# CORE CONNECTIONS INTEGRATED II
## Parent Guide with Extra Practice

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Introduction to the Parent Guide with Extra Practice

Welcome to the Core Connections Integrated II Parent Guide with Extra Practice. The purpose of this guide is to assist you should your child need help with homework or the ideas in the course. We believe all students can be successful in mathematics as long as they are willing to work and ask for help when they need it. We encourage you to contact your child’s teacher if your student has additional questions that this guide or other resources do not answer.

This guide was written to address the major topics in each chapter of the textbook. Each section begins with a title bar and the lesson(s) in the book that it addresses. In many cases the explanation box at the beginning of the section refers you to one or more Math Notes boxes in the student text for additional information about the fundamentals of the idea. Detailed examples follow a summary of the concept or skill and include complete solutions. The examples are similar to the work your child has done in class. Additional problems, with answers, are provided for your child to practice.

There will be some topics that your child understands quickly and some concepts that may take longer to master. The big ideas of the course take time to learn. This means that students are not necessarily expected to master a concept when it is first introduced. When a topic is first introduced in the textbook, there will be several problems to do for practice. Subsequent lessons and homework assignments will continue to practice the concept or skill over weeks and months so that mastery will develop over time.

Practice and discussion are required to understand mathematics. When your child comes to you with a question about a homework problem, often you may simply need to ask your child to read the problem and then ask them what the problem is asking. Reading the problem aloud is often more effective than reading it silently. When you are working problems together, have your child talk about the problems. Then have your child practice on their own.

Below is a list of additional questions to use when working with your child. These questions do not refer to any particular concept or topic. Some questions may or may not be appropriate for some problems.

- What have you been doing in class or during this chapter that might be related to this problem? Let’s look at your notebook, class notes, and Learning Log. Do you have them?
- Were the other members of your team having difficulty with this as well? Can you call your study partner or someone from your study team?
- Have you checked the online homework help (homework.cpm.org)?
- What have you tried? What steps did you take?
- What did not work? Why did it not work?
- Which words are most important? Why? What does this word/phrase tell you?
- What do you know about this part of the problem?
- Explain what you know right now.
- What is unknown? What do you need to know to solve the problem?
- How did the members of your study team explain this problem in class?
- What important examples or ideas were highlighted by your teacher?
- How did you organize your information? Do you have a record of your work?
- Can you draw a diagram or sketch to help you?
- Have you tried making a list, looking for a pattern, etc.?
- What is your estimate/prediction?
- Is there a simpler, similar problem we can do first?
If your student has made a start at the problem, try these questions:

- What do you think comes next? Why?
- What is still left to be done?
- Is that the only possible answer?
- Is that answer reasonable? Are the units correct?
- How could you check your work and your answer?

If you do not seem to be making any progress, you might try these questions.

- Let’s look at your notebook and class notes. Do you have them?
- Were you listening to your team members and teacher in class? What did they say?
- Did you use the class time working on the assignment? Show me what you did.
- Were the other members of your team having difficulty with this as well? Can you call your study partner or someone from your study team?

This is certainly not a complete list; you will probably come up with some of your own questions as you work through the problems with your child. Ask any question at all, even if it seems too simple to you.

To be successful in mathematics, students need to develop the ability to reason mathematically. To do so, students need to think about what they already know and then connect this knowledge to the new ideas they are learning. Many students are not used to the idea that what they learned yesterday or last week will be connected to today’s lesson. Too often students do not have to do much thinking in school because they are usually just told what to do. When students understand that connecting prior learning to new ideas is a normal part of their education, they will be more successful in this mathematics course (and any other course, for that matter). The student’s responsibilities for learning mathematics include the following:

- Actively contributing in whole class and study team work and discussion.
- Completing (or at least attempting) all assigned problems and turning in assignments in a timely manner.
- Checking and correcting problems on assignments (usually with their study partner or study team) based on answers and solutions provided in class and online.
- Asking for help when needed from their study partner, study team, and/or teacher.
- Attempting to provide help when asked by other students.
- Taking notes and using his/her toolkit when recommended by the teacher or the text.
- Keeping a well-organized notebook.
- Not distracting other students from the opportunity to learn.

Assisting your child to understand and accept these responsibilities will help them to be successful in this course, develop mathematical reasoning, and form habits that will help them become a life-long learner.
ADDITIONAL SUPPORT

Consider these additional resources for assisting students with the CPM Educational Program:

- **This Core Connections Integrated II Parent Guide with Extra Practice**
  This booklet can be downloaded free of charge at cpm.org. It can also be purchased at shop.cpm.org.

- **CPM Homework Help website at homework.cpm.org**
  A variety of complete solutions, hints, and answers are provided. Some problems refer back to other similar problems. The homework help is designed to assist students to be able to do the problems but not necessarily do the problems for them.

- **Checkpoints**
  The student text has Checkpoint materials to assist students with skills they should master. The checkpoints are numbered to align with the chapter in the text. For example, the topics in Checkpoint 5A and Checkpoint 5B should be mastered while students complete Chapter 5.

- **Resource Pages**
  The resource pages referred to in the student text can be found at cpm.org.

- **Previous tests**
  Many teachers allow students to examine their own tests from previous chapters in the course. Even if they are not allowed to bring these tests home, a student can learn much by analyzing errors on past tests.

- **Math Notes in the student text**
  The Closure section at the end of each chapter has a list of the Math Notes in that chapter. Note that relevant Math Notes are sometimes found in other chapters than the one currently being studied.

- **Glossary and Index in the student text**

- **“Answers and Support” table**
  The “What Have I Learned” questions (in the Closure section at the end of each chapter) are followed by an “Answers and Support” table that indicates where students can get more help with the problems.

- **After-school assistance**
  Some schools have after-school or at-lunch support programs for students. Ask the teacher.

- **Other students**
  Consider asking your child to obtain the contact information for a couple other students in class.

- **Parent Guides with Extra Practice from previous courses**
  If your student needs help with the concepts from previous courses that are necessary preparation for this class, the Core Connections Integrated I Parent Guides with Extra Practice is available for free download at cpm.org.

Many of these resources can also be found in the student eBook.
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Principles of Counting
Geometric shapes such as polygons occur in many places. In these lessons students look at polygons more closely, noticing similarities and differences among them. They identify certain characteristics of specific polygons and classify polygons using Venn diagrams. Students review geometric vocabulary and notation.

Example 1

Decide which polygons from the Polygon Graphic Organizer belong in each section of the Venn diagram below.

Circle #1 represents all polygons from the Polygon Graphic Organizer that have exactly two pairs of parallel sides. There are four polygons that have this characteristic: the rectangle, the square, the rhombus, and the parallelogram. These polygons will be contained in Circle #1. Circle #2 represents the polygons that are equilateral, that is, polygons that have sides all of the same length. There are five polygons with this characteristic: the regular hexagon, the equilateral triangle, the square, the rhombus, and the regular pentagon. All five of these polygons will be completely contained in Circle #2. There are two polygons that are on both lists: the square and the rhombus. These two polygons are equilateral and they have exactly two pairs of parallel sides. Thus the square and the rhombus must be placed in the region that is in both circles, which is shaded in the diagram. The other nine polygons from the Polygon Graphic Organizer would be placed outside the circles.
Example 2

Based on the markings of each polygon below, name the polygon using the most specific name possible.

a.  

b.  

c.  

In Lesson 1.1.2 students created a Polygon Graphic Organizer, that is, a resource page showing and describing the attributes of many different polygons. Using terms, definitions, and characteristics they had identified, students described the polygons on the resource page and added appropriate markings. Tick marks are used to indicate sides of equal length. Arrowheads are used to indicate parallel segments.

The figure in part (a) appears to be a square. The markings show that the sides of the quadrilateral are equal in length, but to be a square, it also needs four right angles. The angles in the drawing look like right angles, but maybe they are not quite 90°. They could be 89° and 91°, so without the appropriate markings or other information, we cannot assume the angles are right angles. This quadrilateral with four sides of equal length is called a rhombus.

The small box in the corner of the triangle in part (b) tells us that angle is a right angle (measures 90°), so this is a right triangle. A triangle with two sides that are the same length is called an isosceles triangle. Putting both of these facts together, this polygon is an isosceles right triangle.

The arrowheads on the two sides of the quadrilateral in part (c) tell us that those sides are parallel. One pair of parallel sides makes this polygon a trapezoid.
Problems

Place the polygons from your Polygon Graphic Organizer into the appropriate regions of the Venn diagram at right. The conditions that the polygons must meet to be placed in each circle are listed in each problem. Note: Create a new Venn diagram for each problem.

1. Circle #1: Has more than three sides
   Circle #2: Has at least one pair of parallel sides

2. Circle #1: Has fewer than four sides
   Circle #2: Has at least two sides equal in length

3. Circle #1: Is equilateral
   Circle #2: Is equiangular

Each polygon below is missing markings. Add the correct markings so that the polygon represents the name listed. Note: The polygons may not be drawn to scale.

4. A rectangle.

5. A scalene trapezoid.

6. An isosceles right triangle.

7. An equilateral quadrilateral.

Based on the markings, name each polygon below with the most specific name. Note: The polygons are not drawn to scale.

8.

9.

10.
Answers

1. Common to both circles and placed in the overlapping region are: square, rectangle, parallelogram, isosceles trapezoid, trapezoid, right trapezoid, rhombus, and regular hexagon

Only in Circle #1: quadrilateral, kite, and regular pentagon

Only in Circle #2: none

Outside of both circles: scalene triangle, equilateral triangle, isosceles right triangle, isosceles triangle, and scalene right triangle

2. Common to both circles and placed in the overlapping region are: equilateral triangle, isosceles triangle, and isosceles right triangle

Only in Circle #1: scalene triangle and scalene right triangle

Only in Circle #2: square, rectangle, parallelogram, rhombus, kite, regular pentagon, isosceles trapezoid, and regular hexagon

Outside of both circles: quadrilateral, trapezoid, and right trapezoid

3. Common to both circles and placed in the overlapping region are: equilateral triangle, square, regular pentagon, and regular hexagon.

Only in Circle #1: rhombus

Only in Circle #2: rectangle

Outside of both circles: isosceles right triangle, isosceles triangle, scalene right triangle, scalene triangle, parallelogram, quadrilateral, kite, isosceles trapezoid, right trapezoid, trapezoid

4. A parallelogram

5. An isosceles triangle

6. An isosceles trapezoid
Two ways to determine the area of a rectangle are: as the product of its (height) and (base) or as the sum of the areas of individual pieces of the rectangle. For a given rectangle these two areas must be the same, so area as a product = area as a sum. Algebra tiles, and later, area models help students multiply expressions in a visual, concrete manner.

For additional information, see the Math Notes box in Lesson 1.3.1.

Example 1: Using Algebra Tiles

The algebra tile pieces \(x^2 + 6x + 8\) are arranged into a rectangle as shown at right. The area of the rectangle can be written as the product of its base and height or as the sum of its parts.

\[
\begin{align*}
\frac{(x+4)(x+2)}{\text{base} \quad \text{height}} &= \frac{x^2 + 6x + 8}{\text{area}} \\
\text{area as a product} &= \text{area as a sum}
\end{align*}
\]

Example 2: Using Area Models

An area model allows the problem to be organized in the same way as the first example, but without drawing the individual tiles. The rectangle does not have to be drawn accurately or to scale.

Multiply \( (2x+1)(x-3) \).

\[
\begin{array}{c|c|c|c|c}
-3 & -3 & -6x & -3 \\
\hline
x & 2x^2 & x & \\
\hline
2x & +1 & & \\
\hline
\end{array}
\Rightarrow (2x + 1)(x - 3) = 2x^2 - 5x - 3
\]

Area as a product Area as a sum

Note that the factors can be written in either order, so \((x - 3)(2x + 1) = 2x^2 - 5x - 3\) is alternate way to write the equation.
Problems

Write an equation showing area as a product equals area as a sum.

1. \[
\begin{array}{|c|c|c|}
\hline
x^2 & 3x \\
\hline
-5x & -15 \\
\hline
\end{array}
\]

5. \[
\begin{array}{|c|c|}
\hline
18y & -12x \\
\hline
3y & -2x \\
\hline
\end{array}
\]

Multiply.

7. \[(3x + 2)(2x + 7)\]
8. \[(2x - 1)(3x + 1)\]
9. \[(2x)(x - 1)\]
10. \[(2y - 1)(4y + 7)\]
11. \[(y - 4)(y + 4)\]
12. \[(y)(x - 1)\]
13. \[(3x - 1)(x + 2)\]
14. \[(2y - 5)(y + 4)\]
15. \[(3y)(x - y)\]
16. \[(3x - 5)(3x + 5)\]
17. \[(4x + 1)^2\]
18. \[(x + y)(x + 2)\]
19. \[(2y - 3)^2\]
20. \[(x - 1)(x + y + 1)\]
21. \[(x + 2)(x + y - 2)\]

Answers

1. \[(x + 1)(x + 3) = x^2 + 4x + 3\]
2. \[(x + 2)(2x + 1) = 2x^2 + 5x + 2\]
3. \[(x + 2)(2x + 3) = 2x^2 + 7x + 6\]
4. \[(x - 5)(x + 3) = x^2 - 2x - 15\]
5. \[6(3y - 2x) = 18y - 12x\]
6. \[(x + 4)(3y - 2) = 3xy - 2x + 12y - 8\]
7. \[6x^2 + 25x + 14\]
8. \[6x^2 - x - 1\]
9. \[2x^2 - 2x\]
10. \[8y^2 + 10y - 7\]
11. \[y^2 - 16\]
12. \[xy - y\]
13. \[3x^2 + 5x - 2\]
14. \[2y^2 + 3y - 20\]
15. \[3xy - 3y^2\]
16. \[9x^2 - 25\]
17. \[16x^2 + 8x + 1\]
18. \[x^2 + 2x + xy + 2y\]
19. \[4y^2 - 12y + 9\]
20. \[x^2 + xy - y - 1\]
21. \[x^2 + xy + 2y - 4\]
Applications of geometry in everyday settings often involve the measures of angles. In this chapter students investigate and prove angle pair relationships and begin a Theorem Graphic Organizer. The graphic organizer helps students record important relationships that they have proven. Students investigate vertical angles (which are congruent), and angle pairs formed by parallel lines cut by a transversal, such corresponding angles, alternate interior angles, and same-side interior angles.

See the Math Notes box in Lesson 1.3.4 for more information about angle pair relationships.

At the end of this section, students use technology to explore the Triangle Inequality, which determines the restrictions on the possible lengths of the third side of a triangle given the lengths of its other two sides. They also observe that in a given triangle, the largest angle is opposite the longest side, and the smallest angle is opposite the shortest side.

Example 1

In each figure below, determine the measures of angles $a$, $b$, and/or $c$. Justify your answers.

a. 

![Diagram](image)

Part (a): The small box at angle $b$ tells us that angle $b$ is a right angle, so $m\angle b = 90^\circ$. The angle labeled $c$ is a straight angle (it forms a straight line) so $m\angle c = 180^\circ$. To calculate $m\angle a$, realize that $\angle a$ and the $72^\circ$ angle are complementary (they sum to $90^\circ$). Therefore, $m\angle a + 72^\circ = 90^\circ$ which tells us that $m\angle a = 18^\circ$. 

b. 

![Diagram](image)

c. 

![Diagram](image)

d. 

![Diagram](image)
Part (b): First, $m\angle a$ and the $22^\circ$ angle are supplementary because they form a straight angle (line), so the sum of their measures is $180^\circ$. Subtracting from $180^\circ$ we see that $m\angle a = 158^\circ$. Vertical angles are formed when two lines intersect. They are a pair of angles that are opposite (across from) each other where the lines cross. Their angle measures are always equal. Since the $22^\circ$ angle and $\angle b$ are a pair of vertical angles, $m\angle b = 22^\circ$. Similarly, $\angle a$ and $\angle c$ are vertical angles, so $m\angle a = m\angle c = 158^\circ$.

Part (c): This diagram shows two parallel lines (as indicated by the double arrows on the lines) that are intersected by a transversal. $\angle a$ and the $92^\circ$ angle are called alternate interior angles, and since the lines are parallel these angles have equal measures. Therefore, $m\angle a = 92^\circ$.

There are several ways to calculate the remaining angles. One way is to realize that $\angle a$ and $\angle b$ are supplementary. Another uses the fact that $\angle b$ and the $92^\circ$ angle are same-side interior angles, which makes them supplementary because the lines are parallel. Either way gives the same result: $m\angle b = 180^\circ - 92^\circ = 88^\circ$. There is also more than one way to calculate $m\angle c$. $\angle c$ and $\angle b$ are supplementary because they form a straight angle. Alternately, $\angle c$ and the $92^\circ$ angle are corresponding angles, which have equal measures because the lines are parallel. A third way is to see that $\angle a$ and $\angle c$ are vertical angles. With any of these approaches, $m\angle c = 92^\circ$.

Part (d): The sum of the measures of the three angles of a triangle is always $180^\circ$. Therefore $m\angle a + 50^\circ + 97^\circ = 180^\circ$, so $m\angle a = 33^\circ$.

**Example 2**

Can a triangle have sides lengths of:

- a. 3, 4, 5?
- b. 8, 2, 12?

At first, students might think that the lengths of the sides of a triangle can be any three lengths, but that is not so. The Triangle Inequality for three side lengths to form a triangle, the length of any side must be less than the sum of the lengths of the other two sides. For a triangle with the side lengths given in part (a) to exist, all of these inequalities must be true:

$$5 < 3 + 4, \ 3 < 4 + 5, \ \text{and} \ 4 < 5 + 3.$$ 

Since each inequality is true, a triangle with sides lengths 3, 4, and 5 is possible.

In part (b) we need to check if:

$$12 < 8 + 2, \ 8 < 2 + 12, \ \text{and} \ 2 < 12 + 8.$$ 

In this case, only two of the three inequalities are true, namely, the last two. The first inequality is not true so a triangle with side lengths of 8, 2, and 12 is not possible.
Example 3

Longest Side, Largest Angle Conjecture

a. List the angles in order from smallest to largest.

\[
\begin{align*}
\angle b & \text{ is the largest angle since it is opposite the side marked 25 units.} \\
\angle c & \text{ is the smallest angle as it is opposite the 7 unit side.} \\
\end{align*}
\]

Part (a): \( \angle b \) is the largest angle since it is opposite the side marked 25 units. \( \angle c \) is the smallest angle as it is opposite the 7 unit side. \( \angle c, \angle a, \angle b \)

b. List the sides in order from shortest to longest.

Part (b): Side \( c \) is the longest side (it is opposite the 105° angle) and side \( b \) is the shortest side (it is opposite the 25° angle). \( b, a, c \)

Problems

Use the geometric properties and theorems you have learned to solve for \( x \) in each diagram.

1. \[
\begin{align*}
\text{60°} & \\
\text{75°} & \quad \text{x} \\
\end{align*}
\]

2. \[
\begin{align*}
\text{100°} & \\
\x & \quad \x \\
\end{align*}
\]

3. \[
\begin{align*}
\text{60°} & \\
\text{4x + 10°} & \quad \x \\
\end{align*}
\]

4. \[
\begin{align*}
\text{45°} & \\
\text{3x} & \quad \x \\
\end{align*}
\]

5. \[
\begin{align*}
\text{68°} & \\
\text{5x + 12°} & \quad \x \\
\end{align*}
\]

6. \[
\begin{align*}
\text{30°} & \\
\text{19x + 3°} & \quad \x \\
\end{align*}
\]

7. \[
\begin{align*}
\text{142°} & \\
\text{20x + 2°} & \quad \text{38°} \\
\end{align*}
\]

8. \[
\begin{align*}
\text{128°} & \\
\text{7x + 3°} & \quad \text{32°} \\
\end{align*}
\]

9. \[
\begin{align*}
\text{23°} & \\
\text{58°} & \quad \x \\
\end{align*}
\]

10. \[
\begin{align*}
\text{18°} & \\
\text{5x + 36°} & \quad \text{9x} \\
\end{align*}
\]

11. \[
\begin{align*}
\text{5x - 18°} & \\
\text{9x} & \quad \x \\
\end{align*}
\]

12. \[
\begin{align*}
\text{13x + 2°} & \\
\text{15x - 2°} & \quad \x \\
\end{align*}
\]

13. \[
\begin{align*}
\text{45°} & \\
\text{2x + 5°} & \quad \x \\
\end{align*}
\]

14. \[
\begin{align*}
\text{7x - 4°} & \\
\text{5x + 8°} & \quad \x \\
\end{align*}
\]
Use what you know about angle measures to determine the values of $x$, $y$, or $z$.

15. $x + 5^\circ$
   $4x$

16. $x + 13^\circ$
   $2x + 7^\circ$
   $5x$

17. $y$
   $6x - 4^\circ$
   $4x - 6^\circ$

18. $x - 7^\circ$
   $3x - 3^\circ$
   $y$

19. $z$
   $28^\circ$
   $100^\circ$

20. $z$
   $30^\circ$
   $20^\circ$

21. In the triangle in problem 16, list the side lengths in order from shortest to longest.

Can the triangle have side lengths of:

22. 1, 2, 3?  
23. 7, 8, 9?

24. 4.5, 2.5, 6?  
25. 9.5, 1.25, 11.75?
Answers

1. \( x = 45^\circ \)  
2. \( x = 40^\circ \)  
3. \( x = 12.5^\circ \)  
4. \( x = 15^\circ \)

5. \( x = 20^\circ \)  
6. \( x = 3^\circ \)  
7. \( x = 7^\circ \)  
8. \( x = 7^\circ \)

9. \( x = 81^\circ \)  
10. \( x = 9^\circ \)  
11. \( x = 15.6^\circ \)  
12. \( x = 2^\circ \)

13. \( x = 65^\circ \)  
14. \( x = 7 \frac{1}{6}^\circ \)  
15. \( x = 35^\circ \)  
16. \( x = 20^\circ \)

17. \( x = 19^\circ, y = 110^\circ \)  
18. \( x = 25^\circ, y = 90^\circ \)

19. \( x = 28^\circ, y = 52^\circ, z = 80^\circ \)  
20. \( x = 150^\circ, y = 160^\circ, z = 130^\circ \)

21. \( \angle A = 33^\circ, \angle B = 100^\circ, \angle C = 47^\circ \), so \( \overline{CB}, \overline{AB}, \overline{AC} \).

22. no  
23. yes  
24. yes  
25. no
Two triangles are congruent if there is a sequence of rigid transformations that carries one onto the other. Two triangles are also congruent if all three pairs of corresponding angles and all three pairs of corresponding sides are congruent. In these lessons, students review ways to prove triangles congruent in fewer steps, by using five triangle congruence theorems. They are SSS ≅, ASA ≅, AAS ≅, SAS ≅, and HL ≅, illustrated below.

Note: “S” stands for “side” and “A” stands for “angle.” HL ≅ is only used with right triangles. The “H” stands for “hypotenuse” and the “L” stands for leg. The pattern for HL in the diagram appears to be “SSA”, but this arrangement is NOT one of the theorems, because SSA is only valid for right triangles.

See the Math Notes box in Lesson 2.1.1.
Example 1

Determine whether or not each pair of triangles must be congruent. Base each decision on the markings, not on appearances. Justify each answer with a triangle congruence theorem.

a. In part (a), the triangles are congruent by the SAS $\equiv$ theorem. The triangles are also congruent in part (b), this time by the SSS $\equiv$ theorem. In part (c), the triangles are congruent by the AAS $\equiv$ theorem. Part (d) shows a pair of triangles that are not necessarily congruent. The first triangle displays an ASA arrangement, while the second triangle displays an AAS arrangement. The triangles could still be congruent, but based on the given markings, we cannot conclude that they definitely are congruent. The triangles in part (e) are right triangles and the markings fit the HL $\equiv$ theorem. Lastly, in part (f), the triangles are congruent by the ASA $\equiv$ theorem.
Example 2

Using the information given in the diagram at right, decide if the triangles are congruent. If you claim the triangles are congruent, create a flowchart justifying your answer.

\[ \triangle ABD \cong \triangle CBD \] by the SAS \( \cong \) theorem. Note: If you only see “SA,” observe that \( BD \) is congruent to itself. The Reflexive Property justifies stating that something is equal or congruent to itself.

Problems

Decide if each of the following pairs of triangles must be congruent. If so, state the triangle congruence theorem that supports your conclusion.

1. 
2. 
3. 

4. 
5. 
6. 

7. 
8. 
9. 

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Use the triangle congruence theorems to decide whether or not each pair of triangles must be congruent. Base your decision on the markings, not on appearances. Justify your answer.
In each diagram below, are any triangles congruent? If so, prove it.

26. 

27. 

28. 

29. 

30. 

31. 

Answers

1. \(\triangle ABC \cong \triangle DEF\) by ASA
2. \(\triangle GIH \cong \triangle LJK\) by SAS
3. \(\triangle PNM \cong \triangle PNO\) by SSS
4. \(\overline{QS} \cong \overline{QS}\), so \(\triangle QRS \cong \triangle QTS\) by HL
5. The triangles are not necessarily congruent.
6. \(\triangle ABC \cong \triangle DFE\) by ASA or AAS
7. \(\overline{GI} \cong \overline{GI}\), so \(\triangle GHI \cong \triangle JIG\) by SSS
8. \(\parallel\) lines \(\Rightarrow\) alternate interior angles \(\cong\) can be used twice, so \(\triangle KLN \cong \triangle NMK\) by ASA
9. Vertical angles \(\cong\), so \(\triangle POQ \cong \triangle ROS\) by SAS
10. Vertical angles and/or \(\parallel\) lines \(\Rightarrow\) alternate interior angles \(\cong\), so \(\triangle TUX \cong \triangle VWX\) by ASA or AAS
11. No, the lengths of the hypotenuses of the triangles are different.
12. By the Pythagorean Theorem, \(EG = IH\) and \(EH = IG\), so \(\triangle EGH \cong \triangle JHG\) by SSS
13. Sum of angles of triangle = 180°, but since the congruent angles do not correspond, the triangles are not congruent.
14. \(AF + FC = FC + CD\), so \(\triangle ABC \cong \triangle DEF\) by SSS
15. \(\overline{XZ} \cong \overline{XZ}\), so \(\triangle WXZ \cong \triangle YXZ\) by AAS
16. \(\triangle ABC \cong \triangle EDC\) by AAS
17. \(\triangle PQS \cong \triangle PRS\) by AAS or HL \(\equiv\) with \(\overline{PS} \cong \overline{PS}\) by the Reflexive Property.
18. $\triangle VXW \cong \triangle ZXY$ by ASA $\cong$, with $\angle VXW \cong \angle ZXY$ because vertical angles are $\cong$.

