(2)(53)}}{2(2)}
\]

\[
= \frac{20 \pm \sqrt{400 - 424}}{4}
\]

\[
= \frac{20 \pm \sqrt{-24}}{4}
\]

\[
= \frac{20 \pm 2i \sqrt{6}}{4}
\]

\[
= \frac{10 \pm i \sqrt{6}}{2}
\]

\[
= 5 \pm \frac{\sqrt{6}}{2}i
\]

Because the equation \(0 = 2x^2 - 20x + 53\) has complex solutions, this means that the graph of the related function \(y = 2x^2 - 20x + 53\) does not cross the \(x\)-axis. Verify this with your graphing tool.

Problems

Simplify the following expressions.

1. \((6 + 4i) - (2 - i)\)
2. \(8i - \sqrt{-16}\)
3. \((-3)(4i)(7i)\)
4. \((5 - 7i)(-2 + 3i)\)
5. \((3 + 2i)(3 - 2i)\)
6. \((6 - 5i)(6 + 5i)\)
7. \(\sqrt{-49}\)
8. \((8i)^2\)
9. \((i - 3)^2\)
10. \((3 + 4i) + (7 - 2i)\)
11. \((5i)(2i)^2\)
12. \((4 + 9i)(1 - i)\)

Solve the following quadratic equations.

13. \(0 = 3x^2 + 5x + 4\)
14. \(x^2 + 2x = -5\)
15. \(8 = -x^2 - x\)
16. \(6x^2 + 5x + 3 = 0\)
17. \(-4x = x^2 + 4\)
18. \(2x^2 + 2x + 5 = 0\)
Answers

1. $4 + 5i$
2. $4i$
3. $84$
4. $11 + 29i$
5. $13$
6. $61$
7. $7i$
8. $-64$
9. $8 - 6i$
10. $10 + 2i$
11. $-20i$
12. $13 + 5i$

13. $x = \frac{-5 \pm \sqrt{5^2 - 4(3)(4)}}{2(3)} = \frac{-5 \pm \sqrt{25}}{6}$

14. $x = -1 \pm 2i$
15. $x = \frac{-1 \pm i\sqrt{31}}{2} = \frac{1}{2} \pm \frac{\sqrt{31}}{2}i$
16. $x = \frac{-5 \pm i\sqrt{47}}{12} = -\frac{5}{12} \pm \frac{\sqrt{47}}{12}i$

17. $x = -2$
18. $x = \frac{-1 \pm 3i}{2} = -\frac{1}{2} \pm \frac{3}{2}i$