19. $\triangle TEA \cong \triangle SAE$ by SSS $\cong$, with $\overline{EA} \cong \overline{EA}$ by the Reflexive Property.

20. $\triangle KLB \cong \triangle EBL$ by HL $\cong$, with $\overline{BL} \cong \overline{BL}$ by the Reflexive Property.


26. Yes

27. Yes

28. Yes

29. Yes

30. Not necessarily.

Counterexample:

31. Yes
CONVERSES 2.1.3 and 2.1.4

A conditional statement is a sentence in “If …, then …” form. “If all of the sides are equal in length, then the triangle is equilateral” is an example of a conditional statement. Conditional statements can be abbreviated by creating an arrow diagram. When the clause after the “if” in a conditional statement (called the hypothesis) changes places with the clause after the “then” (called the conclusion), the new statement is called the converse of the original. If the conditional statement is true, the converse is not necessarily true, and vice versa.

In Lesson 2.1.4 students use what they have learned about angle measures to create proofs by contradiction. For additional information see the Math Notes box in Lesson 2.2.1.

Example 1

Read each conditional statement below. Write it as an arrow diagram and state whether or not it is true. Then write the converse of the statement and state whether or not the converse is true.

a. If a triangle is equilateral, then it is equiangular.

b. If \( x = 4 \), then \( x^2 = 16 \).

c. If \( ABCD \) is a square, then \( ABCD \) is a parallelogram.

Part (a): \( \Delta \) is equilateral \( \Rightarrow \) \( \Delta \) is equiangular

The converse is: If a triangle is equiangular, then it is equilateral. This statement and the original conditional statement are both true.

In part (b), the conditional statement is true and the arrow diagram is: \( x = 4 \) \( \Rightarrow \) \( x^2 = 16 \).

The converse of this statement, “If \( x^2 = 16 \), then \( x = 4 \),” is not necessarily true because \( x \) could equal \(-4\).

In part (c), the arrow diagram is: \( ABCD \) is a square \( \Rightarrow \) \( ABCD \) is a parallelogram.

This statement is true, but the converse, “If \( ABCD \) is a parallelogram, then \( ABCD \) is a square,” is not necessarily true. Any parallelogram that does not have four equal sides or four right angles is not a square.
Problems

Rewrite each conditional statement below as an arrow diagram and state whether or not it is true. Then write the converse of the statement and state whether or not the converse is true.

1. If an angle is a straight angle, then the angle measures 180°.

2. If a triangle is a right triangle, then the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.

3. If the measures of two angles of one triangle are equal to the measures of two angles of another triangle, then the measures of the third angles are also equal.

4. If one angle of a quadrilateral is a right angle, then the quadrilateral is a rectangle.

5. If two angles of a triangle have equal measures, then the two sides of the triangle opposite those angles have equal length.

Use the method of proof by contradiction to justify each of your conclusions to problems 6 and 7 below.

6. Nik scored 40 points lower than Tess on their last math test. The scores could range from 0 to 100 points. Could Tess have scored a 30 on this test? Justify using a proof by contradiction.

7. Jolie claims that a triangle can have two right angles. Prove her wrong! Justify your answer with a proof by contradiction.

Answers

1. Conditional: True

   ![Diagram](image)

   Converse: If an angle measures 180°, then it is a straight angle. True.

2. Conditional: True

   ![Diagram](image)

   Converse: If the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse, then the triangle is a right triangle. True.
3. Conditional: True

Converse: If the measures of one pair of corresponding angles of two triangles are equal, then the measures of the two other pairs of corresponding angles are also equal. False.

4. Conditional: False

Converse: If a quadrilateral is a rectangle, then one angle is a right angle. True, in fact, all four angles are right angles.

5. Conditional: True

Converse: If two sides of a triangle are equal in length, then the two angles opposite those sides are equal in measure. True.

6. Assume that Tess scored 30 points. Then Nik’s score was 30 – 40 = –10, which is impossible. So Tess cannot have a score of 30 points.

7. Assume that a triangle has two right angles. Using the Triangle Sum Theorem, the measure of the third angle must be zero. However, this is impossible, so a triangle cannot have two right angles. OR: If a triangle has two 90° angles, the two sides that intersect with the side between them would be parallel and never meet to complete the triangle, as shown in the figure.
SIMILARITY 2.2.1 and 2.2.2

In this section students focus on comparing geometric shapes. They begin by dilating shapes: enlarging them as one might on a copy machine. When students compare the original and enlarged shapes closely, they discover that the shape of the figure remains exactly the same (this means the angle measures of the enlarged figure are equal to those of the original figure), but the size changes (the lengths of the sides increase). Although the size changes, the lengths of the corresponding sides all have a constant ratio, known as the scale factor, or ratio of similarity.

See the Math Notes boxes in Lessons 2.2.2 and 2.3.1 for more information about dilations and similar figures.

Example 1

Enlarge the figure at right from the origin by a factor of 3.

Students used rubber bands to create a dilation (enlargement) of several shapes. We can do this using a grid and slope triangles. Create a right triangle so that the segment from the origin to point A (2, 4), is the hypotenuse, one leg lies on the positive x-axis, and the other leg connects point A to the endpoint of the first leg at (2, 0). This triangle is called a slope triangle since it represents the slope of the hypotenuse from (0, 0) to vertex A. Add two more slope triangles exactly like this one along the ray from (0, 0) through point A as shown in the figure at right. Using three triangles creates an enlargement by a factor of 3 and gives us the new point A’ at (6, 12). Repeat this process for the other two vertices, forming a new slope triangle for each vertex.

This will give us new points B’ at (12, –6) and C’ at (–12, –12). Connecting points A’, B’, and C’, we form a new triangle that is an enlargement of the original triangle by a factor of 3, as shown at left. Notice that the sides of the dilated triangle are parallel to the sides of the original triangle.
Example 2

The two quadrilaterals at right are similar. What parts are equal? Can you determine the lengths of any of the unlabeled sides?

Since the quadrilaterals are similar, we know that all the corresponding angles have the same measure. This means that \( m\angle A = m\angle A' \), \( m\angle B = m\angle B' \), \( m\angle C = m\angle C' \), and \( m\angle D = m\angle D' \). In addition, the corresponding sides are proportional, which means the ratio of corresponding sides is a constant. To determine the ratio, we need to know the lengths of one pair of corresponding sides. From the diagram we see that \( \overrightarrow{AD} \) corresponds to \( \overrightarrow{A'D'} \). Since these sides correspond, we can write \( \frac{\overrightarrow{AD}}{\overrightarrow{A'D'}} = \frac{4}{6} \).

Therefore, the ratio of similarity is \( \frac{4}{6} \), or \( \frac{2}{3} \). We can use this value to calculate the length of another side if we know the length of its corresponding side.

\[
\begin{align*}
\frac{AB}{A'B'} &= \frac{4}{6} \\
\frac{BC}{B'C'} &= \frac{4}{6} \\
\frac{CD}{C'D'} &= \frac{4}{6}
\end{align*}
\]

Example 3

The pair of shapes at right is similar \((ABCDEF \sim UVWXYZ)\). Label the second figure correctly to reflect the similarity statement. Assume the second figure is drawn to scale.

Since the polygons are similar, this means that their corresponding angles have equal measure. When we write a similarity statement, we write the letters so that the corresponding angles match up. By the similarity statement, we must have \( m\angle A = m\angle U \), \( m\angle B = m\angle V \), \( m\angle C = m\angle W \), \( m\angle D = m\angle X \), \( m\angle E = m\angle Y \), and \( m\angle F = m\angle Z \).

The smaller figure is labeled at right. If it is difficult to tell which original angle corresponds to its enlargement or reduction, try rotating the figures so that they have the same orientation.
Problems

1. Copy the figure below onto graph paper and then enlarge it by a factor of 2.

2. Create a figure similar to the one below with a scale factor of 0.5.

For each pair of similar figures below, determine the ratio of similarity for large:small. Diagrams are drawn roughly to scale; you can assume that sides that look longer are longer.

3.

4.

5.

6.

7.

8.

For each pair of similar figures, state the ratio of similarity. Then use it to calculate the value of $x$. Diagrams are drawn roughly to scale; you can assume that sides that look longer are longer.

9.

10.

11.

12.

13.

14.
15. The shadow of a statue is 20 feet long, while the shadow of a student is 4 ft long. If the student is 6 ft tall, how tall is the statue?

Each pair of figures below is similar. Use what you know about similarity to solve for \(x\). Diagrams are drawn roughly to scale; you can assume that sides that look longer are longer.

16. \[ \begin{array}{c}
\text{\includegraphics[width=2in]{triangle.png}} \end{array} \]

17. \[ \begin{array}{c}
\text{\includegraphics[width=2in]{rectangle.png}} \end{array} \]

18. \[ \begin{array}{c}
\text{\includegraphics[width=2in]{shapes.png}} \end{array} \]

19. \[ \begin{array}{c}
\text{\includegraphics[width=2in]{shapes2.png}} \end{array} \]

Solve for the missing lengths in the pairs of similar figures below.

20. \( \triangle ABC \sim \triangle PQR \)

21. \( \triangle JKL \sim \triangle WXY \)

22. \( \triangle STU \sim \triangle MNP \)

23. \( \triangle DAV \sim \triangle ISW \)
24. $ABCDE \sim FGHIJ$

25. $\Delta ABC \sim \Delta DBE$

Answers

1. 

2. 

3. $\frac{4}{3}$

4. $\frac{5}{1}$

5. $\frac{2}{1}$

6. $\frac{24}{7}$

7. $\frac{6}{1}$

8. $\frac{15}{8}$

9. $\frac{7}{8} \; ; \; x = 32$

10. $\frac{2}{1} \; ; \; x = 72$

11. $\frac{1}{3} \; ; \; x = 15$

12. $\frac{5}{6} \; ; \; x = 15$

13. $\frac{4}{5} \; ; \; x = 20$

14. $\frac{3}{2} \; ; \; x = 16.5$

15. 30 ft

16. $x = 12$

17. $x = 9$

18. $x = 0.8$

19. $x = \frac{40}{3} \approx 13.33$

20. $x = 7.5$

21. $x = 1.25$

22. $x = 16$

23. $x = 3.7$

24. $x = 13.5$

25. $x = 12$
CONDITIONS FOR TRIANGLE SIMILARITY 2.3.1 – 2.3.4

When two figures are related by a series of transformations (including dilations), they are similar. Another way to check for similarity is to measure all the angles and sides of two figures. In this section students develop conditions to shorten the process. These are the AA Triangle Similarity Condition (AA ~), the SAS Triangle Similarity Condition (SAS ~), and the SSS Triangle Similarity Condition (SSS ~). The first condition states that if two pairs of corresponding angles are congruent, then the triangles are similar. The second condition states that if two pairs of corresponding side lengths have the same ratio, and their included angles are congruent, then the triangles are similar. The third condition states that if all three pairs of corresponding side lengths have the same ratio, then the triangles are similar. Additionally, students find that if similar figures have a ratio of similarity of 1, then the shapes are congruent, that is, they have the same size and shape. Students use flowcharts in this section to help organize their information and make logical conclusions about similar triangles. Now students are able to use similar triangles to determine side lengths, perimeters, heights, and other measurements.

See the Math Notes box in Lesson 2.3.2 for more information about similar triangles.

Example 1

Based on the given information, is each pair of triangles similar? If they are similar, write the similarity statement. Justify your answer completely.

a. 

b. 

c. 

d. 

e. 

f. 

We will use the three similarity conditions to test whether or not the triangles are similar.

In part (a), we have the lengths of the three sides, so it makes sense to check whether the SSS ~ holds true. Write the ratios of the corresponding side lengths and compare them to see if they are the same, as shown at right. Each ratio reduces to 3, so they are equal. Therefore, \( \triangle TES \sim \triangle AWK \) by SSS ~.

The measurements given in part (b) suggest we look at SAS ~. \( \angle A \) and \( \angle R \) are the included angles. Since they are both right angles, they have equal measures. Now we need to check that the corresponding sides lengths have the same ratio, as shown at right.

Although the triangles display the SAS ~ pattern and the included angles have equal measures, the triangles are not similar because the corresponding side lengths do not have the same ratio.

In part (c), we are given the measures of two angles of each triangle, but not corresponding angles. \( m\angle K = 55^\circ = m\angle N \) which is one pair of corresponding angles. For AA ~, we need two pairs of congruent angles. If we use the fact that the measures of the three angles of a triangle add up to 180\(^\circ\), we can calculate the measures of \( \angle O \) and \( \angle E \) as shown at right. Now we see that all pairs of corresponding angles have equal measures, so \( \triangle POK \sim \triangle EMN \) by AA ~.

Part (d) shows the SAS ~ pattern and we can see that the included angles have equal measures, \( m\angle G = m\angle H \). We also need the ratio of the corresponding side lengths to be equal. Since the two fractions are equal (the second reduces to the first), the corresponding side lengths have the same ratio. Therefore, \( \triangle YUG \sim \triangle IOH \) by SAS ~.

In part (e), we see that the included angles have equal measures, \( m\angle B = m\angle N \). Since \( \frac{45}{15} = \frac{9}{3} = \frac{3}{1} \), the corresponding sides are proportional. Therefore, \( \triangle BOX \sim \triangle NTE \) by SAS ~.

In part (f), we only have one pair of angles that are congruent (the right angles), but those angles are not between the sides with known lengths. However, we can calculate the lengths of the third sides using the Pythagorean Theorem.

\[
8^2 + (IL)^2 = 10^2 \\
64 + (IL)^2 = 100 \\
(IL)^2 = 36 \\
IL = 6
\]

\[
12^2 = (AB)^2 = 20^2 \\
144 + (AB)^2 = 400 \\
(AB)^2 = 256 \\
AB = 16
\]

Now that we know all three sides, we can check to see if the triangles are similar by SSS ~. Since the ratios of the corresponding sides are the same, \( \triangle ELI \sim \triangle BZA \) by SSS ~.
Example 2

Using the information given in the diagram at right, decide if the triangles are similar or congruent. If you claim the triangles are similar or congruent, create a flowchart justifying your answer.

\[ \triangle WXV \sim \triangle ZYV \] by the AA ~ theorem. The triangles are not necessarily congruent; they could be congruent, but since we only have information about angles, we cannot conclude anything else.

There is more than one way to justify this conclusion. There is another pair of alternate interior angles (\( \angle X \) and \( \angle Y \)) that are congruent that we could have used rather than the vertical angles, or we could have used that pair along with the vertical angles.

Example 3

In the diagram at right, \( \overline{AY} \parallel \overline{HP} \). Decide whether or not there are any similar triangles in the figure. Justify your answer with a flowchart.

Can you determine the length of \( \overline{AY} \)? Justify your answer.

Recalling information we studied in earlier chapters, the parallel lines give us congruent angles. In this figure, we have two pairs of corresponding angles that are congruent: \( \angle PHR \cong \angle YAR \) and \( \angle HPR \cong \angle AYR \). Because two pairs of corresponding angles are congruent, we know the triangles are similar: \( \triangle PHR \sim \triangle YAR \) by AA ~. Since the triangles are similar, the lengths of corresponding sides are proportional (i.e., have the same ratio). This means we can write the solution at right.

We can justify this result with a flowchart as well. The flowchart at right organizes and states what is written above.
Problems

Each pair of figures below is similar. Write a correct similarity statement and solve for x for each pair of figures. Diagrams are drawn roughly to scale; you can assume that sides that look longer are longer.

1.

2.

3.

4.

Determine if each pair of triangles is similar. If they are similar, justify your answer.

5.

6.

7.

8.

9.

10.
Decide if each pair of triangles is similar. If they are similar, write a correct similarity statement and justify your answer.

11. \[
\begin{align*}
13. \\
7 & \quad 14
\end{align*}
\]

12. \[
\begin{align*}
10 & \quad 15 \\
7 & \quad 12
\end{align*}
\]

13. \[
\begin{align*}
50^\circ & \quad 40^\circ
\end{align*}
\]

14. \[
\begin{align*}
50 & \quad 40 \\
20 & \quad 10
\end{align*}
\]

15. \[
\begin{align*}
25 & \quad 20
\end{align*}
\]

16. \[
\begin{align*}
\text{Diagram}
\end{align*}
\]

17. \[
\begin{align*}
O & \quad 16^\circ \\
B & \quad 82^\circ \\
X & \quad N
\end{align*}
\]

18. \[
\begin{align*}
O & \quad 16^\circ \\
N & \quad 82^\circ \\
A & \quad A
\end{align*}
\]

19. \[
\begin{align*}
A & \quad 12 \\
M & \quad 15
\end{align*}
\]

20. \[
\begin{align*}
A & \quad 13 \\
M & \quad 10 \\
L & \quad E
\end{align*}
\]
Using the information given in each diagram below, decide if any triangles are congruent, similar but not congruent, or not similar. If you claim the triangles are congruent or similar, create a flowchart justifying your answer.

21. 

22. 

23. 

24. 

25. In the figure at right \( \overline{AB} \parallel \overline{DE} \). Is \( \triangle ABC \) similar to \( \triangle EDC \)? Use a flowchart to organize and justify your answer.

26. Standing 4 feet from a mirror resting on the flat ground, Palmer, whose eye height is 5 feet, 9 inches, can see the reflection of the top of a tree. He measures the mirror to be 24 feet from the tree. How tall is the tree? Draw a picture to help solve the problem.
Answers

1. $\triangle ABCDEF \sim \triangle UZYXWV$, $x = 3.75$
2. $\triangle RECT \sim \triangle NGLA$, $x = 8$
3. $\triangle MNS \sim \triangle RCH$, $x = 72$
4. $\triangle LACEY \sim \triangle ITHOM$, $x = 16.5$
5. AA ~
6. SSS ~
7. AA ~
8. SAS ~
9. not ~
10. not ~
11. SAS ~ or SSS ~
12. not ~
13. AA ~
14. SSS ~
15. AA ~
16. AA ~
17. $\triangle BOX \sim \triangle NCA$ by AA ~
18. The triangles are not similar because the sides are not proportional.
   \[
   \frac{12}{15} = \frac{18}{22.5} = 0.8, \quad \frac{10}{13} = 0.76
   \]
19. $\triangle ALI \sim \triangle MES$ by SAS ~
20. The triangles are not similar. On $\triangle SAM$, the $60^\circ$ is included between the two given sides, but on $\triangle UEL$ the angle is not included.
21. $\triangle DAV \sim \triangle ISV$ by SAS ~
22. $\triangle LUN$ and $\triangle HTC$ are not necessarily similar based on the markings.
23. $\triangle SAP \sim \triangle SJE$ by AA ~
24. \( \triangle KRS \cong \triangle ISR \) by HL

\( \triangle KRS \) and \( \triangle ISR \) are right triangles.

Given

\( KR \cong IS \)

Given

\( RS \cong RS \)

Refl. Prop.

\( \triangle KRS \cong \triangle ISR \)

HL \( \cong \)

25. \( \overline{AB} \parallel \overline{DE} \)

Given

\( \angle ACB \cong \angle ECD \)

Vertical angles \( \cong \)

\( \angle ABC \cong \angle EDC \)

\( \parallel \) lines \( \Rightarrow \) alt. int. angles \( \cong \)

\( \triangle ABC \sim \triangle ECD \)

AA \( \sim \)

Note: There is more than one way to solve this problem. Corresponding angles could have been used twice rather than mentioning vertical angles.

26. A sketch of the situation and how it translates into a diagram with triangles is shown at right. \( \triangle PFM \sim \triangle TRM \) by AA \( \sim \).

The proportion is:

\[
\frac{x}{5.75} = \frac{24}{4}
\]

\( 4x = 138 \)

\( x = 34.5 \)

Therefore, the tree is 34.5 feet tall.
Although the definition of probability is simple, calculating a particular probability can sometimes be tricky. When calculating the probability of flipping a coin and having it come up tails, there are only two possible outcomes and one successful outcome. But what if neither the total number of outcomes nor the total number of successes is obvious? In such cases, an accurate way to count the number of outcomes is needed. In these lessons, students look at three models that can be used to list all possible outcomes (called a sample space): a systematic list, a tree diagram, and an area model. Each different model has its strengths and weaknesses, and is more useful in different situations.

See the Math Notes boxes in Lessons 2.1.4 and 3.1.4 for more information about calculating probabilities and using probability models.

Example 1

As Ms. Dobby prepares the week’s lunch menu for the students, she has certain rules that she must follow. She must have a protein and a vegetable at each lunch. She has four choices for protein: chicken, fish, beef, and tofu. Her list of choices for vegetables is a bit larger: peas, carrots, broccoli, corn, potatoes, and beets. Considering just the protein and the vegetable choices, if she randomly chooses one type of protein and one type of vegetable, what is the probability that the first lunch she makes will have fish or a green vegetable?

To determine the probability of a lunch with fish or a green vegetable, we need to know how many different lunch menus are possible. Then we need to count the number of lunch menus that have fish or a green vegetable. To count all of the possible lunch menus, we will make a systematic list, pairing each protein with a vegetable in an organized way.

<table>
<thead>
<tr>
<th>Chicken</th>
<th>Fish</th>
<th>Beef</th>
<th>Tofu</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chicken and peas</td>
<td>Fish and peas</td>
<td>Beef and peas</td>
<td>Tofu and peas</td>
</tr>
<tr>
<td>Chicken and carrots</td>
<td>Fish and carrots</td>
<td>Beef and carrots</td>
<td>Tofu and carrots</td>
</tr>
<tr>
<td>Chicken and broccoli</td>
<td>Fish and broccoli</td>
<td>Beef and broccoli</td>
<td>Tofu and broccoli</td>
</tr>
<tr>
<td>Chicken and corn</td>
<td>Fish and corn</td>
<td>Beef and potatoes</td>
<td>Tofu and corn</td>
</tr>
<tr>
<td>Chicken and potatoes</td>
<td>Fish and potatoes</td>
<td>Beef and potatoes</td>
<td>Tofu and potatoes</td>
</tr>
<tr>
<td>Chicken and beets</td>
<td>Fish and beets</td>
<td>Beef and beets</td>
<td>Tofu and beets</td>
</tr>
</tbody>
</table>

From this list we can count the total number of lunch menus: 24. Then we count the number of lunch menus with fish or a green vegetable (peas or broccoli). There are twelve such menus. Therefore the probability of the first lunch menu having fish or a green vegetable is \( \frac{12}{24} = \frac{1}{2} \).
Example 2

What is the probability of flipping a fair coin four times and having tails come up exactly two of those times?

Since each flip has only two possible outcomes, this information can be organized in a tree diagram. The first flip has only two possibilities: heads (H) or tails (T). From each branch, we split again into H or T. We do this for each flip of the coin. The final number of branches at the end tells us the total number of outcomes after four flips of the coin. In this example there are 16 possible outcomes. Using the “paths” along the branches there are six ways to get exactly two Ts. The path consisting of HTHT is highlighted. The others are HHTT, HTTH, THHT, THTH, and TTHH. Thus the probability of flipping a coin four times and having T come up exactly two times is \(\frac{6}{16} = \frac{3}{8}\).

Example 3

Romeo the rat is going to randomly run through a maze to hopefully find a block of cheese. The floor plan of the maze is shown at right, with the cheese to be placed in either section A or section B. If every time Romeo comes to a split in the maze he is equally likely to choose any path in front of him, what is the probability he ends up in section A?

It is useful to construct an area model to represent this situation.

Start with a square. When Romeo comes to the first branch in the maze, he has two choices: a top path and a bottom path. We represent this in the square by splitting it into two same-size (equally likely) pieces.

Then consider what happens if Romeo chooses the bottom path. If he chooses the bottom path, he comes to another split with two choices, which are each equally likely. On the area model we show this by splitting the bottom rectangle into two equally likely sections, shown at right.

With one branch, Romeo will end up in section A; with the other branch he will end up in B. We indicate this by putting the letters in the regions representing these outcomes.
Now consider the top path. If Romeo takes the top path at the first split, he quickly comes to another split where again he has a choice of a top path or a bottom path. Once again we split the top rectangle into two same-size rectangles since each path is equally likely. If Romeo takes the lower path, he will end up in section A. We indicate this by writing an A in one of the two new regions as shown at right.

If Romeo takes the upper path, he comes to another split. Again, both paths are equally likely. This means the last section of the square needs to be cut into two equal parts. One of the paths will lead directly to section A, the other to section B.

To calculate the probability, we must determine what part of the total area are the sections marked A. We do this by calculating the fraction of the area of each part. The length of each side of each rectangle is shown on the exterior of the square, while the area is written within the region. The probability of getting into section A is represented by the shaded portion of the square. The area of the shaded region is:

\[
A = \frac{1}{4} + \frac{1}{4} + \frac{1}{8} = \frac{2}{8} + \frac{4}{8} + \frac{1}{8} = \frac{5}{8}
\]

Therefore the probability of Romeo wandering into section A is \( \frac{5}{8} \). This means the probability that he wanders into section B is \( \frac{3}{8} \) since the sum of both probabilities must be 1.

**Problems**

1. If Keisha has four favorite shirts (one blue, one green, one red, and one yellow) and two favorite pairs of pants (one black and one brown), how many different favorite outfits can she make? If Keisha selects a shirt and pants randomly, what is the probability that she chooses an outfit with a blue or red shirt and black pants?

2. Each morning Aaron starts his day with either orange juice or apple juice followed by cereal, toast, or scrambled eggs. How many different morning meals are possible for Aaron? If Aaron randomly selects a juice and breakfast food, what is the probability that he has orange juice and toast or eggs?

3. Eliza likes to make daily events into games of chance. For instance, before she went to buy ice cream at the local ice cream parlor, she created two spinners. The first has her three favorite flavors while the second has “cone” and “dish.” Eliza will order whatever comes up on the spinners. What is the probability that she will be eating tutti fruitti ice cream from a dish?
4. Barty is going to flip a coin three times. What is the probability that he will see at least two tails?

5. Mr. Fudge rolls two fair dice. What is the probability that the sum will be 4 or less?

6. Welcome to another new game show, “Spinning for Luck!” As a contestant, you will be spinning two wheels. Each wheel is divided into equal sectors. The first wheel determines a possible dollar amount that you could win. The second wheel is the “multiplier”. You will multiply the results of your spins to determine the amount you will win. Unfortunately, you could owe money if your multiplier lands on -2!
   a. What is the probability of winning $100 or more?
   b. What is the probability of owing $100 or more?

For problems 7 through 10, a bag contains the polygons shown below. If you reach in and randomly pull out a polygon, what is the probability that you select:

7. A polygon with at least one right angle?

8. A polygon with an acute angle?

9. A polygon with at least one pair of parallel sides?

10. A triangle?

For each question that follows, use an area model or a tree diagram to compute the desired probability.

For problems 11 through 13 use the spinners at right.

11. If each spinner is spun once, what is the probability that both spinners show blue?

12. If each spinner is spun once, what is the probability that both spinners show the same color?

13. If each spinner is spun once, what is the probability of getting a red-blue combination?

14. A pencil box has three yellow pencils, one blue pencil, and two red pencils. There are also two red erasers and one blue. If you randomly choose one pencil and one eraser, what is the probability of getting the red-red combination?
15. Sally’s mother has two bags of candy but she says that Sally can only have one piece. Bag #1 has 70% orange candies and 30% red candies. Bag #2 has 10% orange candies, 50% white candies, and 40% green candies. Sally’s eyes are covered and she chooses one bag and pulls out one candy. What is the probability that she chooses an orange candy?

16. You roll a die and flip a coin. What is the probability of rolling a number less than 5 on the die and flipping tails on the coin?

17. A spinner is evenly divided into eight sections—three are red, three are white, and two are blue. If the spinner is spun twice, what is the probability of getting the same color twice?

18. You and your friend have just won a chance to collect a million dollars. You place the money in one room at right and then your friend has to randomly walk through the maze. In which room should you place the money so that your friend will have the best chance of finding the million dollars?

19. Calculate the probability of randomly entering each room in the maze shown at right.
   a. \( P(A) \)
   b. \( P(B) \)
   c. \( P(C) \)

20. The weather forecast shows a 60% chance of rain. If it does not rain then there is an 80% chance of going to the beach. What is the probability of going to the beach?

21. A baseball player gets a hit 40% of the time if the weather is nice but only 20% of the time if it is cold or windy. The weather forecast shows a 70% chance of being nice, 20% chance of being cold, and 10% chance of being windy. What is the probability of that the baseball player will get a hit?

22. If students have their assignments done on time there is an 80% chance of earning a good grade in the class. If the assignments are finished during class or late then there is only a 30% chance of earning a good grade in the class. If the assignments are not done at all then there is only a 5% chance of earning a good grade in the class. In a certain class, 50% of the students have the assignments completed on time, 40% finish during class, and 10% do not do their assignments. If a student is selected at random, what is the probability that student has a good grade?
**Answers**

1. Eight different outfits. $\frac{2}{8} = \frac{1}{4}$

2. Six meals. $\frac{2}{6} = \frac{1}{3}$

3. $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$

4. $\frac{1}{2}$ (See the tree diagram in Example 2.)

5. $\frac{6}{36} = \frac{1}{6}$

6. a: $\frac{5}{12}$ b: $\frac{1}{12}$

7. $\frac{2}{5}$  8. $\frac{3}{5}$  9. $\frac{2}{5}$  10. $\frac{2}{5}$

11. $\frac{1}{12}$  12. $\frac{9}{24} = \frac{3}{8}$  13. $\frac{7}{24}$  14. $\frac{2}{9}$

15. $\frac{2}{5}$  16. $\frac{1}{3}$  17. $\frac{11}{32}$  18. $P(B) = \frac{5}{9}$

19. $\frac{11}{18}, \frac{5}{18}, \frac{2}{18}$  20. 0.32  21. 0.34  22. 0.525
In Lesson 3.1.5 students investigate expected value by analyzing different games. Ultimately, students develop a method for calculating expected value. Note that students sometimes think that the expected value must actually be one of the possible outcomes of a game or situation. It does not have to be. Expected value is a calculation of the average expected result for one play if the game is played many times.

See the Math Notes box in Lesson 3.2.1 for more information about expected value.

Example 1

The spinner at right is divided into different sections, each assigned a different point value. The three smaller sections are congruent. If you were to spin the spinner 100 times, how many times would you expect to get each of the different point values? What is the expected value of this spinner?

The angle of each sector is what determines the probability of the spinner landing in that region. Therefore the probability of landing on 6 points is $\frac{1}{2}$ because that region takes up half of the spinner. The other half of the circle is divided into three equal parts, each taking up $\frac{1}{6}$ of the whole spinner ($\frac{1}{3}$ of $\frac{1}{2}$). Now that we know the probabilities, we can determine how many times we would expect each of the values to occur. Since the probability of getting 6 points is $\frac{1}{2}$, we expect that about 50 of the 100 spins (half) will land on 6 points. Similarly, since the probability of landing on 1 point (or 2 or 3 points) is $\frac{1}{6}$, we expect about $\frac{1}{6}$ of the 100 spins to land on each of those, or about 16 or 17 times. If the total number of spins is 100, we can expect on average about 50 of them to be 6 points, $16 \frac{2}{3}$ to be 1 point, $16 \frac{2}{3}$ to be 2 points, and $16 \frac{2}{3}$ to be 3 points. (Note: These are estimates, not exact or guaranteed.) Using these values, after 100 spins, the player would have about $50(6) + 16 \frac{2}{3}(1) + 16 \frac{2}{3}(2) + 16 \frac{2}{3}(3) = 400$ points.

It is expected that a player earns 400 points in 100 spins, or an average 4 points per spin. So the expected value is 4 points. Note: 4 points is the expected value for this spinner even though it is NOT one of the possible outcomes.
Example 2

A $3 \times 3$ grid of nine congruent squares is painted various colors. Six of the small squares are painted red while three are painted blue. For $1.00$ a player can throw a dart at the grid. If the player hits a blue square, they win $2.00$. Is this a fair game? Justify your answer.

The expected value is found by summing the products of the amounts that can be won and their probabilities. In this problem, each game costs $1.00$ to play. If the dart lands on a red square, the player loses $1.00$ (the value is $-1$). The probability of hitting a red square is $\frac{6}{9} = \frac{2}{3}$. However, if the player hits a blue square, the player receives $2.00$, which is a gain of only $1.00$ (because he paid $1.00$ to play). As shown in the calculations at right, the expected value of this game is $-\frac{1}{3}$. Therefore, this is not a fair game; it favors the person running the game.
Problems

The spinners below have different point values assigned to different regions. What is the expected value for each spinner? (Assume that regions that appear to be congruent, are congruent.)

1. 
2. 
3. 
4. 
5. 
6. 
7. For $0.40 a player gets one dart to throw at a board that looks like the diagram at right. The board is a square, measuring 1 foot along each side. The circle has a diameter of six inches. For each dart that lands in the circle, the player wins $0.75. Is this game fair? Justify your answer.

Answers

1. 2.5  
2. 4  
3. $3 \frac{2}{3}$  
4. 3  
5. 5.5  
6. 4.75  
7. Not fair because the expected value is about –$0.25.
In the second section of Chapter 3, different slope triangles are explored for a given line or segment. For each line, the slope remains constant no matter where the slope triangle is drawn on that line or how large or small each slope triangle is. All the slope triangles for a given line are similar. These similar slope triangles can be used to write proportions to calculate side lengths and angle measures. This constant slope ratio is known as the tangent (trigonometric) relationship. Later in the section, the tangent button on a calculator is used to determine measurements in application problems.

See the Math Notes boxes in Lessons 3.2.2, 3.2.4, and 3.2.5 for more information about slope angles and the tangent ratio.

**Example 1**

The line graphed at right passes through the origin. Draw in three different slope triangles for the line. For each triangle, what is the slope ratio, \( \frac{\Delta y}{\Delta x} \)? What is true about all three ratios?

Note: \( \Delta x \) (delta x) and \( \Delta y \) (delta y) are read “change in \( x \)” and “change in \( y \),”

A slope triangle is a right triangle that has its hypotenuse on the line that contains it. This means that the two legs of the right triangle are parallel to the axes: one leg runs vertically, the other horizontally. There are infinitely many slope triangles that can be drawn, but it is easiest if the triangles that have their vertices on lattice points (that is, their vertices have integer coordinates). The length of the horizontal leg is \( \Delta x \) and the length of the vertical leg is \( \Delta y \). In the diagram at right there are three possible slope triangles. For the smallest triangle, \( \Delta x = 3 \) (the length of the horizontal leg), and \( \Delta y = 2 \) (the length of the vertical leg). Therefore \( \frac{\Delta y}{\Delta x} = \frac{2}{3} \).

In the medium-sized triangle, \( \Delta x = 6 \) and \( \Delta y = 4 \), which means \( \frac{\Delta y}{\Delta x} = \frac{4}{6} \).

Lastly, the leg lengths of the largest triangle are \( \Delta x = 15 \) and \( \Delta y = 10 \), so \( \frac{\Delta y}{\Delta x} = \frac{10}{15} \).

These slope ratios are all equivalent, so no matter where the slope triangles for this line is drawn, the slope ratio remains constant.
In Lesson 3.2.2 students connect specific slope ratios of slope triangles to their related angles and record these values in a Trig Table Graphic Organizer (Lesson 3.2.2 Resource Page). They use this information to calculate missing side lengths and angle measures of right triangles. In Lesson 3.2.4, students learn to use the tangent button on their calculator to calculate these ratios and solve for missing information.

**Example 2**

Write an equation and use the tangent function on your calculator to calculate the missing side length in each triangle.

a. 

![Diagram](image)

b. 

![Diagram](image)

When using the tangent button on a calculator to solve these problems, be sure that the calculator is in degree mode and not radian mode. Since we know that the slope ratio depends on the angle, we can use the angle measure and the tangent function on a calculator to determine unknown lengths of the triangles.

The tangent of an angle in a right triangle is the ratio \( \frac{\text{opposite leg}}{\text{adjacent leg}} \).

This allows us to write and solve the equations as shown below.

Part (a): \[
\tan 20^\circ = \frac{22}{w} \\
w \tan 20^\circ = 22 \\
w = \frac{22}{\tan 20^\circ} \\
w \approx 60.44
\]

Part (b): \[
\tan 62^\circ = \frac{q}{9.6} \\
9.6(\tan 62^\circ) = q \\
q \approx (9.6)(1.88) \approx 18.05
\]
**Example 3**

Talula is standing 117 feet from the base of the Washington Monument in Washington, D.C. She uses her clinometer to measure the angle of elevation to the top of the monument to be 78°. If Talula’s eye height is 5 feet, 3 inches, what is the height of the Washington Monument?

With all problems representing an everyday situation, it is helpful to draw a diagram of what the problem is describing. In this situation, Talula looking up at the top of a monument. We know how far away Talula is standing from the monument, we know her eye height, and we know the angle of elevation of her line of sight.

We use this information to draw a diagram, as shown at right. Then we write an equation using the tangent ratio and solve for $x$.

We add the “eye height” to the value of $x$ to determine the height of the Washington Monument. Write the answer to the nearest foot.

$$549.9 + 5.25 \approx 555.15, \text{ or about } 555 \text{ feet}$$

**Problems**

For each line, draw several slope triangles. Then calculate the slope ratios.

1. 
2. 
3. 
4.
Calculate the measures indicated by the variables. It may be helpful to rotate the triangle so that it resembles a slope triangle. If you write a tangent equation, use the tangent button on your calculator not your Trig Table Graphic Organizer to solve the equation. Note: Some calculations require the Pythagorean Theorem. For lengths, write your answer to the nearest hundredth. For angles, write your answer to the nearest degree.

5. 

6. 

7. 

8. 

9. 

10. 

11. A ladder leaning against a wall makes a 75° angle with the ground. The base of the ladder is 5.0 feet from the wall. How high up the wall does the ladder reach? Write your answer with the appropriate precision.

12. Davis and Tess are 30 feet apart when Tess lets go of her helium-filled balloon, which rises straight up into the air. (It is a windless day.) After 4 seconds, Davis uses his clinometer to sight the angle of elevation to the balloon at 35°. If Davis’ eye height is 4 feet, 6 inches, what is the height of the balloon after 4 seconds? Write your answer with the appropriate precision.
Answers

1. In each case the slope ratio is \( \frac{4}{1} = 4 \).

2. The slope ratio is \( \frac{5}{3} = \frac{4}{2} = \frac{3}{1} = \frac{1}{\frac{1}{4}} \).

3. The slope ratio is \( \frac{5}{3} \).

4. The slope ratio is \( \frac{1}{4} \).

5. \( \tan 28^\circ = \frac{z}{14} \), \( z \approx 7.44 \)

6. \( \tan 70^\circ = \frac{3z}{m} \), \( m \approx 1.16 \)

7. \( \tan 33^\circ = \frac{y}{210} \), \( y \approx 136.38 \)

8. \( c \approx 119.67 \) (Pythagorean Theorem)

9. \( x = 12.25 \)

10. \( \tan 15^\circ = \frac{w}{47} \), \( w \approx 12.59 \)

11. \( \tan 75^\circ = \frac{h}{5} \); The ladder reaches about 18.7 feet up the wall.

12. \( \tan 35^\circ = \frac{h}{30} \), \( h \approx 21 + 4.5 \approx 25.5 \); After 4 seconds the balloon is about 25.5 feet above the ground.
In Lessons 4.1.1 through 4.1.4 students factor quadratic expressions. This prepares them for solving quadratic equations in Chapter 5.

In Chapter 1 students used algebra tiles to build models of quadratic expressions. They then moved from physical tiles to area models, which can more easily represent large numbers of tiles and negative tiles. In the diagram below, the length and width of the rectangle are \((x + 2)\) and \((x + 4)\). Since \((\text{base})(\text{height}) = \text{area}\), the area of the rectangle can be expressed as a product, \((x + 2)(x + 4)\). The small sections of the rectangle also make up its area, so the area can be expressed as a sum, \(4x + 8 + x^2 + 2x\), or \(x^2 + 6x + 8\).

Area as a product = area as a sum, thus students wrote \((x + 2)(x + 4) = x^2 + 6x + 8\).

The length and width of the rectangle, \((x + 2)\) and \((x + 4)\), are factors of the quadratic expression \(x^2 + 6x + 8\), since the product of \((x + 2)\) and \((x + 4)\) is \(x^2 + 6x + 8\). Therefore, the factored form of \(x^2 + 6x + 8\) is \((x + 2)(x + 4)\).

For a more detailed example of the method used by students to factor quadratic expressions, see the Math Notes box in Lesson 4.1.4. For additional information, see the Math Notes boxes in Lessons 4.1.1 and 4.1.2. For additional examples and more practice, see the Checkpoint 7 materials in the student text.

**Example 1**

Factor \(x^2 + 7x + 12\).

Sketch an area model.

Place the \(x^2\) and the 12 along one diagonal.

Since the products of the diagonals must be the same, identify two terms whose product is \(12x^2\) and whose sum is \(7x\). In this case, \(3x\) and \(4x\). (Students are familiar with this situation as a “diamond problem” from Chapter 1.)

Write these terms along the other diagonal. Either term can go in either diagonal space.

Determine the base and height of the large outer rectangle by using the areas of the small pieces and determining the greatest common factor of each row and each column.

Write the sum as a product (factored form). \(x^2 + 7x + 12 = (x + 3)(x + 4)\)
Example 2

Factor $x^2 + 7x - 30$.

Sketch an area model.
Place the $x^2$ and the $-30$ along one diagonal.

Since the products of the diagonals must be the same, identify two terms whose product is $-30x^2$ and whose sum is $7x$. In this case, $-3x$ and $10x$.

Write these terms along the other diagonal. Either term can go in either diagonal space.

Determine the base and height of the large outer rectangle by using the areas of the small pieces and determining the greatest common factor of each row and each column.

Write the sum as a product (factored form).

$$x^2 + 7x - 30 = (x - 3)(x + 10)$$

Example 3

Factor $x^2 - 15x + 56$.

Sketch an area model.
Place the $x^2$ and the $56$ along one diagonal.

Since the products of the diagonals must be the same, identify two terms whose product is $56x^2$ and whose sum is $-15x$. Write these terms as the other diagonal.

Determine the base and height of the large outer rectangle by using the areas of the small pieces and determining the greatest common factor of each row and each column.

Write the sum as a product (factored form).

$$x^2 - 15x + 56 = (x - 7)(x - 8)$$
Example 4

Factor $12x^2 - 19x + 5$.

Sketch an area model.

Place the $12x^2$ and the 5 along one diagonal.

Since the products of the diagonals must be the same, identify two terms whose product is $60x^2$ and whose sum is $-19x$. Write these terms as the other diagonal.

Determine the base and height of the rectangle. Check the signs of the factors.

Write the sum as a product (factored form). $12x^2 - 19x + 5 = (3x - 1)(4x - 5)$

Example 5

Factor $3x^2 + 21x + 36$.

Note: If a common factor appears in all the terms, it should be factored out first. For example, $3x^2 + 21x + 36 = 3(x^2 + 7x +12)$.

Then $x^2 + 7x + 12$ can be factored as in Example 1. $x^2 + 7x + 12 = (x + 3)(x + 4)$.

Then, since the expression $3x^2 + 21x + 36$ has a factor of 3, $3x^2 + 21x + 36 = 3(x^2 + 7x + 12) = 3(x + 3)(x + 4)$.

Problems

Factor the following expressions.

1. $x^2 + 5x + 6$  
2. $2x^2 + 5x + 3$  
3. $3x^2 + 4x + 1$  
4. $3x^2 + 30x + 75$

5. $x^2 + 15x + 44$  
6. $x^2 + 7x + 6$  
7. $2x^2 + 22x + 48$  
8. $x^2 + 4x - 32$

9. $4x^2 + 12x + 9$  
10. $24x^2 + 22x - 10$  
11. $x^2 + x - 72$  
12. $3x^2 - 20x - 7$

13. $x^3 - 11x^2 + 28x$  
14. $2x^2 + 11x - 6$  
15. $2x^2 + 5x - 3$  
16. $x^2 - 3x - 10$

17. $4x^2 - 12x + 9$  
18. $3x^2 + 2x - 5$  
19. $6x^2 - x - 2$  
20. $9x^2 - 18x + 8$
Answers

1. \((x + 2)(x + 3)\) 2. \((x + 1)(2x + 3)\) 3. \((3x + 1)(x + 1)\) 4. \(3(x + 5)(x + 5)\)
5. \((x + 11)(x + 4)\) 6. \((x + 6)(x + 1)\) 7. \(2(x + 8)(x + 3)\) 8. \((x + 8)(x - 4)\)
9. \((2x + 3)(2x + 3)\) 10. \(2(3x - 1)(4x + 5)\) 11. \((x - 8)(x + 9)\) 12. \((x - 7)(3x + 1)\)
13. \(x(x - 4)(x - 7)\) 14. \((x + 6)(2x - 1)\) 15. \((x + 3)(2x - 1)\) 16. \((x - 5)(x + 2)\)
17. \((2x - 3)(2x - 3)\) 18. \((3x + 5)(x - 1)\) 19. \((2x + 1)(3x - 2)\) 20. \((3x - 4)(3x - 2)\)
Although most factoring problems can be done with area models, there are two special factoring patterns that, if recognized, can be done by sight. The two patterns are known as the **Difference of Squares** and **Perfect Square Trinomials**. The general patterns are as follows:

**Difference of Squares**: \[ a^2x^2 - b^2 = (ax + b)(ax - b) \]

**Perfect Square Trinomial**: \[ a^2x^2 + 2abx + b^2 = (ax + b)^2 \]

**Example 1**

<table>
<thead>
<tr>
<th>Difference of Squares</th>
<th>Perfect Square Trinomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x^2 - 49 = (x + 7)(x - 7))</td>
<td>(x^2 - 10x + 25 = (x - 5)^2)</td>
</tr>
<tr>
<td>(4x^2 - 25 = (2x - 5)(2x + 5))</td>
<td>(9x^2 + 12x + 4 = (3x + 2)^2)</td>
</tr>
<tr>
<td>(x^2 - 36 = (x + 6)(x - 6))</td>
<td>(x^2 - 6x + 9 = (x - 3)^2)</td>
</tr>
<tr>
<td>(9x^2 - 1 = (3x - 1)(3x + 1))</td>
<td>(4x^2 + 20x + 25 = (2x + 5)^2)</td>
</tr>
</tbody>
</table>

**Example 2**

Sometimes factoring out a common factor reveals one of the special cases:

\[8x^2 - 50y^2 \Rightarrow 2(4x^2 - 25y^2) \Rightarrow 2(2x + 5y)(2x - 5y)\]

\[12x^2 + 12x + 3 \Rightarrow 3(4x^2 + 4x + 1) \Rightarrow 3(2x + 1)^2\]
Problems

Factor each difference of squares.
1. \(x^2 - 16\)  
2. \(x^2 - 25\)  
3. \(64m^2 - 25\)  
4. \(4p^2 - 9q^2\)  
5. \(9x^2 - 49\)  
6. \(x^4 - 25\)  
7. \(64 - y^2\)  
8. \(144 - 25p^2\)  
9. \(9x^4 - 4\)

Factor each perfect square trinomial.
10. \(x^2 + 4x + 4\)  
11. \(y^2 + 8y + 16\)  
12. \(m^2 - 10m + 25\)  
13. \(x^2 - 8x + 16\)  
14. \(a^2 + 8ab + 16b^2\)  
15. \(36x^2 + 12x + 1\)  
16. \(25x^2 - 30x + 9\)  
17. \(9x^2 - 6x + 1\)  
18. \(49x^2 + 1 + 14x\)

Factor completely.
19. \(9x^2 - 16\)  
20. \(9x^2 + 24x + 16\)  
21. \(9x^2 - 36\)  
22. \(2x^2 + 8x + 8\)  
23. \(3x^2 + 30x + 75\)  
24. \(8x^2 - 72\)  
25. \(4x^3 - 9x\)  
26. \(4x^2 - 8x + 4\)  
27. \(2x^2 + 8\)

Answers

1. \((x + 4)(x - 4)\)  
2. \((x + 5)(x - 5)\)  
3. \((8m + 5)(8m - 5)\)  
4. \((2p + 3q)(2p - 3q)\)  
5. \((3x + 7)(3x - 7)\)  
6. \((x^2 + 5)(x^2 - 5)\)  
7. \((8 + y)(8 - y)\)  
8. \((12 + 5p)(12 - 5p)\)  
9. \((3x^2 + 2)(3x^2 - 2)\)  
10. \((x + 2)^2\)  
11. \((y + 4)^2\)  
12. \((m - 5)^2\)  
13. \((x - 4)^2\)  
14. \((a + 4b)^2\)  
15. \((6x + 1)^2\)  
16. \((5x - 3)^2\)  
17. \((3x - 1)^2\)  
18. \((7x + 1)^2\)  
19. \((3x + 4)(3x - 4)\)  
20. \((3x + 4)^2\)  
21. \((9x + 2)(x - 2)\)  
22. \(2(x + 2)^2\)  
23. \(3(x + 5)^2\)  
24. \((8x + 3)(x - 3)\)  
25. \(x(2x + 3)(2x - 3)\)  
26. \(4(x - 1)^2\)  
27. \(2(x^2 + 4)\)
In Chapter 4, two more trigonometric ratios are introduced: sine and cosine. Both of them are used with acute angles of right triangles, similar to the tangent ratio. Using the diagram below:

\[
\sin \theta = \frac{\text{opposite leg}}{\text{hypotenuse}} \quad \cos \theta = \frac{\text{adjacent leg}}{\text{hypotenuse}}
\]

From Chapter 3:

\[
\tan \theta = \frac{\text{opposite leg}}{\text{adjacent leg}}
\]

Note: If the other acute angle in the triangle is used, then the names of the legs switch places. The opposite leg is always across the triangle from the acute angle being used.

See the Math Notes boxes in Lessons 4.2.2 and 4.2.4, and Checkpoint 8 for additional information.

**Example 1**

Use the sine ratio to calculate the length of the unknown side in each triangle below.

a. 

b.

The sine of the angle is the ratio \( \frac{\text{opposite leg}}{\text{hypotenuse}} \).

For part (a), use the 78° angle as \( \theta \). From the 78° angle, determine which side of the triangle is the opposite leg and which side is the hypotenuse. The hypotenuse is always the longest side, and it is always opposite the right angle. In this case, the hypotenuse is 18. From the 78° angle, the side labeled \( x \) is the opposite leg. Now write the equation at shown at right and solve it.

\[
\sin 78^\circ = \frac{x}{18} \quad (\text{opposite hypotenuse})
\]

\[
18 \sin 78^\circ = x
\]

\[
x \approx 17.61
\]

In part (b), from the 42° angle, the opposite leg is \( x \) and the hypotenuse is 16. Write and solve the equation as shown at right. Note: In most cases, it is most efficient to solve the equation for \( x \), and then use your calculator to complete the calculations, as shown in these examples.

\[
\sin 42^\circ = \frac{x}{16}
\]

\[
16 \sin 42^\circ = x
\]

\[
x \approx 10.71
\]
Example 2

a. 

b. 

Just as before, set up an equation using the cosine ratio, \( \frac{\text{adjacent leg}}{\text{hypotenuse}} \). Remember that you can always rotate the page, or trace and rotate the triangle, if the figure’s orientation is confusing. The key to solving these problems is recognizing which side is adjacent, which is opposite, and which is the hypotenuse. For part (a), the angle is 25°, so write and solve the equation at right.

\[
\cos 25^\circ = \frac{x}{4} \quad \Rightarrow \quad 4 \cos 25^\circ = x \\
x \approx 3.63
\]

In part (b), using the 62° angle, the adjacent leg is 13 and the hypotenuse is \( x \). This time, the variable is in the denominator. This adds one more step to the solution.

\[
\cos 62^\circ = \frac{13}{x} \\
x \cos 62^\circ = 13 \\
x = \frac{13}{\cos 62^\circ} \approx 27.69
\]
Example 3

In each triangle below, use the inverse trigonometry buttons on your calculator to calculate the measure of the angle $\theta$ to the nearest hundredth of a degree.

a.  
\[ \begin{array}{c} 5 \quad 13 \\ \theta \end{array} \]

b.  
\[ \begin{array}{c} 12 \\ \theta \quad 8 \end{array} \]

c.  
\[ \begin{array}{c} 14 \\ \theta \quad 7 \end{array} \]

d.  
\[ \begin{array}{c} 42 \\ \theta \quad 30 \end{array} \]

For each of these problems you must decide whether you will be using sine, cosine, or tangent to calculate the value of $\theta$. In part (a), 5 is the leg that is opposite angle $\theta$, and 13 is the hypotenuse. Therefore, we use the sine ratio. For the best accuracy, enter the ratio not its decimal approximation.

To calculate the value of $\theta$, find the button on the calculator that says sin$^{-1}$. (Note: Calculator sequences shown are for most graphing calculators. Some calculators use a different order of keystrokes.) This is the $[\text{SIN}^{-1}]$ button, and when a ratio is entered, this button tells you the measure of the angle that has that sine ratio. Here we calculate $\sin^{-1}\left(\frac{5}{13}\right) \approx 22.62^\circ$ by entering $\text{2nd} \, \text{SIN} \, 5 \div 13 \, \text{ENTER}$. Be sure to use parentheses as shown.

\[ \sin \theta = \frac{5}{13} \]
\[ \sin \theta \approx 0.385 \]

In part (b), 8 is the adjacent leg for $\theta$ and 12 is the hypotenuse. This combination of sides fits the cosine ratio. Use the $[\text{COS}^{-1}]$ button to calculate the measure of $\theta$ by entering the following sequence on the calculator: $\text{2nd} \, \text{COS} \, 8 \div 12 \, \text{ENTER}$.

\[ \cos \theta = \frac{8}{12} \]
\[ \cos \theta \approx 0.667 \]
\[ \theta = \cos^{-1}\left(\frac{8}{12}\right) \]
\[ \theta \approx 48.19^\circ \]

In part (c), using $\theta$, 7 is the opposite leg and 14 is the adjacent leg. These two sides fit the tangent ratio. As before, use the $[\text{TAN}^{-1}]$ button on the calculator.

\[ \tan \theta = \frac{7}{14} = 0.5 \]
\[ \tan \theta = 0.5 \]
\[ \theta = \tan^{-1} 0.5 \approx 26.57^\circ \]

In part (d), 42 is the leg opposite angle $\theta$ while 30 is the length of the adjacent leg. Use the tangent ratio to calculate the value of $\theta$.

\[ \tan \theta = \frac{42}{30} = 1.4 \]
\[ \tan \theta = 1.4 \]
\[ \theta = \tan^{-1} (1.4) \approx 54.46^\circ \]
Example 4

Kennedy is standing on the end of a rope that is 40 feet long and threaded through a pulley. The rope is holding a large metal ball 18 feet above the floor. Kennedy slowly slides her feet closer to the pulley to lower the ball. When the ball hits the floor, what angle ($\theta$) does the rope make with the floor where it is under her foot?

As always, we will start by drawing a picture of the situation. The first picture shows the beginning situation, before Kennedy has started lowering the ball. The second picture shows the situation once the ball has reached the floor. We want to determine the angle $\theta$. You should see a right triangle emerging, made of the rope and the floor. The 40-foot rope makes up two sides of the triangle: 18 feet is the length of the leg opposite $\theta$, and the rest of the rope, 22 feet of it, is the hypotenuse. With this information, draw one more picture. This one will show the simple triangle that represents this situation.

From $\theta$, we have the lengths of the opposite leg and the hypotenuse. This tells us to use the sine ratio. The rope makes an angle of about $55^\circ$ with the floor.

\[
\sin \theta = \frac{18}{22} \\
\theta = \sin^{-1} \frac{18}{22} \\
\theta \approx 54.9^\circ
\]

Problems

Use the $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ button on your calculator to calculate the value of $\theta$ to the nearest hundredth of a degree.

1. 

2. 

3. 

4. 

5. 

6. 

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Use trigonometric ratios to solve for the variable in each figure below. Write each answer to the nearest tenth.

7. \( h = 15 \)  
   \( 38^\circ \)

8. \( 8 \)  
   \( h \)  
   \( 26^\circ \)

9. \( 23 \)  
   \( 49^\circ \)

10. \( 37 \)  
    \( 41^\circ \)

11. \( y \)  
    \( 15^\circ \)  
    \( 38 \)

12. \( 55^\circ \)  
    \( 43 \)

13. \( 15 \)  
    \( 38^\circ \)

14. \( z \)  
    \( 52^\circ \)  
    \( 18 \)

15. \( w \)  
    \( 38^\circ \)  
    \( 23 \)

16. \( w \)  
    \( 38^\circ \)  
    \( 15 \)

17. \( 38 \)  
    \( 15^\circ \)  
    \( x \)

18. \( 29^\circ \)  
    \( 91 \)

19. \( 5 \)  
    \( x \)  
    \( 7 \)

20. \( 9 \)  
    \( 7 \)

21. \( 12 \)  
    \( y \)  
    \( 18 \)

22. \( v \)  
    \( 78 \)  
    \( 88 \)
Draw a diagram and use trigonometric ratios to solve each of the following problems. Be sure to round your answers appropriately given the accuracy of the original measurements.

23. Nell’s kite has a 350-foot string. When it is completely out, Ian measures the angle to be 47.5°. How far does Ian need to walk to be directly under the kite?

24. Mayfield High School’s flagpole is 15 feet high. Using a clinometer, Tamara measured an angle of 11.3° to the top of the pole. Tamara is 62 inches tall. How far from the flagpole is Tamara standing?

25. Tamara took another sighting of the top of the flagpole from a different position. This time the angle is 58.4°. If everything else is the same, how far from the flagpole is Tamara standing?

26. Standing 140 feet from the base of a building, Alejandro uses his clinometer to site the top of the building. The reading on his clinometer is 42°. If his eyes are 6 feet above the ground, how tall is the building?

27. An 18-foot ladder rests against a wall. The base of the ladder is 8 feet from the wall. What angle does the ladder make with the ground?
Answers

1. \( \sin \theta = \frac{13}{19}, \quad \theta \approx 43.17^\circ \)
2. \( \tan \theta = \frac{24}{8}, \quad \theta \approx 71.57^\circ \)
3. \( \cos \theta = \frac{53}{68}, \quad \theta \approx 38.79^\circ \)
4. \( \tan \theta = \frac{34}{23}, \quad \theta \approx 55.92^\circ \)
5. \( \sin \theta = \frac{35}{38}, \quad \theta \approx 37.12^\circ \)
6. \( \tan \theta = \frac{2.54}{3.35}, \quad \theta \approx 51.37^\circ \)
7. \( h = 15 \sin 38^\circ \approx 9.2 \)
8. \( h = 8 \sin 26^\circ \approx 3.5 \)
9. \( x = 23 \cos 49^\circ \approx 15.1 \)
10. \( x = 37 \cos 41^\circ \approx 27.9 \)
11. \( y = 38 \tan 15^\circ \approx 10.2 \)
12. \( y = 43 \tan 55^\circ \approx 61.4 \)
13. \( z = \frac{15}{\sin 38^\circ} \approx 24.4 \)
14. \( z = \frac{18}{\sin 52^\circ} \approx 22.8 \)
15. \( w = \frac{23}{\cos 38^\circ} \approx 29.2 \)
16. \( w = \frac{15}{\cos 38^\circ} \approx 19.0 \)
17. \( x = \frac{38}{\tan 15^\circ} \approx 141.8 \)
18. \( x = \frac{91}{\tan 29^\circ} \approx 164.2 \)
19. \( x = \tan^{-1} \left( \frac{5}{7} \right) \approx 35.5^\circ \)
20. \( u = \tan^{-1} \left( \frac{2}{9} \right) \approx 37.9^\circ \)
21. \( y = \tan^{-1} \left( \frac{12}{18} \right) \approx 33.7^\circ \)
22. \( y = \tan^{-1} \left( \frac{78}{88} \right) \approx 41.6^\circ \)
23. \[ \begin{array}{c}
350 \text{ ft} \\
47.5^\circ \\
\end{array} \]
\[ \cos 47.5^\circ = \frac{d}{350} \]
\[ d = 350(\cos 47.5^\circ) \approx 236.456 \]
or about 236 ft
24. \[ \begin{array}{c}
62 \text{ in} \\
15 \text{ ft} \\
11.3^\circ \\
\end{array} \]
15 feet = 180 inches
180" - 62" = 118" = \( h \)
\[ \tan 11.3^\circ = \frac{118}{x}, \quad x \tan 11.3^\circ = 118 \]
\[ x = \frac{118}{\tan 11.3^\circ} \approx 591 \text{ inches} \]
or about 49 ft 4 in
25. \[ \begin{array}{c}
62 \text{ in} \\
58.4^\circ \\
15 \text{ ft} \\
\end{array} \]
\[ h = 118^n, \quad \tan 58.4^\circ = \frac{118}{x}, \quad x \tan 58.4^\circ = 118, \quad x = \frac{118}{\tan 58.4^\circ} \]
\[ x \approx 73 \text{ inches or about 6 ft 1 in} \]
26. \( \tan 42^\circ = \frac{h}{140} \), \( h = 126 \text{ ft} \)
then adding on the height to Alejandro’s eyes, \( 126 + 6 \approx 132 \) ft
27. \( \cos \theta = \frac{8}{18}, \quad \theta = 63.6^\circ \)
The graph of a quadratic function, a parabola, is a symmetrical curve. Its highest or lowest point is called the vertex. The graph of a parabola can be created by using an equation in the form \( y = ax^2 + bx + c \). In previous lessons students graphed parabolas by substituting values for \( x \) and then evaluating \( y \). This can be a tedious process. Another method for graphing a parabola is to determine the \( x \)-intercepts first, and then solve for the vertex and/or the \( y \)-intercept. To determine the \( x \)-intercepts, substitute 0 for \( y \) and solve the quadratic equation, \( 0 = ax^2 + bx + c \). One method for solving a quadratic equation is to factor and use the Zero Product Property. This method uses two ideas:

1. When a product is equal to zero, then at least one of the factors must be zero.
2. Some quadratic expressions can be factored into the product of two simple binomials.

For additional information see the Math Notes box in Lesson 5.1.4.

**Example 1**

Determine the \( x \)-intercepts of the parabola \( y = x^2 + 6x + 8 \). Then find the \( y \)-intercept and sketch the parabola.

First, substitute \( y = 0 \):

\[ 0 = x^2 + 6x + 8 \]

Then factor the quadratic expression.

\[ (x + 4)(x + 2) = 0 \]

Set each factor equal to 0.

\[ (x + 4) = 0 \text{ or } (x + 2) = 0 \]

Solve each equation for \( x \).

\[ x = -4 \text{ or } x = -2 \]

Find the \( y \)-intercept by substituting \( x = 0 \) and evaluating.

\[ y = (0)^2 + 6(0) + 8 = 8 \]

The graph is a parabola with \( x \)-intercepts at \((-4, 0)\) and \((-2, 0)\) and a \( y \)-intercept at \((0, 8)\).
Example 2

Use the $x$-intercepts and the vertex to sketch the graph of $-2x^2 + 7x + 15 = y$.

Substitute $y = 0$. 

$$-2x^2 + 7x + 15 = 0$$

Factor the quadratic expression. 

$$(-2x - 3)(x - 5) = 0$$

Set each factor equal to 0. 

$$(-2x - 3) = 0 \quad \text{or} \quad (x - 5) = 0$$

Solve each equation for $x$. 

$$-2x = 3 \quad \text{or} \quad x = 5$$

$$x = -\frac{3}{2} \quad \text{or} \quad x = 5$$

To find the vertex, average the $x$-intercepts and use that $x$-value to calculate the corresponding $y$-value. The average of the $x$-intercepts is 

$$\frac{5 + (-1.5)}{2} = \frac{3.5}{2} = 1.75.$$ 

Substituting this into the original equation yields: 

$$-2(1.75)^2 + 7(1.75) + 15 = 21.125.$$ 

Thus the vertex is located at $(1.75, 21.125)$. Use the vertex and the $x$-intercepts at $(-\frac{3}{2}, 0)$ and $(5, 0)$ to sketch the graph of the parabola.

Example 3

Sketch a graph of $9x^2 - 6x + 1 = y$.

Substitute $y = 0$. 

$$9x^2 - 6x + 1 = 0$$

Factor the quadratic expression. 

$$(3x - 1)(3x - 1) = 0$$

Solve each equation for $x$. Notice that the factors are the same so there will be only one solution. 

$$(3x - 1) = 0 \quad \text{or} \quad (3x - 1) = 0$$

$$3x = 1 \quad \text{or} \quad x = \frac{1}{3}$$

Substitute $x = 0$ into the original equation to determine that the $y$-intercept is $(0, 1)$. The graph is a parabola that has a single $x$-intercept at $(\frac{1}{3}, 0)$ and $y$-intercept at $(0, 1)$. 

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Problems

Use the Zero Product Property to determine the $x$-intercepts for the graph of each quadratic function.

1. $y = x^2 + x - 20$
2. $y = -x^2 - 7x - 12$
3. $y = 3x^2 - 7x - 6$
4. $y = 3x^2 + 11x + 10$
5. $y = 6x^2 + 5x - 4$
6. $y = -x^2 + 2x + 8$
7. $y = 6x^2 - x - 15$
8. $y = 4x^2 + 12x + 9$
9. $y = 2x^2 + 8x + 6$

Determine the $x$- and $y$-intercepts of the graph of each of the following quadratic functions and then sketch the graph.

10. $y = x^2 + 4x + 3$
11. $y = x^2 + 5x - 6$
12. $y = 2x^2 - 7x - 4$
13. $y = -3x^2 - 10x + 8$
14. $y = 16x^2 - 25$

Answers

1. $(-5, 0)$ and $(4, 0)$
2. $(-4, 0)$ and $(-3, 0)$
3. $(-\frac{2}{3}, 0)$ and $(3, 0)$
4. $\left(-\frac{5}{3}, 0\right)$ and $\left(-\frac{5}{3}, 0\right)$
5. $\left(-\frac{4}{3}, 0\right)$ and $\left(\frac{1}{2}, 0\right)$
6. $(4, 0)$ and $(-2, 0)$
7. $\left(-\frac{3}{2}, 0\right)$ and $\left(\frac{5}{3}, 0\right)$
8. $(0, 2)$
9. $(0, -4)$
10. $(-1, 0)$ and $(-3, 0)$
11. $\left(-\frac{3}{2}, 0\right)$
12. $(-1, 0)$ and $(-3, 0)$
13. $x$-intercepts: $(-1, 0), (-3, 0)$
   $y$-intercept: $(0, 3)$
14. $x$-intercepts: $(-6, 0), (1, 0)$
   $y$-intercept: $(0, -6)$
15. $x$-intercepts: $(-0.5, 0), (4, 0)$
   $y$-intercept: $(0, -4)$
16. $x$-intercepts: $(-4, 0), \left(\frac{2}{3}, 0\right)$
   $y$-intercept: $(0, 8)$
17. $x$-intercepts: $\left(-\frac{5}{4}, 0\right), \left(\frac{5}{4}, 0\right)$
   $y$-intercept: $(0, -25)$
SOLVING BY COMPLETING THE SQUARE  

A quadratic equation in the form \((ax - p)^2 = q\) is in **perfect square form**. There are two solutions to a perfect square equation like \(x^2 = 25\), because \((5)^2 = 25\) and \((-5)^2 = 25\). However, students often forget to consider both solutions, because the radical sign only refers to the principal square root, or the positive square root. Thus, the fact that \(\sqrt{x^2} = |x|\) is used in the solution process.

**Example 1**

\[
\begin{align*}
(x - 3)^2 &= 25 \\
\sqrt{(x - 3)^2} &= \sqrt{25} \\
|x - 3| &= 5 \\
x - 3 &= 5 \text{ or } x - 3 = -5 \\
x &= 8 \text{ or } -2 
\end{align*}
\]

**Example 2**

\[
\begin{align*}
(x - 3)^2 &= 5 \\
\sqrt{(x - 3)^2} &= \sqrt{5} \\
|x - 3| &= \sqrt{5} \\
x - 3 &= \sqrt{5} \text{ or } x - 3 = -\sqrt{5} \\
x &= 3 \pm \sqrt{5}
\end{align*}
\]

The process of rewriting a quadratic equation in standard form, \(ax^2 + bx + c = 0\), into perfect square form is called **completing the square**. This process is illustrated in the following examples.

See the Math Notes box in Lesson 5.2.4 for more information.

**Example 1**  Solve \(x^2 + 4x + 4 = 25\).

The left side is already a perfect square trinomial, as demonstrated by the algebra tiles.

Factor the left side to write it as a perfect square.

Then solve the perfect square equation.

Simplify.

\[
\begin{align*}
(x + 2)(x + 2) &= 25 \\
(x + 2)^2 &= 25 \\
\sqrt{(x + 2)^2} &= \sqrt{25} \\
|x + 2| &= 5 \\
x + 2 &= 5 \text{ or } x + 2 = -5 \\
x &= 3 \text{ or } x = -7
\end{align*}
\]
**Example 2** Solve \( x^2 + 8x + 10 = 0 \).

First, rewrite the equation so that the constant term is on the other side of the equal sign.

Make a partial algebra tile square for the left side of the equation. Then determine how many unit tiles are needed to complete the square.

In this example, 16 unit tiles are needed, so add 16 unit tiles to complete the square.

To keep the equation balanced, add 16 to both sides.

Factor the left side to write it as a perfect square, and solve.

\[
(x + 4)^2 = 6
\]

\[
|x + 4| = \sqrt{6}
\]

\[
x + 4 = \sqrt{6} \quad \text{or} \quad x + 4 = -\sqrt{6}
\]

\[
x = -4 \pm \sqrt{6}
\]

\[
x = -1.55 \quad \text{or} \quad x = -6.45
\]

**Example 3** Solve \( x^2 + 5x + 2 = 0 \).

Rewrite the equation with the constant term on the other side.

We need to make \( x^2 + 5x \) into a perfect square.

Imagine building a square for \( x^2 + 5x \) using algebra tiles. The five \( x \)-tiles must be equally divided, so the sides of the square will be \( x + 2.5 \) and \( x + 2.5 \).

There should be \((2.5)^2 = 6.25\) tiles in the upper right corner to complete the square. Add 6.25, or \(\frac{25}{4}\), to each side of the equation.

Factor the left side to write it as a perfect square, and solve.

\[
x = -\frac{5 \pm \sqrt{17}}{2} = -0.44, -4.56
\]
Problems
Solve each quadratic equation by completing the square.

1. $(x + 2)^2 = 3$
2. $y^2 - 6y + 9 = 25$
3. $x^2 + 2x - 3 = 0$
4. $x^2 + 8x = 5$
5. $x^2 + 6x + 2 = 0$
6. $x^2 + 10x - 75 = 0$
7. $x^2 - 6x + 2 = 0$
8. $y^2 + 5y = 14$
9. $x^2 + 6x + 1 = -10$
10. $x^2 - 2x - 3 = 0$
11. $x^2 + 4x = 3$
12. $x^2 + 3x = 3$
13. $x^2 - 3x - 13.75 = 0$
14. $x^2 - x - 3 = 0$
15. $x^2 - 10x + 12 = 6$

Answers
1. $x = -2 \pm \sqrt{3}$
2. $x = 8$ or $-2$
3. $x = 1$ or $-3$
4. $x = -4 \pm \sqrt{21}$
5. $x = -3 \pm \sqrt{7}$
6. $x = 5$ or $-15$
7. $x = 3 \pm \sqrt{7}$
8. $x = -7$ or $2$
9. no real solution
10. $x = -1$ or $3$
11. $x = -2 \pm \sqrt{7}$
12. $x = \frac{-3 \pm \sqrt{21}}{2}$
13. $x = -2.5$ or $5.5$
14. $x = \frac{1 \pm \sqrt{13}}{2}$
15. $x = 5 \pm \sqrt{19}$
USING THE QUADRATIC FORMULA

When a quadratic equation is not factorable, another method is needed to solve for \( x \).
You can always complete the square to solve a quadratic equation, but that can be challenging when the coefficients are large. The Quadratic Formula can be used to solve any quadratic equation, no matter how complicated.

The solution(s) to any quadratic equation \( ax^2 + bx + c = 0 \) are:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

The \( \pm \) symbol is read as “plus or minus.” It is shorthand notation that tells you to calculate the formula twice, once with + and again with – to get both \( x \)-values.

To use the formula, the quadratic equation must be written in standard form: \( ax^2 + bx + c = 0 \). This is necessary to correctly identify the values of \( a, b \), and \( c \). Once the equation is in standard form and equal to 0, \( a \) is the coefficient of the \( x^2 \)-term, \( b \) is the coefficient of the \( x \)-term and \( c \) is the constant term.

For additional information, see the Math Notes box in Lesson 5.2.5.

Example 1

Solve \( 2x^2 - 5x - 3 = 0 \).

Identify \( a, b \), and \( c \). Watch your signs.

\[ a = 2, \ b = -5, \ c = -3 \]

Write the Quadratic Formula.

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

Substitute \( a, b \), and \( c \) into the formula and do the initial calculations.

\[
x = \frac{5 \pm \sqrt{(-5)^2 - 4(2)(-3)}}{4}.
\]

Simplify the \( \sqrt{} \).

\[
x = \frac{5 \pm \sqrt{49}}{4}.
\]

Calculate both values of \( x \).

\[ x = \frac{5 + 7}{4} = \frac{12}{4} = 3 \quad \text{or} \quad x = \frac{5 - 7}{4} = \frac{-2}{4} = -\frac{1}{2} \]

The solutions are \( x = 3 \) or \( x = -\frac{1}{2} \).
Example 2

Solve \(3x^2 + 5x + 1 = 0\).

Identify \(a\), \(b\), and \(c\). \(a = 3, \ b = 5, \ c = 1\)

Write the Quadratic Formula. 

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Substitute \(a\), \(b\), and \(c\) into the formula and do the initial calculations.

\[ x = \frac{-5 \pm \sqrt{25 - 12}}{6} \]

Simplify the \(\sqrt{}\).

\[ x = \frac{-5 \pm \sqrt{13}}{6} \]

The solutions are \(x = \frac{-5 + \sqrt{13}}{6} \approx -0.23\) or \(x = \frac{-5 - \sqrt{13}}{6} \approx -1.43\).

Example 3

Solve \(25x^2 - 20x + 4 = 0\).

Identify \(a\), \(b\), and \(c\). \(a = 25, \ b = -20, \ c = 4\)

Write the Quadratic Formula. 

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Substitute \(a\), \(b\), and \(c\) into the formula and do the initial calculations.

\[ x = \frac{20 \pm \sqrt{400 - 400}}{50} \]

Simplify the \(\sqrt{}\).

\[ x = \frac{20 \pm \sqrt{0}}{50} \]

This quadratic equation has only one solution: \(x = \frac{2}{5}\).
Example 4

Solve \( x^2 + 4x = -10 \).

Rewrite the equation in standard form. \( x^2 + 4x + 10 = 0 \)

Identify \( a, b, \) and \( c \). \( a = 1, \ b = 4, \ c = 10 \)

Write the Quadratic Formula. \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)

Substitute \( a, b, \) and \( c \) into the formula and do the initial calculations.

\[ x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(10)}}{2(1)} \]
\[ x = \frac{-4 \pm \sqrt{16 - 40}}{2} \]

Simplify the \( \sqrt{\text{.}} \).

\[ x = \frac{-4 \pm \sqrt{-24}}{2} \]

There are no real numbers that can be squared to give \(-24\); therefore this quadratic equation has no real solutions.

Problems

Solve each equation by using the Quadratic Formula.

1. \( x^2 - x - 2 = 0 \)  
2. \( x^2 - x - 3 = 0 \)  
3. \(-3x^2 + 2x + 1 = 0 \)
4. \(-2 - 2x^2 = 4x \)  
5. \( 7x = 10 - 2x^2 \)  
6. \(-6x^2 - x + 6 = 0 \)
7. \( 6 - 4x + 3x^2 = 8 \)  
8. \( 4x^2 + x - 1 = 0 \)  
9. \( x^2 - 5x + 3 = 0 \)
10. \( 0 = 10x^2 - 2x + 3 \)  
11. \( x(-3x + 5) = 7x - 10 \)  
12. \((5x + 5)(x - 5) = 7x \)

Identify the error in each of the following solutions. Then write a correct solution to the problem.

13. Solve: \( 3x^2 + 6x + 1 = 0 \)
   \[ a = 3, \ b = 6, \ c = 1 \]
   \[ x = \frac{-6 \pm \sqrt{6^2 - 4(3)(1)}}{2(3)} \]
   \[ = \frac{6 \pm \sqrt{36 - 12}}{6} \]
   \[ = \frac{6 \pm \sqrt{24}}{6} \]
   \[ = 1 \pm \frac{\sqrt{24}}{6} \]

14. Solve: \(-2x^2 + 7x + 5 = 0 \)
   \[ a = -2, \ b = 7, \ c = 5 \]
   \[ x = \frac{-7 \pm \sqrt{7^2 - 4(-2)(5)}}{2(-2)} \]
   \[ = \frac{-7 \pm \sqrt{49 - 40}}{-4} \]
   \[ = \frac{-7 \pm 3}{-4} \]
   \[ = \frac{-4}{-4} = 1 \text{ or } x = \frac{-10}{-4} = 2.5 \]
Answers

1.  \( x = 2 \) or \(-1\)  
2.  \( x = \frac{1+\sqrt{13}}{2} \)  
\( \approx 2.30 \) or \(-1.30\)  
3.  \( x = -\frac{1}{3} \) or \(1\)  

4.  \( x = -1\)  
5.  \( x = \frac{-7+\sqrt{129}}{4} \)  
\( \approx 1.09 \) or \(-4.59\)  
6.  \( x = \frac{1+\sqrt{145}}{-12} \)  
\( \approx -1.09 \) or \(0.92\)  

7.  \( x = \frac{4+\sqrt{40}}{6} = \frac{2+\sqrt{10}}{3} \)  
\( \approx 1.72 \) or \(-0.39\)  
8.  \( x = \frac{-1+\sqrt{17}}{8} \)  
\( \approx 0.39 \) or \(-0.64\)  
9.  \( x = \frac{5+\sqrt{13}}{2} \)  
\( \approx 4.30 \) or \(0.70\)  

10. no real solution  
11. \( x = \frac{2+\sqrt{124}}{-6} = \frac{1+\sqrt{31}}{-3} \)  
\( \approx -2.19 \) or \(1.52\)  
12. \( x = \frac{27+\sqrt{1229}}{10} \)  
\( \approx 6.21 \) or \(-0.81\)  

13. The formula starts with “\(-b\)”; the negative sign was left off. \( x = -1 \pm \frac{\sqrt{24}}{6} \)  
14. Under the radical, “\(-4ac\)” should equal + 40. \( x = \frac{-7+\sqrt{89}}{-4} \)
COMPLEX NUMBERS

Complex numbers arise when trying to solve some equations such as $x^2 + 1 = 0$, which has no real solution. The equation does, however, have a complex solution.

The imaginary number $i$ is defined to be $\sqrt{-1}$, so $i^2 = -1$. When $i$ is multiplied by a real number, the result is another imaginary number, such as $2i, 3i$, and $i\sqrt{2}$. When an imaginary number is added to a real number, the result is called a complex number. Complex numbers are written in the form $a + bi$, where $a$ and $b$ are real numbers.

For additional information see the Historical Note and Math Notes box in Lesson 5.2.6.

Example 1

Use the definition of $i$ to simplify each of the following expressions.

a. $3 + \sqrt{-16}$

b. $(3 + 4i) + (-2 - 6i)$

c. $(4i)(-5i)$

d. $(8 - 3i)(8 + 3i)$

When simplifying, remember that $i = \sqrt{-1}$ and $i^2 = -1$.

Part (a): $3 + \sqrt{-16} = 3 + \sqrt{16}\sqrt{-1} = 3 + 4i$. This is the simplest form; the real and imaginary parts of the complex number cannot be combined.

Part (b): Combine like terms: $(3 + 4i) + (-2 - 6i) = 1 - 2i$.

Part (c): Use the Commutative Property to rearrange the expression: $(4i)(-5i) = (4)(-5)(i)(i) = -20i^2 = -20(-1) = 20$.

Part (d): Use the Distributive Property or an area model to compute this product.

\[
\begin{array}{cc}
8 & -3i \\
8 & 64 \\
24i & -24i \\
3i & 9 \\
\end{array}
\]

\[(8 - 3i)(8 + 3i) = 8(8) + 8(3i) - 3i(8) - 3i(3i) = 64 + 24i - 24i - 9(i^2) = 73 + 3i\]
Example 2

Solve the equation below using the Quadratic Formula. Explain what the solution tells you about the graph of the related function.

\[ 2x^2 - 20x + 53 = 0 \]

In this example, \( a = 2 \), \( b = -20 \), and \( c = 53 \).

Solution:

\[
x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(2)(53)}}{2(2)}
\]
\[
= \frac{20 \pm \sqrt{400 - 424}}{4}
\]
\[
= \frac{20 \pm \sqrt{-24}}{4}
\]
\[
= \frac{20 \pm 2i \sqrt{6}}{4}
\]
\[
= \frac{10 \pm i \sqrt{6}}{2}
\]
\[
= 5 \pm \frac{\sqrt{6}}{2}i
\]

Because the equation \( 0 = 2x^2 - 20x + 53 \) has complex solutions, this means that the graph of the related function \( y = 2x^2 - 20x + 53 \) does not cross the \( x \)-axis. Verify this with your graphing tool.

Problems

Simplify the following expressions.

1. \((6 + 4i) - (2 - i)\)
2. \(8i - \sqrt{-16}\)
3. \((-3)(4i)(7i)\)
4. \((5 - 7i)(-2 + 3i)\)
5. \((3 + 2i)(3 - 2i)\)
6. \((6 - 5i)(6 + 5i)\)
7. \(\sqrt{-49}\)
8. \((8i)^2\)
9. \((i - 3)^2\)
10. \((3 + 4i) + (7 - 2i)\)
11. \((5i)(2i)^2\)
12. \((4 + 9i)(1 - i)\)

Solve the following quadratic equations.

13. \(0 = 3x^2 + 5x + 4\)
14. \(x^2 + 2x = -5\)
15. \(8 = -x^2 - x\)
16. \(6x^2 + 5x + 3 = 0\)
17. \(-4x = x^2 + 4\)
18. \(2x^2 + 2x + 5 = 0\)
Answers

1. $4 + 5i$  
2. $4i$  
3. $84$

4. $11 + 29i$  
5. $13$  
6. $61$

7. $7i$  
8. $-64$  
9. $8 - 6i$

10. $10 + 2i$  
11. $-20i$  
12. $13 + 5i$

13. $x = \frac{-5 \pm \sqrt{5^2 - 4(3)(4)}}{2(3)} = \frac{-5 \pm \sqrt{23}}{6}$

14. $x = -1 \pm 2i$  
15. $x = \frac{-1 \pm \sqrt{31}}{2} = \frac{1}{2} \pm \frac{\sqrt{31}}{2} i$  
16. $x = \frac{-5 \pm \sqrt{47}}{12} = -\frac{5}{12} \pm \frac{\sqrt{47}}{12} i$

17. $x = -2$  
18. $x = \frac{-1 \pm 3i}{2} = -\frac{1}{2} \pm \frac{3}{2} i$
There are two special right triangles that occur often in mathematics: the $30^\circ$-$60^\circ$-$90^\circ$ triangle and the $45^\circ$-$45^\circ$-$90^\circ$ triangle. By AA~, all $30^\circ$-$60^\circ$-$90^\circ$ triangles are similar, and all $45^\circ$-$45^\circ$-$90^\circ$ triangles are similar. Consequently, for each type of triangle, the side lengths are proportional. The sides of these triangles follow these patterns. (To understand and generate these patterns, it can be helpful to think of a $30^\circ$-$60^\circ$-$90^\circ$ triangle as half of an equilateral triangle, and of a $45^\circ$-$45^\circ$-$90^\circ$ triangle as half of a square.)

Another useful tool when working with right triangles is to recognize when side lengths of right triangles are Pythagorean Triples. The lengths 3, 4, and 5 form a Pythagorean Triple because the three side lengths are integers and satisfy the Pythagorean Theorem. The sides of all triangles similar to the $3:4:5$ triangle will have lengths in the same ratio that form Pythagorean Triples ($6:8:10, 9:12:15$, etc.). Another common Pythagorean Triple is $5:12:13$.

For additional information see the Math Notes box in Lesson 6.1.3.

**Example 1**

Use the $30^\circ$-$60^\circ$-$90^\circ$ and $45^\circ$-$45^\circ$-$90^\circ$ triangle patterns to determine the lengths of the unlabeled sides of each triangle below.

a.

b.

c.

d.

e.

f.
Part (a): This is a 30°-60°-90° triangle. The pattern shows that the hypotenuse is twice the length of the short leg. Since the short leg has a length of 6, the hypotenuse has a length of 12. The long leg is the length of the short leg times $\sqrt{3}$, so the long leg has a length of $6\sqrt{3}$.

(Note: You can also use the Pythagorean Theorem to solve for the last side once you know the lengths of two sides.)

Part (b): This is also a 30°-60°-90° triangle, but this time the length of the hypotenuse is known. Following the pattern, this means the length of the short leg is half the hypotenuse: 7. As before, multiply the length of the short leg by $\sqrt{3}$ to get the length of the long leg: $7\sqrt{3}$.

Part (c): This is a 45°-45°-90° triangle. (You can verify that the missing angle measure is 45° by remembering that the sum of the angles of a triangle is 180°.) The legs of a 45°-45°-90° triangle are equal in length (it is isosceles) so the length of the missing leg is also 5. To calculate the length of the hypotenuse, multiply the length of a leg by $\sqrt{2}$. Therefore the hypotenuse has length $5\sqrt{2}$.

Part (d): This is another 30°-60°-90° triangle. This time the length of the long leg is given. To calculate the length of the short leg, divide the length of the longer leg by $\sqrt{3}$. Therefore, the length of the short leg is 8. To calculate the length of the hypotenuse, double the length of the short leg, so the hypotenuse is 16.

Part (e): This is a 45°-45°-90° triangle with the length of the hypotenuse is given. To determine the lengths of the legs (which are equal in length), divide the length of the hypotenuse by $\sqrt{2}$. Therefore, each leg has length 6.

Part (f): This is another 45°-45°-90° triangle with the length of the hypotenuse given. In the part (e), when given the length of the hypotenuse, we divided by $\sqrt{2}$ to calculate the length of the legs. Do the same thing here.

\[ \frac{10}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2} \]

Note: Multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$ is called rationalizing the denominator. It is a technique for removing the radical from the denominator. The Math Notes box in Lesson 6.1.1 describes this process.
Example 2

Use what you know about Pythagorean Triples and similar triangles to fill in the missing lengths of sides below.

a.  

b.  

c.  

There are a few common Pythagorean Triples that students should recognize; 3:4:5, 5:12:13, 8:15:17, and 7:24:25 are the most common. If you forget about a particular triple or do not recognize one, you can always determine the unknown side by using the Pythagorean Theorem if two of the sides are given.

Part (a): This triangle is similar to the 3 : 4 : 5 triangle. The length of the hypotenuse is 500.

Part (b): Each leg has a length that is a multiple of 4. Knowing this, you can rewrite the lengths as 48 = 4(12), and 20 = (4)(5). This is to the 5 : 12 : 13 triangle. The length of the hypotenuse is 4(13) = 52.

Part (c): Do not let the decimal bother you. In fact, since we are working with Pythagorean Triples and their multiples, double both sides to create a similar triangle. This eliminates the decimal and makes a similar triangle with a leg length of 24 and a hypotenuse of 25. This makes it easier to recognize the triple as 7 : 24 : 25. Since the sides of this triangle are half of the 7 : 24 : 25 triangle, the length of the leg corresponding to 7 is 3.5.
**Problems**

Identify the special triangle relationships. Then solve for $x$, $y$, or both.

1. 
   \[ \begin{align*}
   x &= 16 \\
   y &= 60^\circ
   \end{align*} \]

2. 
   \[ \begin{align*}
   x\_\text{side} &= 8\sqrt{2} \\
   x\_\text{angle} &= 45^\circ
   \end{align*} \]

3. 
   \[ \begin{align*}
   x &= 5 \\
   y &= 12
   \end{align*} \]

4. 
   \[ \begin{align*}
   x\_\text{side} &= 1000 \\
   x\_\text{angle} &= 60^\circ
   \end{align*} \]

5. 
   \[ \begin{align*}
   x\_\text{side} &= 6 \\
   x\_\text{angle} &= 45^\circ
   \end{align*} \]

6. 
   \[ \begin{align*}
   x\_\text{side} &= 12 \\
   x\_\text{angle} &= 45^\circ
   \end{align*} \]

7. 
   \[ \begin{align*}
   x\_\text{side} &= 11 \\
   x\_\text{angle} &= 30^\circ
   \end{align*} \]

8. 
   \[ \begin{align*}
   x\_\text{side} &= 22.5\sqrt{3} \\
   x\_\text{angle} &= 60^\circ
   \end{align*} \]

9. 
   \[ \begin{align*}
   x\_\text{side} &= 16 \\
   x\_\text{angle} &= 30^\circ
   \end{align*} \]

10. 
    \[ \begin{align*}
    x\_\text{side} &= 50 \\
    x\_\text{angle} &= 14
    \end{align*} \]

**Answers**

1. \( x = 8\sqrt{3} \), \( y = 8 \)
2. \( x = y = 8 \)
3. \( x = 13 \)
4. \( y = 800 \)
5. \( x = 6 \), \( y = 6\sqrt{2} \)
6. \( x = y = \frac{12}{\sqrt{2}} = 6\sqrt{2} \)
7. \( x = 11\sqrt{3} \), \( y = 22 \)
8. \( x = 45 \), \( y = 22.5 \)
9. \( x = 34 \)
10. \( x = 48 \)
FRACTIONAL EXPONENTS

6.1.4

A fractional exponent is equivalent to an expression with roots or radicals.

For \( x \neq 0 \) and \( n \neq 0 \), \( x^{m/n} = (x^m)^{1/n} = \sqrt[n]{x^m} \) or \( x^{m/n} = (x^{1/n})^m = (\sqrt[n]{x})^m \).

Fractional exponents can also be used to solve equations containing exponents. For additional information, see the Math Notes box in Lesson 6.2.1.

Example 1

Rewrite each expression in radical form and simplify if possible.

a. \( 16^{5/4} \)

Solution:

\[
\begin{align*}
16^{5/4} &= (16^{1/4})^5 \\
&= (\sqrt[4]{16})^5 \\
&= (2)^5 \\
&= 32
\end{align*}
\]

b. \( (-8)^{2/3} \)

\[
\begin{align*}
(-8)^{2/3} &= ((-8)^{1/3})^2 \\
&= (\sqrt[3]{-8})^2 \\
&= (-2)^2 \\
&= 4
\end{align*}
\]

Example 2

Simplify each expression. Answer should contain no parentheses and no negative exponents.

a. \( (144x^{-12})^{1/2} \)

Using the Laws of Exponents:

\[
\begin{align*}
(144x^{-12})^{1/2} &= \left( \frac{144}{x^{12}} \right)^{1/2} \\
&= \sqrt{\frac{144}{x^{12}}} \\
&= \frac{12}{x^6}
\end{align*}
\]

b. \( \left( \frac{8x^7y^3}{x} \right)^{-1/3} \)

\[
\begin{align*}
\left( \frac{8x^7y^3}{x} \right)^{-1/3} &= \left( \frac{8x^{7-1}y^3}{x} \right)^{1/3} \\
&= \left( \frac{8x^6y^3}{x} \right)^{1/3} \\
&= \sqrt[3]{\frac{1}{8x^6y^3}} \\
&= \frac{1}{2x^2y}
\end{align*}
\]
Problems

Rewrite each expression as at least three different equivalent expressions and then simplify.

1. \((64)^{2/3}\)
2. \(16^{-1/2}\)
3. \((-27)^{1/3}\)

Simplify the following expressions. Your final expressions should contain no negative exponents and no parentheses.

4. \(\left( \frac{3}{5x} \right)^{-2}\)
5. \((36^{1/2}x^4)(16x^3)\)
6. \(\left( \frac{x^7y^3}{x} \right)^{1/3}\)

7. \((16a^8b^{12})^{3/4}\)
8. \(\frac{144^{1/2}x^{-3}}{(16^{3/4}x^7)^0}\)
9. \(\frac{a^{2/3}b^{-3/4}c^{7/8}}{a^{-1/3}b^{1/4}c^{1/8}}\)

Answers

Example answers are given for problems 1–3; other answers are possible.

1. \(\sqrt[3]{64^2}, (64^2)^{1/3}, (\sqrt[3]{64})^2, 16\)
2. \(\frac{1}{(16)^{1/2}} \cdot \sqrt[4]{16}, \frac{1}{\sqrt[4]{16}}, \frac{1}{4}\)
3. \(\left( (-3)^3 \right)^{1/3}, \sqrt[3]{-1}, \frac{1}{\sqrt[3]{27}}, \frac{1}{\sqrt[3]{-27}}, -3\)
4. \(\frac{25x^2}{9}\)
5. \(96x^7\)
6. \(x^2y\)
7. \(8a^6b^9\)
8. \(\frac{12}{x^3}\)
9. \(\frac{ac^{3/4}}{b}\)
QUADRILATERALS AND PROOF  

By tracing and reflecting triangles to form quadrilaterals, students discover properties about quadrilaterals. More importantly, they develop a method to prove that what they have observed is true. Students are already familiar with using flowcharts to organize information, and in this section they learn how to organize proofs in a two-column format. Since they develop their conjectures by reflecting triangles, their proofs rely heavily on the triangle congruence theorems from Chapter 2. Once students prove that their conjectures are true, they can use the theorems in later problems.

See the Math Notes boxes in Lessons 7.1.1, 7.1.5, and 7.2.1 as well as the Theorem Graphic Organizer for additional information.

Example 1

ABCD at right is a parallelogram. Use the definition of parallelogram and theorems about angles and triangles to prove that the opposite sides of ABCD are congruent.

By the definition of parallelogram, the opposite sides of ABCD are parallel. Whenever there are parallel lines with a transversal, you can look for pairs of congruent angles. Also, congruent triangles can help to prove congruent sides. Drawing in diagonal AC, as shown at right, creates a transversal for parallel sides AB and DC as well as for AD and BC. It also creates two triangles.

Since AB \parallel CD , \angle BAC \cong \angle DCA because alternate interior angles formed by parallel lines are congruent. Similarly, since AD \parallel CB , \angle DAC \cong \angle BCA. Also, AC \equiv CA by the Reflexive Property. This is indicated in the diagram at right.

Using these three pairs of congruent, corresponding parts we can conclude that \triangle BAC \cong \triangle DCA by ASA \cong.

Since the triangles are congruent, we know that all of their corresponding parts are also congruent. In particular, AB \equiv CD and AD \equiv CB , which proves that the opposite sides of ABCD are congruent.

This is organized in a flowchart proof on the following page.
As a flowchart proof, this argument could be presented as shown at right.

Example 2

In the figure at right, if \( \overline{AI} \) is the perpendicular bisector of \( \overline{DV} \), is \( \triangle DAV \) isosceles? Prove your conclusion using the two-column proof format.

Before starting a two-column proof, it is helpful to think about what is being proved. To prove that a triangle is isosceles, we must show that \( \overline{DA} \equiv \overline{VA} \) because an isosceles triangle has two sides congruent. By showing that \( \triangle AID \equiv \triangle AVI \), we can then conclude that this pair of sides is congruent.

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \overline{AI} ) is the perpendicular bisector of ( \overline{DV} )</td>
<td>Given</td>
</tr>
<tr>
<td>( \overline{DI} \equiv \overline{VI} )</td>
<td>Definition of bisector</td>
</tr>
<tr>
<td>( \angle DIA ) and ( \angle VIA ) are right angles</td>
<td>Definition of perpendicular</td>
</tr>
<tr>
<td>( \angle DIA \equiv \angle VIA )</td>
<td>All right angles are congruent</td>
</tr>
<tr>
<td>( \overline{AI} \equiv \overline{AI} )</td>
<td>Reflexive Property</td>
</tr>
<tr>
<td>( \triangle DAI \equiv \triangle VAI )</td>
<td>SAS ( \equiv )</td>
</tr>
<tr>
<td>( \overline{DA} \equiv \overline{VA} )</td>
<td>( \equiv \angle s \rightarrow \equiv ) parts</td>
</tr>
<tr>
<td>( \triangle DAV ) is isosceles</td>
<td>Definition of isosceles</td>
</tr>
</tbody>
</table>
Problems

Be sure to justify your answers below. Each part is a new problem.

For problems 1–4 use the parallelogram at right.
1. If \( CD = 12 \) and \( CB = 10 \), what is the perimeter of \( ABCD \)?
2. If \( CT = 9 \), determine \( AT \).
3. If \( m \angle CDA = 60^\circ \), calculate \( m \angle CBA \) and \( m \angle BAD \).
4. If \( AT = 4x - 7 \) and \( CT = -x + 13 \), solve for \( x \).

For problems 5–8 use the rhombus at right.
5. If \( PS = \sqrt{6} \), what is the perimeter of \( PQRS \)?
6. If \( PQ = 3x + 7 \) and \( QR = -x + 17 \), solve for \( x \).
7. If \( m \angle PSM = 22^\circ \), calculate \( m \angle RSM \) and \( m \angle SPQ \).
8. If \( m \angle PMQ = 4x - 5 \), solve for \( x \).

For problems 9–12 use the quadrilateral at right.
9. If \( WX = YZ \) and \( WZ = XY \), must \( WXYZ \) be a rectangle?
10. If \( m \angle WZY = 90^\circ \), must \( WXYZ \) be a rectangle?
11. If the conditions in problems 9 and 10 are both true, must \( WXYZ \) be a rectangle?
12. If the conditions in problems 9 and 10 are true, \( WY = 15 \), and \( WZ = 9 \), what are \( YZ \) and \( XZ \)?

For problems 13–16 use the kite at right.
13. If \( m \angle XWZ = 95^\circ \), determine \( m \angle XYZ \).
14. If \( m \angle WZY = 110^\circ \) and \( m \angle WXY = 40^\circ \), calculate \( m \angle ZWX \).
15. If \( WZ = 5 \) and \( WT = 4 \), calculate \( ZT \).
16. If \( WT = 4 \), \( TZ = 3 \), and \( TX = 10 \), then what is the perimeter of \( WXYZ \)?
17. If \( \overline{PQ} \cong \overline{RS} \) and \( \overline{QR} \cong \overline{SP} \), is \( PQRS \) a parallelogram? Prove your answer using the format of your choice.
For problems 18–20, use the figure at right. Base your decision on the markings, not appearances.

18. Is \( \triangle ABCD \cong \triangle EDC \)? Prove your answer.

19. Is \( AB \cong ED \)? Prove your answer.

20. Is \( AB \cong DC \)? Prove your answer.

21. If \( NIFH \) is a parallelogram, is \( ES \cong ET \)? Prove your answer.

22. If \( DSIA \) is a parallelogram and \( IA \cong IV \), is \( \angle D \cong \angle V \)? Prove your answer.

Answers

1. 44 units
2. 9 units
3. \( 60^\circ, 120^\circ \)
4. 4
5. \( 4\sqrt{6} \)
6. 2.5
7. \( 22^\circ, 136^\circ \)
8. 23.75
9. no
10. no
11. yes
12. 12, 15
13. \( 95^\circ \)
14. \( 105^\circ \)
15. 3
16. \( 10 + 4\sqrt{29} \)

17. Yes. \( QS \cong SQ \) (Reflexive Prop.). This fact, along with the given information means that \( \triangle PQS \cong \triangle RSQ \) (SSS \( \cong \)). The corresponding parts are also congruent, so \( \angle PQS \cong \angle RSQ \) and \( \angle PSQ \cong \angle RQS \). These angles are alternate interior angles, so both pairs of opposite sides are parallel. Therefore, \( PQRS \) is a parallelogram. As a flowchart proof, this can be presented as shown at right.
18. Yes. Since the lines are parallel, the alternate interior angles are congruent so $\angle BDC \equiv \angle ECD$. Also, $\overline{DC} \equiv \overline{CD}$ (Reflexive Prop.) so the triangles are congruent by SAS $\equiv$. As a flowchart proof, this can be presented as shown at right.

19. Not necessarily (there is not enough information to prove).

20. Not necessarily (there is not enough information to prove).

21. Yes. Because $\text{NIFH}$ is a parallelogram, we know several things. First, $\angle TNE \equiv \angle SFE$ (alternate interior angles) and $\overline{NE} \equiv \overline{EF}$ (diagonals of a parallelogram bisect each other). Also, $\angle TEN \equiv \angle SEF$ because vertical angles are congruent. This gives us $\triangle NTE \equiv \triangle FSE$ by ASA $\equiv$. Therefore, the corresponding parts of the triangle are congruent, so $\overline{ES} \equiv \overline{ET}$.

As a flowchart proof, this argument could be presented as shown at right.

22. Since $\text{DSIA}$ is a parallelogram, $\overline{DS} \parallel \overline{AI}$, and therefore $\angle D \equiv \angle IAV$ (corresponding angles). Also, since $\overline{IA} \equiv \overline{IV}$, $\triangle IAV$ is isosceles, so $\angle IAV \equiv \angle V$ because the base angles of an isosceles triangle are congruent. These two angle congruence statements allow us to use substitution or the Transitive Property to conclude that $\angle D \equiv \angle V$. As a flowchart proof, this can be presented as shown at right.
The probability of one event occurring, knowing that another event has already occurred is called a conditional probability. Two-way tables are useful for representing conditional probability situations.

For additional information see the Math Notes boxes in Lessons 7.2.2 and 7.2.3.

Example 1

For the spinners at right, assume that the smaller sections of spinner #1 are half the size of the larger section and for spinner #2 assume that the smaller sections are one-third the size of the larger section.

a.  Draw a diagram for spinning each spinner once.

b.  What is the probability of getting the same color twice?

c.  If you know you got the same color twice, what is the probability it was red?

The diagram for part (a) is shown at right. Note that the boxes do not need to be to scale.

The circled boxes indicate getting the same color, so the total probability for part (b) is: \( \frac{1}{8} + \frac{1}{12} + \frac{1}{24} = \frac{6}{24} = \frac{1}{4} \).

For part (c), both red (RR) is \( \frac{1}{8} \) out of the \( \frac{1}{4} \) from part (b). Therefore, the probability that both spins were red knowing that you got the same color twice is \( \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{8} \cdot \frac{4}{1} = \frac{1}{2} \).
Example 2

A soda company conducted a taste test for three different kinds of soda that it makes. It surveyed 200 people in each age group about their favorite flavor and the results are shown in the table below.

<table>
<thead>
<tr>
<th>Age</th>
<th>Soda A</th>
<th>Soda B</th>
<th>Soda C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Under 20</td>
<td>30</td>
<td>44</td>
<td>126</td>
</tr>
<tr>
<td>20 to 39</td>
<td>67</td>
<td>75</td>
<td>58</td>
</tr>
<tr>
<td>40 to 59</td>
<td>88</td>
<td>78</td>
<td>34</td>
</tr>
<tr>
<td>60 and over</td>
<td>141</td>
<td>49</td>
<td>10</td>
</tr>
</tbody>
</table>

a. What is the probability that a participant chose Soda C or was under 20 years old?

b. What is the probability that Soda A was chosen?

c. If Soda A was chosen, what is the probability that the participant was 60 years old or older?

For part (a), using the Addition Rule:

\[ P(C \text{ or } < 20) = P(C) + P(< 20) - P(C \text{ and } < 20) = \frac{228}{800} + \frac{200}{800} - \frac{126}{800} = \frac{302}{800} = 0.3775 \text{, or about 38\%.} \]

For part (b), add the number of participants selecting Soda A:

\[ \frac{30 + 67 + 88 + 141}{800} = \frac{326}{800} = 0.4075 \text{, or about 41\%.} \]

For part (c), take only the participants over 60 who selected Soda A out of all those selecting Soda A:

\[ \frac{141}{30 + 67 + 88 + 141} = 0.43 \text{, or about 43\%.} \]

Problems

1. Two six-sided dice are thrown.
   a. How many ways are there to get 7 points?
   b. What is the probability of getting 7 points?
   c. If you got 7 points, what is the probability that the result on one of the dice was a 5?

2. Elizabeth and Scott are playing a game at the state fair that uses two spinners that are shown in the diagrams at right. The player spins both wheels and if the colors match the player wins a prize.
   a. Make a probability diagram for this situation.
   b. What is the probability of winning a prize?
   c. If you won a prize, what is the probability that the matching colors were red?
3. An airline wants to determine if passengers not checking luggage is related to people being on business trips. Data for 1000 random passengers at an airport was collected and summarized in the table below.

<table>
<thead>
<tr>
<th>Traveling for business</th>
<th>Checked Baggage</th>
<th>No Checked Baggage</th>
</tr>
</thead>
<tbody>
<tr>
<td>103</td>
<td>387</td>
<td></td>
</tr>
<tr>
<td>216</td>
<td>294</td>
<td></td>
</tr>
</tbody>
</table>

a. What is the probability that a passenger did not check baggage?
b. If a passenger is traveling for business, what is the probability that the passenger did not check baggage?

4. In Canada, 92% of the households have televisions. 72% of households have televisions and Internet access. What is the probability that a house has Internet given that it has a television?

5. There is a 25% chance that Claire will have to work tonight and cannot study for the big math test. If Claire studies, then she has an 80% chance of earning a good grade. If she does not study, she only has a 30% chance of earning a good grade.

a. Draw a diagram to represent this situation.
b. Calculate the probability of Claire earning a good grade on the math test.
c. If Claire earned a good grade, what is the probability that she studied?

6. A bag contains 4 blue marbles and 2 yellow marbles. Two marbles are randomly chosen (the first marble is NOT replaced before drawing the second one).

a. What is the probability that both marbles are blue?
b. What is the probability that both marbles are yellow?
c. What is the probability of one blue and then one yellow?
d. If you are told that both selected marbles are the same color, what is the probability that both are blue?
7. At Cal’s Computer Warehouse, Cal wants to know the probability that a customer who comes into his store will buy a computer or a printer. He collected the following data during a recent week: 233 customers entered the store, 126 purchased computers, 44 purchased printers, and 93 made no purchase.
   a. Draw a Venn diagram to represent the situation.
   b. Based on this data, what is the probability that the next customer who comes into the store will buy a computer or a printer?
   c. Cal has promised a raise for his salespeople if they can increase the probability that the customers who buy computers also buy printers. For the given data, what is the probability that if a customer bought a computer, he or she also bought a printer?

8. A survey of 200 recent high school graduates found that 170 had driver licenses and 108 had jobs. Twenty-one graduates said that they had neither a driver license nor a job.
   a. Draw a two-way table to represent the situation.
   b. If one of these 200 graduates is randomly selected, what is the probability that he or she has a job and no license?
   c. If the randomly selected graduate is known to have a job, what is the probability that he or she has a license?

9. At McDougal’s Giant Hotdogs 15% of the workers are under 18 years old. The most desirable shift is 4 to 8 p.m. and 80% of the workers under 18 years old have that shift. 30% of the workers who are 18 years or older have the 4 to 8 p.m. shift.
   a. Represent these probabilities in a two-way table.
   b. What is the probability that a randomly selected worker is 18 or over and does not work the 4 to 8 p.m. shift?
   c. What is the probability that a randomly selected worker from the 4 to 8 p.m. shift is under 18 years old?
Answers

1. a. 6 ways  
b. \( \frac{6}{36} = \frac{1}{6} \)  
c. \( \frac{2}{36} + \frac{6}{36} = \frac{1}{3} \)

2. a. See diagram at right.  
b. \( \frac{1}{6} + \frac{12}{12} = \frac{1}{4} \)  
c. \( \frac{1}{6} + \frac{1}{4} = \frac{2}{3} \)

3. a. \( \frac{681}{1000} = 0.681 \)  
b. \( \frac{387}{1034} \approx 0.79 \)

4. \( \frac{72\%}{92\%} = 78\% \)

5. a. See diagram at right.  
b. \( 0.075 + 0.60 = 0.675 = 67.5\% \)  
c. \( \frac{0.60}{0.675} = 0.89 \approx 89\% \)

6. a. \( \frac{4}{6} \cdot \frac{3}{5} = \frac{2}{5} \)  
b. \( \frac{2}{6} \cdot \frac{1}{5} = \frac{1}{15} \)  
c. \( \frac{4}{6} \cdot \frac{2}{5} = \frac{4}{15} \)  
d. \( \frac{5}{2} + \left( \frac{2}{5} + \frac{1}{15} \right) = \frac{6}{7} \)

7. a. computer  
   printer  
   \[ \begin{array}{c} 96 \\ 30 \\ 14 \end{array} \]

8. a. See diagram at right.  
b. \( \frac{9}{200} \)  
c. \( \frac{99}{108} \)

9. a. See diagram at right.  
b. 0.595  
c. \( \frac{0.12}{0.375} = 0.32 \)
After studying triangles and quadrilaterals, students now extend their study to all polygons, with particular attention to regular polygons, which are equilateral and equiangular. Using the fact that the sum of the measures of the angles in a triangle is 180°, students describe a method to determine the sum of the measures of the interior angles of any polygon. Next they explore the sum of the measures of the exterior angles of a polygon. Finally they use the information about the angles of polygons along with their triangle tools to determine the angle measures and areas of regular polygons.

For additional information see the Math Notes boxes in Lessons 8.2.2 and 8.4.1.

Example 1

The figure at right is a hexagon. What is the sum of the measures of the interior angles of a hexagon? Explain how you know. Then write an equation and solve for $x$.

One way to calculate the sum of the interior angles of the hexagon is to divide the polygon into triangles. One way to divide the hexagon into triangles is to draw all of the diagonals from a single vertex, as shown at right. Doing this forms four triangles, each with angle measures summing to 180°.

\[
\frac{m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 + m\angle 5 + m\angle 6 + m\angle 7 + m\angle 8 + m\angle 9 + m\angle 10 + m\angle 11 + m\angle 12}{180^\circ} = 4(180^\circ) = 720^\circ
\]

Note: Students may notice that the number of triangles drawn from a single vertex is always two less than the number of sides. This example illustrates why the sum of the interior angles of a polygon may be calculated using the formula sum of interior angles = $180^\circ(n - 2)$, where $n$ is the number of sides of the polygon.

Now using the sum of the angles, write an equation, and solve for $x$.

\[
(3x + 1^\circ) + (4x + 7^\circ) + (x + 1^\circ) + (3x - 5^\circ) + (5x - 4^\circ) + (2x) = 720^\circ
\]

\[
18x = 720^\circ
\]

\[
x = 40^\circ
\]
**Example 2**

If the sum of the measures of the interior angles of a polygon is $2340^\circ$, how many sides does the polygon have?

Use the formula sum of interior angles $= 180^\circ(n - 2)$ to write an equation and solve for $n$. The solution is shown at right.

Since $n = 15$, the polygon has 15 sides.

It is important to note that if the answer is not a whole number, then either an error was made or there is no polygon with interior angles that sum to the given measure. Since the answer is the number of sides, the answer must be a whole number. Polygons cannot have “7.2” sides!

**Example 3**

What is the measure of an exterior angle of a regular decagon?

A decagon is a 10-sided polygon. The sum of the measures of the exterior angles of any polygon, one at each vertex, is always $360^\circ$, no matter how many sides the polygon has. In this case the ten exterior angles are congruent since the decagon is regular. Therefore, each angle measures $\frac{360^\circ}{10} = 36^\circ$.

**Example 4**

A regular dodecagon (12-sided polygon) has a side length of 8 cm. What is the area of the dodecagon?

Imagine dividing the dodecagon into twelve congruent triangles, radiating from the center, as shown at right. If we determine the area of one of the triangles, then we can multiply it by twelve to get the area of the entire dodecagon.

One of the triangles is enlarged at right. The triangle is isosceles, so drawing a segment from the vertex angle perpendicular to the base is the height. Because the triangle is isosceles, the height bisects the base.

Calculate the sum of all the interior angles of the dodecagon by using the formula $(180^\circ)(12 - 2) = 1800^\circ$. 
Since it is a regular dodecagon, all the interior angles are congruent, so each angle measures \(1800^\circ \div 12 = 150^\circ\). The segments radiating from the center form congruent isosceles triangles, so the base angles of each triangle measure \(75^\circ\) (half of the \(150^\circ\) angle, as shown in the diagram on the previous page). Use trigonometry to calculate the value of \(h\) as shown as right. It is best to use an unrounded value of \(h\) to calculate the area, and then round the answer appropriately at the end of your calculations.

Therefore the area of one of these triangles is: 
\[
A = \frac{1}{2} (8\text{ cm})(14.928\text{ cm}) = 59.713\text{ cm}^2
\]

Multiply the area of one triangle by 12 to get the area of the entire dodecagon. 
\[
A \approx 12(59.713\text{ cm}) = 716.55, \text{ or about } 717\text{ cm}^2
\]

**Problems**

Calculate the measures of the angles in each problem below.

1. The sum of the interior angles of a heptagon (7-gon).
2. The sum of the interior angles of an octagon (8-gon).
3. The measure of each interior angle of a regular dodecagon (12-gon).
4. The measure of each interior angle of a regular 15-gon.
5. The measure of each exterior angle of a regular 17-gon.
6. The measure of each exterior angle of a regular 21-gon.

Solve for \(x\) in each of the figures below.

7. 

8. 

9. 

10. 

Complete each of the following problems.

11. Each exterior angle of a regular \(n\)-gon measures \(16 \frac{4}{11}\)°. How many sides does this \(n\)-gon have?
12. Each exterior angle of a regular \(n\)-gon measures \(13 \frac{1}{3}\)°. How many sides does this \(n\)-gon have?
13. Each interior angle of a regular \(n\)-gon measures \(156^\circ\). How many sides does this \(n\)-gon have?
14. Each interior angle of a regular \(n\)-gon measures \(165.6^\circ\). How many sides does this \(n\)-gon have?
15. What is the area of a regular pentagon with side length 8.0 cm?

16. Calculate the area of a regular hexagon with side length 10.0 ft.

17. Calculate the area of a regular octagon with side length 12.0 m.

18. What is the area of a regular decagon with side length 14.0 in?

19. Using the pentagon at right, write an equation and solve for $x$.

20. What is the sum of the measures of the interior angles of a 14-sided polygon?

21. What is the measure of each interior angle of a regular 16-sided polygon?

22. What is the sum of the measures of the exterior angles of a decagon (10-gon)?

23. Each exterior angle of a regular polygon measures 22.5°. How many sides does the polygon have?

24. Is there a polygon with interior angle measures that add up to 3060°? If so, how many sides does it have? If not, explain why not.

25. Is there a polygon with interior angle measures that add up to 1350°? If so, how many sides does it have? If not, explain why not.

26. Is there a polygon with interior angle measures that add up to 4410°? If so, how many sides does it have? If not, explain why not.

27. In the figure at right, $ABCDE$ is a regular pentagon. Is $EB \parallel DF$? Justify your answer.

28. What is the area of a regular pentagon with a side length of 10 units?

29. What is the area of a regular 15-gon with a side length of 5 units?
Answers

1. 900°  
2. 1080°  
3. 150°  
4. 156°  
5. ≈ 21.2°  
6. ≈ 17.1°  
7. x = 24°  
8. x = 30°  
9. x ≈ 98.2°  
10. x ≈ 31.3°  
11. 22 sides  
12. 27 sides  
13. 15 sides  
14. 25 sides  
15. ≈ 110.1 cm²  
16. ≈ 259.8 ft²  
17. ≈ 695.3 m²  
18. ≈ 1508.1 in²  
19. 19x + 7° = 540°; x ≈ 28.1°  
20. 2160°  
21. 157.5°  
22. 360°  
23. 16 sides  
24. 19 sides  
25. No. The result is not a whole number.  
26. No. The result is not a whole number.  
27. Yes. Since $ABCDE$ is a regular pentagon, the measure of each interior angle is 108°. Therefore, $m\angle DCB = 108°$. Since $\angle DCB$ and $\angle FCB$ are supplementary, $m\angle FCB = 72°$. Thus $\overline{EB} \parallel \overline{DF}$ because alternate interior angles $\angle FCB$ and $\angle EBC$ are congruent.  
28. ≈ 172.0 sq. units  
29. ≈ 441.1 sq. units
In this section, students return to similarity to explore what happens to the area of a figure if it is reduced or enlarged. In Chapter 2, students learned about the ratio of similarity, also called the “scale factor.” If two similar figures have a ratio of similarity of \( \frac{a}{b} \), then the ratio of their perimeters is also \( \frac{a}{b} \), while the ratio of their areas is \( \frac{a^2}{b^2} \).

**Example 1**

The polygons \( P \) and \( Q \) at right are similar.

a. What is the ratio of similarity?

b. What is the perimeter of polygon \( P \)?

c. Use your previous two answers to determine the perimeter of polygon \( Q \).

d. If the area of polygon \( P \) is 20 square units, what is the area of polygon \( Q \)?

The ratio of similarity is the ratio of the lengths of two corresponding sides. In this case, use the side of \( P \) that corresponds to the side of \( Q \) that is labeled with its length. The ratio of similarity is \( \frac{3}{7} \).

To calculate the perimeter of \( P \), add all the side lengths: \( 3 + 6 + 4 + 5 + 3 = 21 \). If the ratio of similarity of the two polygons is \( \frac{3}{7} \), then the ratio of their perimeters is also \( \frac{3}{7} \).

If the ratio of similarity is \( \frac{3}{7} \), then the ratio of the areas is \( \left( \frac{3}{7} \right)^2 = \frac{9}{49} \).

\[
\begin{align*}
\text{perimeter } P &= 3 \cdot 7 \\
\frac{21}{Q} &= \frac{3}{7} \\
3Q &= 147 \\
\text{perimeter } Q &= 49 \text{ units}
\end{align*}
\]

\[
\begin{align*}
\text{area } P &= \left( \frac{3}{7} \right)^2 \\
\frac{20}{Q} &= \frac{9}{49} \\
9Q &= 980 \\
\text{area } Q &\approx 108.89 \text{ square units}
\end{align*}
\]
Example 2

Two rectangles are similar. If the area of one rectangle is 49 square units, and the area of the other rectangle is 256 square units, what is the ratio of similarity between these two rectangles?

Since the rectangles are similar, the ratio of their areas is \( \frac{a^2}{b^2} \) and the ratio of similarity is \( \frac{a}{b} \). Using the given areas, the ratio of their areas is \( \frac{49}{256} \). Therefore we can write:

\[
\frac{a}{b} = \sqrt{\frac{49}{256}} = \frac{\sqrt{49}}{\sqrt{256}} = \frac{7}{16}
\]

The ratio of similarity of the two rectangles is \( \frac{7}{16} \).

Problems

1. If figure A and figure B are similar with a ratio of similarity of \( \frac{5}{4} \), and the perimeter of figure A is 18 units, what is the perimeter of figure B?

2. If figure A and figure B are similar with a ratio of similarity of \( \frac{1}{8} \), and the area of figure A is 13 square units, what is the area of figure B?

3. If figure A and figure B are similar with a ratio of similarity of 6, that is, 6 to 1, and the perimeter of figure A is 54 units, what is the perimeter of figure B?

4. If figure A and figure B are similar and the ratio of their perimeters is \( \frac{12}{6} \), what is their ratio of similarity?

5. If figure A and figure B are similar and the ratio of their areas is \( \frac{32}{9} \), what is their ratio of similarity?

6. If figure A and figure B are similar and the ratio of their perimeters is \( \frac{23}{11} \), does that mean the perimeter of figure A is 23 units and the perimeter of figure B is 11 units? Explain.

Answers

1. 14.4 units 2. 832 sq. units 3. 9 units 4. \( \frac{17}{6} \) 5. \( \frac{\sqrt{72}}{\sqrt{9}} = \frac{4\sqrt{2}}{3} \approx \frac{5.66}{3} \)

6. No, it just tells us the ratio. Figure A could have a perimeter of 46 units and figure B a perimeter of 22 units.
CIRCLES, ARCS, AND SECTORS

By exploring the areas of polygons with many sides, students visualize that as the number of sides of a regular polygon gets very large it approaches the shape of a circle. Students extend what they know about the perimeters and areas of polygons to circles, and confirm the relationships for the circumference \( C \) and area \( A \) of circles.

\[
C = \pi d \quad \text{or} \quad C = 2\pi r \\
A = \pi r^2
\]

In the formulas above, “\( d \)” is the diameter, and “\( r \)” is the radius of the circle. The constant \( \pi \) is by definition the ratio \( \frac{\text{circumference}}{\text{diameter}} \), and it has the same value for any size circle.

Using these formulas, along with ratios, students are able to calculate the perimeters and areas of shapes containing parts of circles.

For additional information see the Math Notes boxes in Lessons 8.4.2 and 8.4.3.

Example 1

The circle at right has radius 8.0 cm. What are the circumference and the area of the circle?

Using the formulas:

\[
\begin{align*}
C &= 2\pi r \\
&= 2\pi(8.0) \\
&= 16\pi \\
&\approx 50.3 \text{ cm}
\end{align*}
\]

\[
\begin{align*}
A &= \pi r^2 \\
&= \pi(8.0)^2 \\
&= 64\pi \\
&\approx 201.1 \text{ sq. cm}
\end{align*}
\]

Example 2

Hermione has a small space on her corner lot that she would like to turn into a patio as shown in the diagram at right. To do this, she wants to do two things. First, she wants to put some decorative edging on the curved part. Second, with the edging in place, she will need to purchase concrete to cover the patio. The edging is sold by the foot. The concrete is sold in bags. Each bag will fill 2.5 square feet to the required depth of four inches.

How much edging and concrete should Hermione buy?
The edging is a portion of the circumference of a circle with its center at point \(O\) and radius 10 feet. The concrete will cover a portion of the area of the circle. The exact fraction of the circle used for the patio can be determined by looking at the measure of the central angle. Since the central angle measures 40° and there are 360° in the whole circle, this portion is \(\frac{40°}{360°} = \frac{1}{9}\) of the circle. If we know the circumference and area of the whole circle, then we can take \(\frac{1}{9}\) of each of those measurements to determine the amount of edging and concrete Hermione needs.

\[
C = \frac{1}{9} (2\pi r) = \frac{1}{9} (2 \cdot \pi \cdot 10) = \frac{20\pi}{9} \approx 6.98 \text{ feet}
\]

\[
A = \frac{1}{9} \pi r^2 = \frac{1}{9} \cdot (10)^2 = \frac{100\pi}{9} \approx 34.91 \text{ square feet}
\]

Hermione should buy 7 feet of edging and 14 bags of concrete (34.91 + 2.5 \approx 13.96 bags) since concrete is sold in full bags only.

**Example 3**

Rubeus’ dog, Fluffy, is tethered to the side of his house at point \(X\) as shown in the diagram at right. If Fluffy’s rope is 18 feet long, how much area does Fluffy have to roam?

Because Fluffy is tethered by a rope, he can only go where the rope can reach. If there were no obstacles, this area would be circular. However, since Fluffy is blocked by the house, the area will be portions of circles.

Fluffy can reach 18 feet to the left and right of point \(X\). This initial area is a semicircle. But, to the right of point \(X\), the rope will bend around the corner of the house, adding a little more area for Fluffy. This smaller area is a quarter of a circle with a radius of 3 feet.

\[
\text{Semicircle:} \quad A = \frac{1}{2} \pi r^2 = \frac{18^2\pi}{2} = \frac{324\pi}{2} = 162\pi \approx 508.94
\]

\[
\text{Quarter circle:} \quad A = \frac{1}{4} \pi r^2 = \frac{3^2\pi}{4} = \frac{9\pi}{4} \approx 7.07
\]

Fluffy has a total of 508.94 + 7.07 \approx 516 square feet in which to run.
Problems

Determine the area of the shaded sector in each circle below. In each case, point $O$ is the center.

1.  
   \[
   \begin{array}{c}
   \text{O} \\
   45^\circ \\
   4
   \end{array}
   \]

2.  
   \[
   \begin{array}{c}
   \text{O} \\
   120^\circ \\
   7
   \end{array}
   \]

3.  
   \[
   \begin{array}{c}
   \text{O} \\
   11
   \end{array}
   \]

4.  
   \[
   \begin{array}{c}
   \text{O} \\
   30^\circ \\
   6
   \end{array}
   \]

5.  What is the arc length of the shaded sector in problem 1?

6.  What is the arc length of the shaded sector in problem 2?

7.  What is the arc length of the shaded sector in problem 3?

8.  What is the arc length of the shaded sector in problem 4?

9.  The shaded region in the figure is called a segment of the circle. It can be found by subtracting the area of $\triangle MIL$ from the area of sector $MIL$. Determine the area of the shaded segment of circle $I$.

10. What is the area of a circular garden with a diameter of 30 feet? Write your answer in exact form.

11. If a circle is inscribed in a square whose diagonal is 8 feet long, what is the area of the circle? Write your answer in exact form.

12. The area of a $60^\circ$ sector of a circle is $10\pi \text{ m}^2$. Determine the radius of the circle. Write your answer in exact form.

13. The area of a sector of a circle with a radius of 5 mm is $10\pi \text{ mm}^2$. Calculate the measure of its central angle.
14. Kennedy and Tess are constructing a racetrack for their horses. The track encloses a field that is rectangular, with two semicircles at each end. A fence must surround this field. How much fencing will Kennedy and Tess need? Round your answer appropriately given precision of measurement.

15. Rubeus has moved his dog Fluffy to a corner of his barn because he wants him to have more room to roam. If Fluffy is tethered at point X on the barn with a 20-foot rope, how much area does Fluffy have to roam? Round your answer appropriately given precision of measurement.

**Answers**

1. \(2\pi \approx 6.28\) sq. units  
2. \(\frac{49}{3} \pi \approx 51.31\) sq. units  
3. \(\frac{363\pi}{4} \approx 285.10\) sq. units  
4. \(3\pi \approx 9.42\) sq. units  
5. \(\pi \approx 3.14\) units  
6. \(\frac{14\pi}{3} \approx 16.66\) units  
7. \(\frac{33\pi}{2} \approx 51.84\) units  
8. \(\pi \approx 3.14\) units  
9. \(\pi - 2 \approx 1.14\) sq. units  
10. \(225\pi\) ft\(^2\)  
11. \(8\pi\) ft\(^2\)  
12. \(2\sqrt{15}\) m  
13. \(144^\circ\)  
14. \(2816 + 302\pi \approx 3765\) meters of fencing  
15. \(200\pi + 100\pi + \frac{25\pi}{4} \approx 962\) square feet
In order for students to be proficient in modeling data and relationships from everyday situations, they must be able to recognize and transform graphs of various functions. Students first model data obtained from weighing discs of different radii. Then they investigate how to transform quadratic functions, discovering ways to shift and vertically stretch parabolas by modifying their equations. Additionally, students learn how to graph a quadratic function without making a table when it is written in graphing form. In future courses, students will transform other types of functions.

For additional information see the Math Notes box in Lesson 9.2.1.

Example 1

The graph of \( f(x) = x^2 \) is shown at right. For each function listed below, explain how its graph differs from the graph at right.

- \( g(x) = -2x^2 \)
- \( h(x) = (x + 3)^2 \)
- \( j(x) = x^2 - 6 \)
- \( k(x) = \frac{1}{4}x^2 \)
- \( l(x) = 3(x - 2)^2 + 7 \)

Every function listed above has something in common: they are all quadratic functions, which means they will form a parabola when graphed. The only differences will be in the direction of opening (upward or downward), the shape (vertically compressed or vertically stretched), and/or the location of the vertex.

The “–2” in \( g(x) = -2x^2 \) does two things to the parabola. The negative sign changes the parabola’s direction so that it opens downward. The “2” stretches the graph vertically, making it appear “skinnier.”

The graph of \( h(x) = (x + 3)^2 \) has the same shape as \( f(x) = x^2 \) and opens upward, but it has a new location: it moves to the left 3 units.

The graph of \( j(x) = x^2 - 6 \) has the same shape as \( f(x) = x^2 \), opens upward, and is shifted down 6 units.

The function \( k(x) = \frac{1}{4}x^2 \) does not shift position, and still opens upward, but the \( \frac{1}{4} \) compresses the parabola vertically, making it appear “wider.”

The last function, \( l(x) = 3(x - 2)^2 + 7 \), combines all of these transformations. The “3” stretches the graph vertically (appear “skinnier”) and opens upward, the “–2” shifts the graph to the right 2 units, and the “+ 7” shifts the graph up 7 units.

These graphs are shown at right. Match each function to the correct parabola.
Example 2

For each of the quadratic functions below, where is the vertex of the parabola?

\[ f(x) = -2(x + 4)^2 + 7 \quad g(x) = 5(x - 8)^2 \quad h(x) = \frac{2}{3} x^2 - \frac{2}{3} \]

For a quadratic function, the vertex is called the locator point. This point gives a starting point for graphing the parabola. The vertex of a quadratic function in graphing form, \( f(x) = a(x - h)^2 + k \), is the point \((h, k)\).

For the function \( f(x) = -2(x + 4)^2 + 7 \), \( h = -4 \) and \( k = 7 \), so the vertex is \((-4, 7)\).

Since \( g(x) = 5(x - 8)^2 \) can also be written \( g(x) = 5(x - 8)^2 + 0 \), the vertex of its graph is \((8, 0)\).

Rewrite \( h(x) = \frac{2}{3} x^2 - \frac{2}{3} \) as \( h(x) = \frac{2}{3} (x - 0)^2 - \frac{2}{3} \) to see that the vertex of its graph is \(\left(0, -\frac{2}{3}\right)\).

Problems

For each quadratic function below, describe the transformation, sketch the graph, and state the vertex.

1. \( y = -2(x - 1)^2 + 3 \)  
2. \( y = (x + 5)^2 - 6 \)
3. \( y = (x + 2)^2 - 25 \)  
4. \( y = 2(x + 6)^2 - 1 \)

Answers

1. \( \text{vertex: } (1, 3) \)
   
   The parabola opens downward, is stretched vertically, and the vertex is shifted to the right 1 unit and up 3 units.

2. \( \text{vertex: } (-5, -6) \)
   
   The vertex of the parabola is shifted to the left 5 units and down 6 units.

3. \( \text{vertex: } (-2, -25) \)

   The vertex of the parabola is moved to the left 2 units and down 25 units.

4. \( \text{vertex: } (-6, -1) \)

   The parabola is stretched vertically and the vertex is moved to the left 6 units and down 1 unit.
In Lesson 9.1.3, students learn that if the equation of a parabola is written in **graphing form**, \( f(x) = (x - h)^2 + k \), then the vertex can be identified as \((h, k)\). For example, for the graph of \( f(x) = (x + 3)^2 - 1 \), the vertex of the parabola is at \((-3, -1)\).

When the equation of the parabola is given in standard form, \( f(x) = ax^2 + bx + c \), then the process of **completing the square** can be used to rewrite the equation in graphing form. Algebra tiles can be used to help visualize the process. To review this process see *Solving by Completing the Square* in Chapter 5 of this Parent Guide with Extra Practice.

**Example 1**

Complete the square to rewrite \( f(x) = x^2 + 5x + 2 \) in graphing form. Identify the vertex and y-intercept, and sketch a graph of the parabola.

Move the constant term to the other side of the equation: \( f(x) - 2 = x^2 + 5x \)

Identify what needs to be added to both sides of the equation to make \( x^2 + 5x \) into a perfect square by taking half of the \( x \)-term coefficient and squaring it: \( \left( \frac{5}{2} \right)^2 = \frac{25}{4} \)

Add \( \frac{25}{4} \) to both sides of the equation: \( f(x) - 2 + \frac{25}{4} = x^2 + 5x + \frac{25}{4} \)

Simplify the left side and factor the trinomial on the right side: \( f(x) + \frac{17}{4} = \left( x + \frac{5}{2} \right)^2 \)

This can be rewritten as: \( f(x) = \left( x + \frac{5}{2} \right)^2 - \frac{17}{4} \).

The function is now in graphing form. The vertex is \( \left( -\frac{5}{2}, -\frac{17}{4} \right) \) or \((-2.5, -4.25)\).

The y-intercept is where \( x = 0 \).

Substitute \( x = 0 \) into the original equation: \( f(0) = 0^2 + 5(0) + 2 = 2 \); the y-intercept is \((0, 2)\).

Use the vertex and y-intercept to sketch the graph.
Example 2

Complete the square to rewrite \( f(x) = x^2 + 8x + 15 \) in graphing form, then identify the vertex of the parabola.

\[
\begin{align*}
  f(x) &= x^2 + 8x + 15 \\
  f(x) - 15 &= x^2 + 8x \\
  f(x) - 15 + 16 &= x^2 + 8x + 16 \\
  f(x) + 1 &= x^2 + 8x + 16 \\
  f(x) + 1 &= (x + 4)^2 \\
  f(x) &= (x + 4)^2 - 1
\end{align*}
\]

Move the constant term to the other side.
Square half the coefficient of the \( x \)-term \((\frac{8}{2})^2\), and add it to both sides.
Simplify the left side of the equation.
Factor the right side of the equation.
Move the constant term back to the right side.

Therefore, the equation in graphing form is \( f(x) = (x + 4)^2 - 1 \). The vertex is at \((h, k) = (-4, -1)\).

Problems

Complete the square to rewrite the equation of each function in graphing form. Then state the vertex of the parabola.

1. \( f(x) = x^2 + 6x + 7 \)  
2. \( f(x) = x^2 + 4x + 11 \)
3. \( f(x) = x^2 + 10x \)  
4. \( f(x) = x^2 + 7x + 2 \)
5. \( f(x) = x^2 - 6x + 9 \)  
6. \( f(x) = x^2 + 3 \)
7. \( f(x) = x^2 - 4x \)  
8. \( f(x) = x^2 + 2x - 3 \)
9. \( f(x) = x^2 + 5x + 1 \)  
10. \( f(x) = x^2 - \frac{1}{2}x \)

Answers

1. \( f(x) = (x + 3)^2 - 2; \ (-3, -2) \)  
2. \( f(x) = (x + 2)^2 + 7; \ (-2, 7) \)
3. \( f(x) = (x + 5)^2 - 25; \ (-5, -25) \)  
4. \( f(x) = (x + 3.5)^2 - 10.25; \ (-3.5, -10.25) \)
5. \( f(x) = (x - 3)^2; \ (3, 0) \)  
6. \( f(x) = x^2 + 3; \ (0, 3) \)
7. \( f(x) = (x - 2)^2 - 4; \ (2, -4) \)  
8. \( f(x) = (x + 1)^2 - 4; \ (-1, -4) \)
9. \( f(x) = (x + \frac{5}{2})^2 - \frac{21}{4}; \ (-\frac{5}{2}, -\frac{21}{4}) \)  
10. \( f(x) = (x - \frac{1}{6})^2 - \frac{1}{36}; \ (\frac{1}{6}, -\frac{1}{36}) \)
SOLVING QUADRATIC INEQUALITIES

There are several methods for solving quadratic inequalities in one variable, but one method that works for many kinds of inequalities is to change the inequality to an equation, solve it, and then graph the solution(s) on a number line. The solution(s) to the equation, called boundary point(s), divide the number line into regions. By checking a number from each region in the original inequality it can be determined if the numbers in that region are solutions. The boundary points are included in (≥ or ≤) or excluded from (> or <) the solution depending on the inequality sign.

For additional information see the Math Notes box in Lesson 9.2.2.

Example 1  Solve: $x^2 - 3x - 18 < 0$
Change to an equation and solve.

$x^2 - 3x - 18 = 0$
$(x - 6)(x + 3) = 0$
$x = 6$ or $x = -3$ (the boundary points)

Choosing $x = -4, 0, 7$ to test in the original inequality, $x = -4$ is false, $x = 0$ is true, and $x = 7$ is false.

The solution is all numbers greater than $-3$ and less than 6, written as $-3 < x < 6$.

Example 2  Solve: $m^2 - 3 ≥ 1$
Change to an equation and solve.

$m^2 - 3 = 1$
$m^2 = 4$
$m = ±2$ (the boundary points)

Choosing $m = -3, 0, 3$ to test in the original inequality, $m = -3$ is true, $m = 0$ is false, and $m = 3$ is true.

The solution is all numbers less than or equal to $-2$ or all numbers greater than or equal to 2, written as $m ≤ -2$ or $m ≥ 2$.

Problems
Solve each inequality.

1. $x^2 + 6x + 8 < 0$
2. $y^2 - 5y > 0$
3. $y^2 - 5y < 0$
4. $x^2 - 3x - 4 < 0$
5. $-x^2 - 9x - 14 < 0$
6. $y^2 ≤ 16$
7. $3x^2 + 7x - 6 ≥ 0$
8. $x^2 + 4x - 8 < 4$
9. $y^2 + 6y + 9 > 0$
10. $x(7x - 26) ≤ 8$
SOLVING SYSTEMS OF EQUATIONS

In this lesson students focus on what a solution of a system of equations means, both algebraically and graphically. Students also apply their knowledge of solving linear systems to solve systems involving quadratic functions.

Example 1

Refer to the graph of the parabola \( y = x^2 - 3x - 10 \) and the line \( y = -2x + 2 \) at right. The parabola and the line cross each other twice, yielding two points of intersection: \((-3, 8)\) and \((4, -6)\).

To determine the points of intersection, you can also solve the system of equations algebraically.

To solve the system of equations, use the Equal Values Method (from Core Connections Integrated I) or substitution.

\[
x^2 - 3x - 10 = -2x + 2
\]

Add and/or subtract like terms from both sides of the equation to make one side of the equation equal to zero, and then factor and use the Zero Product Property to solve for \(x\). (The Quadratic Formula can also be used.)

\[
x^2 - x - 12 = 0
\]

\[
(x - 4)(x + 3) = 0
\]

\[
x = 4 \text{ or } x = -3
\]

Substituting \(x = 4\) into either equation yields \(y = -6\), so \((4, -6)\) is a solution to the system.

Substituting \(x = -3\) into either equation yields \(y = 8\), so \((-3, 8)\) is also a solution to the system.
Example 2

Solve the system of equations at right without graphing. Explain what the solution(s) tell you about the graph of the system.

\[
y = -2(x - 2)^2 + 35 \\
y = -2x + 15
\]

Both equations are written in “\(y = \)” form, so use the substitution method of solving.

\[
-2(x - 2)^2 + 35 = -2x + 15 \\
-2(x - 2)^2 = -2x + 20 \\
-2(x^2 - 4x + 4) = -2x - 20 \\
-2x^2 + 8x - 8 = -2x - 20 \\
-2x^2 + 10x + 12 = 0 \\
x^2 - 5x - 6 = 0 \\
(x - 6)(x + 1) = 0 \\
x = 6 \text{ or } x = -1
\]

Substitute each \(x\)-value into either equation to solve for the corresponding \(y\)-value. You can use either equation. Here we will use the simpler equation.

\[
x = 6, y = -2x + 15 \\
y = -2(6) + 15 \\
y = -12 + 15 = 3 \\
\text{Solution: } (6, 3)
\]

\[
x = -1, y = -2x + 15 \\
y = -2(-1) + 15 \\
y = 2 + 15 = 17 \\
\text{Solution: } (-1, 17)
\]

Lastly, we need to check each point in both equations to verify our work.

\[
(6, 3): \quad y = -2(x - 2)^2 + 35 \\
3 = -2(6 - 2)^2 + 35 \\
3 = -2(16) + 35 \checkmark
\]

\[
(-1, 17): \quad y = -2(x - 2)^2 + 35 \\
17 = -2(-1 - 2)^2 + 35 \\
17 = -2(9) + 35 \checkmark
\]

In solving these two equations with two unknowns, we determined two solutions, both of which check in the original equations. This means that the graphs of the equations, a parabola and a line, intersect in exactly two distinct points. A sketch of the graphs can confirm this conclusion.
Problems

Solve each of the following systems of equations algebraically. What do the solutions tell you about the graph of the system?

1. \( y = -\frac{2}{3}x + 7 \)
   \( 4x + 6y = 42 \)
2. \( y = (x + 1)^2 + 3 \)
   \( y = 2x + 4 \)
3. \( y = -3(x - 4)^2 - 2 \)
   \( y = -\frac{4}{7}x + 4 \)
4. \( x + y = 0 \)
   \( y = (x - 4)^2 - 6 \)
5. \( y = x^2 - 3x + 2 \)
   \( y = x^2 + 3x \)
6. \( y = x^2 - 6x + 9 \)
   \( y = x^2 - 9 \)
7. \( y = 3x^2 - 15x + 23 \)
   \( y = -x^2 + 5x - 1 \)
8. \( y = \frac{1}{2}x^2 - 2x - 2.5 \)
   \( y = -2x^2 - 12x - 16 \)
9. \( y = 4x^2 - 12x + 5 \)
   \( y = -12x^2 + 36x - 31 \)
10. \( y = 0.25x^2 + 2x + 4 \)
    \( y = -0.25x^2 + 4 \)

Answers

1. All real numbers. When graphed, these equations give the same line.
2. \((0, 4)\); The parabola and the line intersect once.
3. No solution. The parabola and the line do not intersect.
4. \((2, -2)\) and \((5, -5)\); The line and the parabola intersect twice.
5. \((\frac{1}{3}, 1 \frac{1}{3})\); The parabolas intersect once.
6. \((3, 0)\); The parabolas intersect once.
7. \((2, 5)\) and \((3, 5)\). The parabolas intersect twice.
8. No solution. The parabolas do not intersect.
9. \((1.5, -4)\). The parabolas intersect once.
10. \((-4, 0)\) and \((0, 4)\). The parabolas intersect twice.
AVERAGE RATE OF CHANGE

In previous math courses students learned that for a linear function, there is a constant rate of change which corresponds to the slope of a line. In Lesson 9.3.1, students learn how to calculate the average rate of change of a nonlinear function over a given interval. The average rate of change is calculated by determining the slope between two points on the graph of the function. If the slope represents a distance per time, then the average rate of change describes the average speed or velocity.

For additional information see the Math Notes box in Lesson 9.3.2.

Example 1

Using the data given in the table at right for an object’s position at given times, calculate the average rate of change for the following time periods:

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>Height (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1.0</td>
<td>120.6</td>
</tr>
<tr>
<td>2.0</td>
<td>176.8</td>
</tr>
<tr>
<td>2.5</td>
<td>181.0</td>
</tr>
<tr>
<td>3.0</td>
<td>169.2</td>
</tr>
<tr>
<td>4.0</td>
<td>97.6</td>
</tr>
</tbody>
</table>

a. 0.0 to 1.0 seconds
b. 1.0 to 2.0 seconds
c. 2.0 to 2.5 seconds
d. 2.5 to 3.0 seconds
e. 3.0 to 4.0 seconds
f. During which time period is the object moving the fastest? When is it moving the slowest? When does it change direction?

For each time period, calculate the slope between the points corresponding to the given intervals. For example, for the time period 0.0 to 1.0 seconds the points are (0.0, 0.0) and (1.0, 120.6).

Remember to include units in the answers.

a. \( \frac{120.6 - 0}{1.0 - 0} = \frac{120.6}{1.0} = 120.6 \text{ m/s} \)
b. \( \frac{176.8 - 120.6}{2.0 - 1.0} = \frac{56.2}{1.0} = 56.2 \text{ m/s} \)
c. \( \frac{181.0 - 176.8}{2.5 - 2.0} = 8.4 \text{ m/s} \)
d. \( \frac{169.2 - 181.0}{3.0 - 2.5} = -23.6 \text{ m/s} \)
e. \( \frac{97.6 - 169.2}{4.0 - 3.0} = -71.6 \text{ m/s} \)
f. The object is moving the fastest during the time period of 0 to 1 second. It is moving the slowest (while traveling upward) between 2.0 to 2.5 seconds. It changes direction between 2.5 and 3.0 seconds.

Notice that when the object is decreasing in height (going downward) the average rate of change (velocity) is negative.
Example 2

Use the graph at right to estimate the average rate of change for the interval:

a. \( x = -1 \) to \( x = 1 \)
b. \( x = 2 \) to \( x = 3 \)
c. \( x = 4 \) to \( x = 5 \)
d. \( x = 6 \) to \( x = 7 \)
e. \( x = 8 \) to \( x = 9 \)

For each of these intervals, determine the corresponding points and then calculate the slope.

a. \((-1, -5)\) and \((1, 4)\), \( \frac{4 - (-5)}{1 - (-1)} = 4.5 \)
b. \((2, 7)\) and \((3, 9)\), \( \frac{9 - 7}{3 - 2} = 2 \)
c. \((4, 10)\) and \((5, 10)\), \( \frac{10 - 10}{5 - 4} = 0 \)
d. \((6, 9)\) and \((7, 7)\), \( \frac{7 - 9}{7 - 6} = -2 \)
e. \((8, 4)\) and \((9, 0)\), \( \frac{0 - 4}{9 - 8} = -4 \)

Problems

Calculate the average rate of change for the given time periods using the data in the table at right.

1. 0 to 5 minutes 2. 10 to 15 minutes
3. 15 to 20 minutes 4. 25 to 30 minutes

Use the graph at right to approximate the average rate of change for the given time periods.

5. \( t = 0 \) to \( t = 1 \) s 6. \( t = 3 \) to \( t = 4 \) s
7. \( t = 4 \) to \( t = 5 \) s 8. \( t = 8 \) to \( t = 9 \) s
9. \( t = 10 \) to \( t = 11 \) s
Answers

1. 13.5 m/min  
2. 8.5 m/min  
3. −1.5 m/min  
4. −11.5 m/min  
5. ≈ 5 ft/s  
6. ≈ 2 ft/s  
7. ≈ 1.5 ft/s  
8. ≈ −2 ft/s  
9. ≈ −4.5 ft/s

**INVERSE FUNCTIONS 9.4.1**

Students explore inverse functions, that is, equations that “undo” the actions of functions.

For example, the function $f(x) = 3x + 1$ performs the following operations on $x$: it multiplies by 3 and then adds 1. The inverse function, named $f^{-1}(x)$, reverses the operations: it subtracts 1 and then divides by 3. Therefore, $f^{-1}(x) = \frac{x-1}{3}$.

**Example 1**

Determine the inverse of each function.

a. $h(x) = \frac{x-6}{3}$  
b. $g(x) = 2(x + 4)$

The function in part (a) subtracts 6 from the input ($x$) and then divides the result by 3. The inverse function reverses this process. Therefore, the inverse function should multiply by 3 and then add 6. The inverse function is $h^{-1}(x) = 3x + 6$.

Test an input value in the original function: For $x = 6$, $h(6) = \frac{6-6}{3} = 0$.

Use the output value as the input to the inverse function: For $x = 0$, $h^{-1}(0) = 3(0) + 6 = 6$.

The inverse function “undoes” the original function, yielding the original input value of 6.

In part (b), the function $g(x)$ adds 4 to the input and then multiplies the result by 2. The inverse function must first divide by 2 and then subtract 4. Therefore, $g^{-1}(x) = \frac{x}{2} - 4$.

To verify this result, test an input value in the original function: $g(1) = 2(1 + 4) = 10$.

Use the output value as the input to the inverse function: $g^{-1}(10) = \frac{10}{2} - 4 = 5 - 4 = 1$.

The inverse function “undoes” the original function, yielding the original input value of 1.
Problems

Write the inverse of each of the following functions.

1. \( f(x) = 8(x - 13) \) 
2. \( f(x) = -\frac{3}{4} x + 6 \)

3. \( f(x) = \frac{5(x+2)}{3} \) 
4. \( f(x) = 2x + 6 \)

5. \( f(x) = \frac{3x+6}{5} \) 
6. \( g(x) = \frac{x}{5} \)

7. \( g(x) = 4(x + 1) - 3 \) 
8. \( j(x) = 2(x + 2) \)

9. \( h(x) = 3x - 4 \) 
10. \( g(x) = 6x + 2 \)

Answers

1. \( f^{-1}(x) = \frac{x}{8} + 13 \) 
2. \( f^{-1}(x) = -\frac{4}{3} x + 8 \)

3. \( f^{-1}(x) = \frac{2}{3} x - 2 \) 
4. \( f^{-1}(x) = \frac{1}{2} x - 3 \)

5. \( f^{-1}(x) = \frac{5x-6}{3} \) 
6. \( g^{-1}(x) = 5x \)

7. \( g^{-1}(x) = \frac{x+3}{4} - 1 \) 
8. \( j^{-1}(x) = \frac{1}{2} x - 2 \)

9. \( h^{-1}(x) = \frac{1}{3} x + \frac{4}{3} \) 
10. \( g^{-1}(x) = \frac{1}{6} x - \frac{1}{3} \)
THE EQUATION OF A CIRCLE

Students have calculated circumferences and areas of circles and parts of circles, and have used circle properties in applications of probability. This section places the circle on a coordinate graph so that the students can derive the equation of a circle.

For additional information see the Math Notes box in Lesson 10.1.2.

Example 1

What is the equation of the circle centered at the origin with radius 5 units?

The key to deriving the equation of a circle is the Pythagorean Theorem. That means a right triangle within the circle needs to be created. First, draw the circle on graph paper. The coordinates of any point on the circle can be thought of as \((x, y)\). Since the endpoints of the radius to that point are \((0, 0)\) and \((x, y)\), the length of the vertical leg can be represented as \(y\) and the length of the horizontal leg as \(x\). If the radius is represented as \(r\), then using the Pythagorean Theorem we can write \(x^2 + y^2 = r^2\). Since the radius is 5, the equation of this circle can be written as \(x^2 + y^2 = 5^2\), or \(x^2 + y^2 = 25\).

Example 2

Graph the circle \((x - 4)^2 + (y + 2)^2 = 49\).

This is a circle with radius 7 units. This circle, however, is not centered at the origin. The general equation of a circle is \((x - h)^2 + (y - k)^2 = r^2\). The center of the circle is represented by \((h, k)\), so in this example the center is \((4, -2)\).
Example 3

What are the center and the radius of the circle $x^2 - 6x + y^2 + 2y - 5 = 0$?

This equation is not in the graphing form, $(x - h)^2 + (y - k)^2 = r^2$, so it is necessary to “complete the square.” To make a perfect square for $x$, add 9 to both sides of the equation to make a perfect square. For $y$, add 1 to both sides of the equation. (Or add a total of 10 to both sides of the equation). Finally, factor the perfect square trinomials to rewrite the equation in graphing form.

$$
(x^2 - 6x + 9) + (y^2 + 2y + 1) - 5 = 10
$$

$$
(x - 3)^2 + (y + 1)^2 = 15
$$

The center is $(3, -1)$ and the radius is $\sqrt{15}$ units.

Problems

1. What is the equation of the circle centered at $(0, 0)$ with a radius of 25?
2. What is the equation of the circle centered at the origin with a radius of 7.5?
3. What is the equation of the circle centered at $(5, -3)$ with a radius of 9?

Graph the following circles.

4. $(x + 1)^2 + (y + 5)^2 = 16$

5. $x^2 + (y - 6)^2 = 36$

6. $(x - 3)^2 + y^2 = 64$

Complete the square to rewrite the equation of each circle in graphing form. Identify the center and the radius of each circle.

7. $x^2 + 6x + y^2 - 4y = -9$

8. $x^2 + 10x + y^2 - 8y = -31$

9. $x^2 - 2x + y^2 + 4y - 11 = 0$

10. $x^2 + 9x + y^2 = 0$
Answers

1. \( x^2 + y^2 = 625 \)

2. \( x^2 + y^2 = 56.25 \)

3. \( (x - 5)^2 + (y + 3)^2 = 81 \)

4. \( (x + 3)^2 + (y - 2)^2 = 4; (-3, 2), r = 2 \) units

5. \( (x + 5)^2 + (y - 4)^2 = 10; (-5, 4), r = \sqrt{10} \approx 3.2 \) units

6. \( (x - 1)^2 + (y + 2)^2 = 16; (1, -2), r = 4 \) units

7. \( (x + 3)^2 + (y - 2)^2 = 4; (-3, 2), r = 2 \) units

8. \( (x + 5)^2 + (y - 4)^2 = 10; (-5, 4), r = \sqrt{10} \approx 3.2 \) units

9. \( (x - 1)^2 + (y + 2)^2 = 16; (1, -2), r = 4 \) units

10. \( (x + \frac{9}{2})^2 + y^2 = \frac{81}{4}; (-\frac{9}{2}, 0), r = \frac{9}{2} \) units
In this section students develop circle tools, which will help them determine the lengths of chords and measures of angle in circles. As with many topics that students have studied in this course, triangles are useful in solving many of these types of problems.

For additional information see the Math Notes boxes in Lessons 10.2.2, 10.2.3, 10.2.4, and 10.2.5.

Example 1

In the circle at right are two chords, $\overline{AB}$ and $\overline{CD}$. Locate the center of the circle and label it $P$.

The chords of a circle (segments with endpoints on the circle) are useful segments. In particular, the diameter is a special chord that passes through the center. The perpendicular bisector of any chord passes through the center of the circle. Therefore to locate the center, construct the perpendicular bisectors of each chord. The perpendicular bisectors will intersect at the center.

There are several ways to construct perpendicular bisectors of the segments. A quick way is to fold the paper so that the endpoints of the chords come together. The crease will be perpendicular to the chord and bisect it. You can also use a compass-and-straightedge construction. In either case, point $P$ shown in the diagram at right is the center of the circle.
Example 2

In $\odot O$ at right, use the given information to calculate the values of $x$, $y$, and $z$.

Connected parts of a circle are called arcs, and any pair of points on a circle breaks the circle into two arcs. The larger arc is called a major arc, and the smaller arc is called a minor arc. The length of an arc can be calculated as a fraction of the circumference of its circle. An arc also has a measure based on the measure of its corresponding central angle. In the diagram at right, $\angle JOE$ is a central angle since its vertex is at the center, $O$. An arc’s measure is the same as the measure of its central angle. Since $JE = 100^\circ$, $m\angle JOE = 100^\circ$, or $x = 100^\circ$.

An angle with its vertex on the circle is called an inscribed angle. Both angles $y$ and $z$ are inscribed angles. Inscribed angles measure half of their intercepted arc. In this case $JE$ is the intercepted arc of both $\angle y$ and $\angle z$. Therefore, $y = z = \frac{1}{2}(100^\circ) = 50^\circ$.

Example 3

In the figure at right, $O$ is the center of the circle, $\overline{TX}$ and $\overline{TB}$ are tangent to $\odot O$, and $m\angle BOX = 120^\circ$. What is the $m\angle BTX$?

A line is tangent to a circle intersects the circle in only one point. Also, a radius drawn to the point of tangency is perpendicular to the tangent line. Therefore, $\overline{OB} \perp \overline{BT}$ and $\overline{OX} \perp \overline{XT}$. One way to solve this problem is to add a segment to the diagram. Adding $\overline{OT}$ creates two triangles. These two right triangles are congruent by HL $\cong$ ($\overline{OB} \cong \overline{OX}$ because they are both radii, and $\overline{OT} \cong \overline{OT}$). Since the corresponding parts of congruent triangles are also congruent, and $m\angle BOX = 120^\circ$, we know that $m\angle BOT = m\angle XOT = 60^\circ$. Since the sum of the angles of a triangle is $180^\circ$, $60^\circ + 90^\circ + m\angle BTO = 180^\circ$. Therefore $m\angle BTO = m\angle XTO = 30^\circ$ and $m\angle BTX = 60^\circ$.

An alternate solution is to note that the two right angles at $B$ and $X$, added to $\angle BOX$, sum to $300^\circ$. Since we know that the angles in a quadrilateral sum to $360^\circ$, $m\angle BTX = 360^\circ - 300^\circ = 60^\circ$. 
Example 4

In the circle at right, $DV = 9$ units, $SV = 12$ units, and $AV = 4$ units. Determine the length of $IV$.

In the diagram, if we draw $SI$ and $DA$ two similar triangles will be formed. (See the Math Notes box in Lesson 10.2.5.) The sides of similar triangles are proportional, so the proportion at right can be written.

Substitute the known lengths, then solve the equation.

Problems

Determine each measure in $\odot P$ if $m \angle WPX = 28^\circ$, $m \angle ZPY = 38^\circ$, and $WZ$ and $XV$ are diameters.

1. $m \widehat{YZ}$ 2. $m \widehat{WX}$ 3. $m \angle VPZ$ 4. $m \widehat{VWX}$

5. $m \angle XPY$ 6. $m \widehat{XY}$ 7. $m \widehat{XWY}$ 8. $m \widehat{WZX}$

In each of the following diagrams, $O$ is the center of the circle. Calculate the value of $x$ and justify your answer.

9. 
10. 
11. 
12. 

13. 
14. 
15. 
16. 

17. 
18. 
19. 
20. 

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In $\odot O$, $m\widehat{WT} = 86^\circ$ and $m\widehat{EA} = 62^\circ$. Calculate:

21. $m\angle EWA$
22. $m\angle WET$
23. $m\angle WES$
24. $m\angle WST$

In $\odot O$, $m\angle EWA = 36^\circ$ and $m\angle WST = 42^\circ$. Calculate:

25. $m\angle WES$
26. $m\angle TW$
27. $m\angle EA$
28. $m\angle TKE$

29. In the diagram at right, $m\widehat{SD} = 92^\circ$, $m\widehat{DA} = 103^\circ$, $m\widehat{AI} = 41^\circ$ and $SW$ is tangent to $\odot O$. What are $m\angle AKD$ and $m\angle VAS$?

30. In the diagram at right, $m\widehat{EK} = 43^\circ$, $\overline{EW} \equiv \overline{KW}$, and $ST$ is tangent to $\odot O$. What are $m\angle WEO$ and $m\angle SEW$?
Answers

1. 38°  
2. 28°  
3. 28°  
4. 180°  
5. 114°  
6. 114°  
7. 246°  
8. 332°  
9. 68°  
10. 73°  
11. 98°  
12. 124°  
13. 50°  
14. 55°  
15. 18°  
16. 27°  
17. 55°  
18. 77°  
19. 35°  
20. 50°  

21. \(\frac{1}{2}(62°) = 31°\)  
22. \(\frac{1}{2}(86°) = 43°\)  

23. \(180° - 43° = 137°\)  
24. \(180° - 137° - 31° = 12°\)  

25. \(180° - 36° - 42° = 102°\)  
26. \(m\angle TEW = 180° - 102° = 78°, 2(78°) = 156°\)  

27. \(2(36°) = 72°\)  
28. \(180° - 36° - 78° = 66°\)  

29. \(m\angle SAD = \frac{1}{2}(92°), m\angle IDA = \frac{1}{2}(41°), 180° - 46° - 20.5° = 113.5°, 180°\)  
\(m\angle VAS = 180° - 46° = 134°\)  

30. \(m\angle EWK = \frac{1}{2}(43°) = 21.5°, m\angle EOK = 43°, so 317° remain for the other angle at O.\)  
\(m\angle WEO = m\angle WKO\) and for \(WEOK, 360° - 21.5° - 317° = 21.5° = m\angle WEO + m\angle WKO,\)  
so \(m\angle WEO = \frac{1}{2}(21.5°) = 10.75°. m\angle SEO = 90°, m\angle WEO = 10.75°, so m\angle SEW = 79.25°.\)
SOLIDS AND RATIOS OF SIMILARITY 11.1.1 – 11.1.3

In this chapter, students examine three-dimensional shapes, known as solids. They review how to determine the surface area and volume of prisms and cylinders, including solids that are slanted, or oblique. They also continue to look at similarity, this time studying similar three-dimensional objects, and discover how the linear scale factor can be used to calculate the ratio of the volumes of similar solids.

For additional information see the Math Notes box in Lesson 11.1.3.

Example 1

A cube has an edge length of 20 cm. What are the surface area and volume of this solid?

To calculate the volume, multiply the area of the base by the height. Since the base is a square, its area is 400 square cm. The height is 20 cm, therefore the volume is (400)(20) = 8000 cubic cm.

To calculate the surface area, determine the sum of the areas of all six faces. Each face is a square and they are all congruent. The area of one square is 400 square cm, and there are six of them. Therefore the surface area is 2400 square cm.

Example 2

The dimensions of the prism at right are shown. What are the volume and surface area of this prism?

A prism is a special type of solid that has two congruent and parallel bases. The volume of a prism is found by multiplying the area of the base by the height of the prism. To understand this process, think of a prism as a stack of cubes. The base area tells you how many cubes are in one layer of the stack. The height tells you how many layers of cubes are in the solid.

In this example, the base is a right triangle, so the area is \( \frac{1}{2}bh \). The top of the prism can be seen as the base.

Base area = \( \frac{1}{2}bh = \frac{1}{2}(6)(8) = 24 \) sq units, so there are 24 cubes in one layer.
Example continued from previous page.

Calculate the volume by multiplying by the height, 12.

\[ V = (\text{Base area}) \cdot h = (24)(12) = 288 \text{ cubic units} \]

The surface area of this prism is calculated by adding together the areas of the faces, including the bases. One way to illustrate the sub-problems is to make sketches of the surfaces.

\[
\text{Surface Area} = 2 \left( \frac{6}{2} \right) + 6 \cdot 12 + \frac{8}{2} \cdot 12 + \frac{12}{2} \cdot \frac{8}{2} \]

All of the surfaces are familiar shapes, namely, triangles and rectangles. The length of the rectangle on the back face (the last rectangle in the pictorial equation above) is needed. Fortunately, that length is also the hypotenuse of the right triangle of the base, so the Pythagorean Theorem can be used.

\[ \text{Surface Area} = 6^2 + 8^2 = \text{?}^2 \quad 36 + 64 = \text{?}^2 \quad \text{?}^2 = 100 \quad \text{?} = \sqrt{100} = 10 \]

Therefore the surface area is:

\[ \text{S.A.} = 2 \left( \frac{1}{2} \cdot 6 \cdot 8 \right) + (6 \cdot 12) + (8 \cdot 12) + (10 \cdot 12) \]
\[ = 48 + 72 + 96 + 120 \]
\[ = 336 \text{ square units} \]

Example 3

The Styrofoam pieces used in packing boxes, known as “shipping peanuts,” are sold in three box sizes: small, medium, and large. The small box has a volume of 1200 cubic inches. The dimensions on the medium box are twice the dimensions of the small box, and the large box has triple the dimensions of the small one. All three boxes are similar prisms. What are the volumes of the medium and large boxes?

Since the boxes are similar, we can use the ratio of similarity to determine the volume of the medium and large boxes without knowing their actual dimensions. When solids are similar with ratio of similarity \( \frac{2}{1} \), the ratio of the areas is \( \left( \frac{2}{1} \right)^2 \) and the ratio of the volumes is \( \left( \frac{2}{1} \right)^3 \). Since the medium box has dimensions twice the small box and the large box has dimensions three times the small box, we can write:

\[
\frac{\text{medium box}}{\text{small box}} = \frac{2}{1} \quad \frac{\text{volume of medium box}}{\text{volume of small box}} = \left( \frac{2}{1} \right)^3 \quad \frac{\text{large box}}{\text{small box}} = \frac{3}{1} \quad \frac{\text{volume of large box}}{\text{volume of small box}} = \left( \frac{3}{1} \right)^3
\]

Solving, \( x = 8 \cdot 1200 \) or volume of the medium box = 9600 cubic inches and \( y = 27 \cdot 1200 \) or volume of large box = 32,400 cubic inches.
Problems

Determine the volume of each solid.

1. ![Volume Problem 1]
2. ![Volume Problem 2]
3. ![Volume Problem 3]
4. ![Volume Problem 4]
5. ![Volume Problem 5]
6. ![Volume Problem 6]
7. ![Volume Problem 7]
8. ![Volume Problem 8]
9. ![Volume Problem 9]

Determine the total surface area of the solids in the previous volume problems.


18. What are the volume and surface area of the solid below?

19. Calculate the volume of the remaining solid after a hole with a diameter of 4 mm is drilled through it.

20. At Cakes R Us, it is possible to buy round cakes in different sizes. The smallest cake has a diameter of 8 inches and a height of 4 inches, and requires 3 cups of batter. Another similar round cake has a diameter of 13 inches. How much batter would this cake require?
21. Two rectangular prisms are similar. The smaller, A, has a height of 4 units while the larger, B, has a height of 6 units.
   a. What is the scale factor from prism A to prism B?
   b. What is the ratio of the lengths of the edges labeled \( x \) and \( y \)?
   c. What is the ratio of their surface areas?
   d. What is the ratio of their volumes?

22. Prism A and prism B are similar with a ratio of similarity of 2:3. If the volume of prism A is 36 cubic units, what is the volume of prism B?

23. If rectangle A and rectangle B have a ratio of similarity of 5:4, what is the area of rectangle B if the area of rectangle A is 24 square units?

24. Rectangle A is similar to rectangle B. The area of rectangle A is 81 square units while the area of rectangle B is 49 square units. What is the ratio of similarity between the two rectangles?

25. If prism A and prism B have a ratio of similarity of 1:4, what is the volume of prism B if the volume of prism A is 83 cubic units?

26. Prism A and prism B are similar. The volume of prism A is 72 cubic units while the volume of prism B is 1125 cubic units. What is the ratio of similarity between these two prisms?

27. Prism A and prism B are similar. The volume of prism A is 27 cubic units while the volume of prism B is approximately 512 cubic units. If the surface area of prism B is 128 square units, what is the surface area of prism A?

28. The corresponding diagonals of two similar trapezoids are in the ratio of 1:7. What is the ratio of their areas?

29. The ratio of the perimeters of two similar parallelograms is 3:7. What is the ratio of their areas?

30. The areas of two circles are in the ratio of 25:16. What is the ratio of their radii?

31. The ratio of the volumes of two similar circular cylinders is 27:64. What is the ratio of the diameters of their similar bases?

32. The surface areas of two cubes are in the ratio of 49:81. What is the ratio of their volumes?
Answers

1. \(48 \text{ m}^3\)  
2. \(540 \text{ cm}^3\)  
3. \(\approx 14967 \text{ ft}^3\)  
4. \(\approx 77.0 \text{ in}^3\)  
5. \(\approx 1508.7 \text{ m}^3\)  
6. \(\approx 157 \text{ m}^3\)  
7. \(72 \text{ ft}^3\)  
8. \(\approx 1045 \text{ cm}^3\)  
9. \(\approx 332.6 \text{ cm}^3\)  
10. \(80 \text{ m}^2\)  
11. \(468 \text{ cm}^2\)  
12. \(\approx 3997 \text{ ft}^2\)  
13. \(\approx 121.0 \text{ in}^2\)  
14. \(\approx 728.3 \text{ m}^2\)  
15. \(\approx 220 \text{ m}^2\)  
16. \(124 \text{ ft}^2\)  
17. \(\approx 570 \text{ cm}^2\)  
18. \(7245 \text{ ft}^3, \approx 2395 \text{ ft}^2\)  
19. \(\approx 1012 \text{ mm}^3\)  
20. \(\approx 13 \text{ cups}\)  
21. a. \(\frac{4}{6} = \frac{2}{3}\)  
   b. \(\frac{x}{y} = \frac{4}{6} = \frac{2}{3}\)  
   c. \(\frac{16}{36} = \frac{4}{9}, \frac{64}{216} = \frac{8}{27}\)  
   d. 375 cu. units  
22. \(121.5 \text{ units}^3\)  
23. \(15.36 \text{ units}^2\)  
24. \(\frac{9}{7}\)  
25. \(5312 \text{ units}^3\)  
26. \(\frac{2}{5}\)  
27. \(\approx 18 \text{ units}^2\)  
28. \(\frac{1}{19}\)  
29. \(\frac{9}{49}\)  
30. \(\frac{5}{4}\)  
31. \(\frac{3}{4}\)  
32. \(\frac{343}{729}\)
Students have already worked with solids, calculating the volume and surface area of prisms and cylinders, and investigating the relationship between the volumes of similar solids. Now these skills are extended to determining the volumes and surface areas of pyramids, cones, and spheres.

For additional information see the Math Notes boxes in Lessons 11.2.2, 11.2.3, and 12.1.1.

**Example 1**

The base of the pyramid at right is a regular hexagon. Using the measurements provided, calculate the surface area and volume of the pyramid.

The volume of any pyramid is \[ V = \frac{1}{3} \text{(base area)(height)} \]. Calculate the surface area the same way as for all solids: calculate the area of each face and base, then add them all together. The lateral faces of this pyramid are all congruent triangles. The base is a regular hexagon. Since the area of the hexagon is needed for both the volume and the surface area, calculate it first.

There are several ways to determine the area of a regular hexagon. One way is to divide the hexagon into six congruent equilateral triangles, each with a side of 8 inches. Calculate the area of one triangle, then multiply by 6 to get the area of the hexagon. The value of \( h \), the height of the triangle, is needed first. Observe that the height divides the equilateral triangle into two congruent 30°-60°-90° triangles. To determine \( h \), use the Pythagorean Theorem or the pattern for a 30°-60°-90° triangle. Using either method \( h = 4\sqrt{3} \) in. Therefore the area of one equilateral triangle is shown at right.

\[
A = \frac{1}{2} \cdot bh = \frac{1}{2} \cdot (8) \cdot (4\sqrt{3}) = 16\sqrt{3} \approx 27.71 \text{ in}^2
\]

The area of the hexagon is \( 6 \cdot 16\sqrt{3} = 96\sqrt{3} \approx 166.28 \text{ in}^2 \).

Then calculate the volume of the pyramid using the formula as shown at right.

\[
V = \frac{1}{3} \text{(base area)(height)} = \frac{1}{3} \cdot (96\sqrt{3}) \cdot (14) = 448\sqrt{3} \approx 776 \text{ in}^3
\]
Next determine the area of one of the triangular faces. These triangles are slanted, and the height of one of them is called a slant height. The problem does not give us the value of the slant height (labeled $c$ at right), but it can calculated based on the information already given.

A cross section of the pyramid at right shows a right triangle in its interior. One leg is labeled $a$, another $b$, and the hypotenuse $c$. The original picture indicates that $a = 14''$. The length of $b$ was calculated previously: it is the height of one of the equilateral triangles in the hexagonal base. Therefore, $b = 4\sqrt{3}''$. To calculate $c$, use the Pythagorean Theorem.

\[ a^2 + b^2 = c^2 \]
\[ 14^2 + (4\sqrt{3})^2 = c^2 \]
\[ 196 + 48 = c^2 \]
\[ c^2 = 244 \]
\[ c = \sqrt{244} = 2\sqrt{61} \approx 15.62'' \]

The base of one of the slanted triangles is 8", the length of the side of the hexagon. Therefore the area of one slanted triangle is $8\sqrt{61} \approx 62.48 \text{ in}^2$ as shown below right.

Since there are six of these triangles, the area of the lateral faces is $6(8\sqrt{61}) = 48\sqrt{61} \approx 374.89 \text{ in}^2$.

Now determine the total surface area: $96\sqrt{3} + 48\sqrt{61} \approx 541.17 \text{ in}^2$.

### Example 2

The cone at right has the measurements shown. What are the lateral surface area and volume of the cone?

The volume of a cone is the same as the volume of any pyramid:

\[ V = \frac{1}{3} \text{(base area)(height)} \]
\[ = \frac{1}{3} \left( \pi r^2 \right) h \]
\[ = \frac{1}{3} \left( \pi \cdot 4^2 \right) \cdot 11 \]
\[ = \frac{1}{3} \cdot 176\pi = \frac{176\pi}{3} \]
\[ \approx 184.3 \text{ cm}^3 \]

Calculating the lateral surface area of a cone is a different matter. Think of a cone as a child’s party hat. Then imagine cutting it apart to make it lay flat. If you did, you would see that the cone is really a sector of a circle—not the circle that makes up the base of the cone, but a circle whose radius is the slant height of the cone, labeled as $l$ at right.
By using ratios the formula for the lateral surface area of the cone can be derived as $SA = \pi rl$, where $r$ is the radius of the base and $l$ is the slant height. In this problem, $r$ is given, but $l$ is not. Determine the length of $l$ by taking a cross section of the cone to create a right triangle. The legs of the right triangle are 11 cm and 4 cm, and $l$ is the hypotenuse. Use the Pythagorean Theorem to calculate $l \approx 11.7$ cm, as shown below right.

Now calculate the lateral surface area:

$$SA = \pi (4)(11.7) \approx 147.1 \text{ cm}^2$$

Example 3

The sphere at right has a radius of 6 feet. Calculate the surface area and the volume of the sphere.

Since spheres are related to circles, the formulas for the surface area and volume have $\pi$ in them. The surface area of a sphere with radius $r$ is $4\pi r^2$. Since the radius of the sphere is 6, $SA = 4\pi (6)^2 = 144\pi \approx 452.39 \text{ ft}^2$. To calculate the volume of the sphere, use the formula $V = \frac{4}{3} \pi r^3$. Therefore,

$$V = \frac{4}{3} \pi (6)^3 = \frac{4 \cdot 216 \pi}{3} = 288\pi \approx 904.78 \text{ ft}^3.$$
Problems

1. The figure at right is a square-based pyramid. Calculate its surface area and its volume.

2. Another pyramid, congruent to the one in the previous problem, is glued to the bottom of the first pyramid, so that their bases coincide. Calculate the surface area and volume of the new solid.

3. A regular pentagon has a side length of 10 inches. Calculate the area of the pentagon.

4. The pentagon of the previous problem is the base of a right pyramid with a height of 18 inches. What is the surface area and volume of the pyramid?

5. What is the total surface area and volume of the cone at right?

6. A cone fits perfectly inside a cylinder as shown. If the volume of the cylinder is $81\pi$ cubic units, what is the volume of the cone?

7. A sphere has a radius of 12 cm. What are the surface area and volume of the sphere?

Determine the volume of each solid.

8. 

9. 

10. 

11. 

12. 

13. 

Determine the total surface area of the figures in the previous volume problems.

21. Problem 12  22. Problem 16

Use the given information to calculate the volume of the cone.

23. radius = 1.5 in  24. diameter = 6 cm  25. base area = $25\pi$
   height = 4 in  height = 5 cm  height = 3

26. base circum. = $12\pi$  27. diameter = 12  28. lateral area = $12\pi$
   height = 10  slant height = 10  radius = 1.5

Use the given information to calculate the lateral area of the cone.

29. radius = 8 in  30. slant height = 10 cm  31. base area = $25\pi$
   slant height = 1.75 in  height = 8 cm  slant height = 6

32. radius = 8 cm  33. volume = $100\pi$  34. volume = $36\pi$
   height = 15 cm  height = 5  radius = 3

Use the given information to calculate the volume of the sphere.

35. radius = 10 cm  36. diameter = 10 cm  37. circumference of
   38. surface area = $256\pi$  39. circumference of
   40. surface area = 100
great circle = $12\pi$
great circle = 20 cm

Use the given information to calculate the surface area of the sphere.

41. radius = 5 in  42. diameter = 12 in  43. circumference of
   44. volume = 250  45. circumference of
   46. volume = $\frac{9\pi}{2}$
great circle = $14$
great circle = $\pi$
Answers

1. $V = 147 \text{ cm}^3$, $SA \approx 184 \text{ cm}^2$
2. $V = 294 \text{ cm}^3$, $SA \approx 270 \text{ cm}^2$
3. $A = 172 \text{ in}^2$
4. $V \approx 1032 \text{ in}^3$, $SA \approx 654 \text{ in}^2$
5. $V = 314 \text{ ft}^3$, $SA = 90 \pi \approx 283 \text{ ft}^2$
6. $27 \pi$ cubic units
7. $SA = 576 \pi \approx 1090 \text{ cm}^2$, $V = 2304 \pi \approx 7238 \text{ cm}^3$
8. 320 in$^3$
9. $100 \pi \approx 314.2 \text{ in}^3$
10. $\approx 610 \text{ cm}^3$
11. $\approx 2.5 \text{ m}^3$
12. 512 in$^3$
13. $\approx 514 \text{ m}^3$
14. $\approx 52.3 \text{ cm}^3$
15. $\frac{20\pi}{3} \approx 21 \text{ cm}^3$
16. $\approx 149 \text{ in}^3$
17. $\approx 229 \text{ in}^2$
18. $90\pi \approx 283 \text{ in}^2$
19. $\approx 478 \text{ cm}^2$
20. $3.6\pi \approx 11.3 \text{ m}^2$
21. 576 in$^2$
22. 193.0 in$^2$
23. $3\pi \approx 9.4 \text{ in}^3$
24. $15\pi \approx 47 \text{ cm}^3$
25. $25\pi \approx 79 \text{ units}^3$
26. $120\pi \approx 377 \text{ units}^3$
27. $96\pi \approx 302 \text{ units}^3$
28. $\approx 18.5 \text{ units}^3$
29. $14\pi \approx 43.98 \text{ in}^2$
30. $60\pi \approx 189 \text{ cm}^2$
31. $30\pi \approx 94 \text{ units}^2$
32. $136\pi \approx 427 \text{ cm}^2$
33. $\approx 224 \text{ units}^2$
34. 117 units$^2$
35. $\frac{4000\pi}{3} \approx 4189 \text{ cm}^3$
36. $\frac{500\pi}{3} \approx 524 \text{ cm}^3$
37. $288\pi \approx 905 \text{ units}^3$
38. $\frac{2048\pi}{3} \approx 2145 \text{ units}^3$
39. $\approx 135. \text{ cm}^3$
40. $\approx 94 \text{ units}^3$
41. $100\pi \approx 314 \text{ in}^2$
42. $144\pi \approx 452 \text{ in}^2$
43. $\approx 62 \text{ units}^2$
44. $\approx 192 \text{ units}^2$
45. $\pi \approx 3 \text{ units}^2$
46. $9\pi \approx 28 \text{ units}^2$
Students take on challenging problems using the Fundamental Counting Principle, permutations, and combinations to compute probabilities. These techniques are essential when the sample space is too large to model or to count.

For additional information see the Math Notes boxes in Lessons 12.1.2, 12.1.3, and 12.1.4.

Example 1

Twenty-three people have entered the pie-eating contest at the county fair. The first place pie-eater (the person eating the most pies in fifteen minutes) wins a pie each week for a year. Second place will receive new baking ware to make his/her own pies, and third place will receive the *Sky High Pies* recipe book. How many different possible top finishers are there?

Since the prizes are different for first, second, and third place, the order of the top finishers matters. Use a decision chart to determine the number of ways winners can finish. How many different people can come in first? Twenty-three. Once first place is “chosen” (that is, removed from the list of contenders) how many people are left to take second place? Twenty-two. This leaves twenty-one possible third-place finishers. Multiply these numbers to determine the number of arrangements: $(23)(22)(21) = 10,626$.

Example 2

Fifteen students are participating in a photo shoot for a layout in the new journal *Mathmaticious*. In how many ways can you arrange:

a. Eight of the students?  

b. Two of them?  

c. Fifteen of them?

A decision chart can be used for each of these situations, but there is another, more efficient method for answering these questions. An arrangement of items in which order matters is called a permutation, and in this case, since changing the order of the students changes the layout, the order matters.

With a permutation, you need to know the total number things to be arranged (in this case $n = 15$ students) and how many will be taken $(r)$ at a time. The formula for a permutation is

$$nPr = \frac{n!}{(n-r)!}.$$  

*Example continues on next page →*
Example continued from previous page.

In part (a), 15 students taken 8 at a time.
The number of permutations is: \(15 \, P_8 = \frac{15!}{(15-8)!} = \frac{15!}{7!} = 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = 259,459,200\)

In part (b) the solution becomes: \(15 \, P_2 = \frac{15!}{(15-2)!} = \frac{15!}{13!} = 15 \cdot 14 = 210\)

Part (c) poses a new problem: \(15 \, P_{15} = \frac{15!}{(15-15)!} = \frac{15!}{0!}\)

What is 0!? “Factorial” means to calculate the product of the integers from the given value down to one. How can 0! be computed? If it equals zero, then there is a problem because part (c) would not have an answer (dividing by zero is undefined). But this situation must have an answer. In fact, if a decision chart were used to determine how many ways the 15 people can line up, we would discover that there are 15! arrangements. Thus it makes sense that \(15 \, P_{15} = 15!\) and therefore 0! = 1. This is another case of mathematicians defining elements of mathematics to fit their needs. 0! is defined to equal 1 so that other mathematics makes sense.

Example 3

In the annual homecoming parade, three students get to ride on the lead float. Seven students are being considered for this coveted position. How many ways can three students be chosen for this honor?

All three students who are selected will ride on the lead float, but whether they are the first, second, or third student selected does not matter. In a case where the order of the selections does not matter, the situation is called a combination. This means that if the students were labeled A, B, C, D, E, and F, choosing A, B, and then C would be essentially the same as choosing B, C, and then A. In fact, all the arrangements of A, B, and C counted as being the same. This makes the number of combinations much smaller than the number of permutations. The symbol for a combination is \(n \, C_r\) where \(n\) is the total number of items under consideration, and \(r\) is the number of items chosen. It is often read as “\(n\) choose \(r\).” This situation is \(7 \, C_3\), or 7 choose 3.

The formula for a combination is similar to the formula for a permutation, but the similar groups must be divided out.

\[n \, C_r = \frac{n!}{(n-r)! \, r!}\]

So: \(7 \, C_3 = \frac{7!}{(7-3)! \, 3!} = \frac{7!}{4! \, 3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 35\)
Problems

Simplify the following expressions.

1. \(10!\) 
2. \(\frac{10!}{3!}\) 
3. \(\frac{35!}{30!}\) 
4. \(\frac{88!}{87!}\) 
5. \(\frac{72!}{70!}\) 
6. \(\frac{65!}{62!3!}\) 
7. \(8P_2\) 
8. \(15P_0\) 
9. \(9P_9\) 
10. \(12C_4\) 
11. \(5C_0\) 
12. \(32C_{32}\)

Solve the following problems.

13. In how many ways can you arrange the letters from the word “KAREN”?
14. In how many ways can you arrange the letters from the word “KAREN” if you want the arrangement to begin with a vowel?
15. All standard license plates in Alaska start with three letters followed by three digits. If repetition is allowed, how many different license plates are there?
16. For $3.99, The Creamery Ice Cream Parlor will put any three different flavored scoops, out of their 25 flavors of ice cream, into a bowl. How many different “bowls” are there? (Note: A bowl of chocolate, strawberry, and vanilla is the same bowl as a bowl of chocolate, vanilla, and strawberry.)
17. Suppose those same three scoops of ice cream are on a cone. Now how many arrangements are there? (Note: Ice cream on a cone must be eaten “top down” because you cannot eat the bottom or middle scoop while keeping the cone intact.)
18. A normal deck of playing cards contains 52 cards. How many five-card poker hands can be made?
19. How many ways are there to make a full house (three of one kind, two of another)?
20. What is the probability of getting a full house (three of one kind of card and two of another)? Assume a standard deck and no wild cards.

For problems 21–25, a bag contains 36 marbles. There are twelve blue marbles, eight red marbles, seven green marbles, five yellow marbles, and four white marbles. Without looking, you reach into the bag and pull out eight marbles. What is the probability you pull out:

21. All blue marbles? 
22. Four blue and four white marbles?
23. Seven green and one yellow marble? 
24. At least one red and at least two yellow?
25. No blue marbles?
Answers

1. 3,628,800  
2. 604,800  
3. 38,955,840  
4. 88  
5. 5,112  
6. 43,680  
7. 56  
8. 1  
9. 362,880  
10. 495  
11. 1  
12. 1  
13. $5! = 120$  
14. $2(4!) = 48$  
15. $(26)(26)(26)(10)(10)(10) = 17,576,000$  
16. $25C_3 = 2300$  
17. $25P_3 = 13,800$ (On a cone, order matters!)  
18. $52C_5 = 2,598,960$  
19. This is tricky and tough! There are 13 different “types” of cards: twos, threes, fours, ..., Jacks, Queens, Kings, and Aces. We need to choose which of the 13 we want three of ($13C_1$). Once we choose what type (for example, we pick Jacks) then we need to choose which three out of the four to take ($4C_3$). Then from the remaining 12 types, we choose which type to have two of ($12C_1$). Then again we need to choose which two out of the four ($4C_2$). This gives us $(13C_1)(4C_3)(12C_1)(4C_2) = 3,744$.  
20. We already calculated the numbers we need in problems 18 and 19 so: $\frac{3,744}{2,598,960} \approx 0.0014$.  
21. Each time we reach in and pull out 8 marbles, order does not matter. The number of ways to do this is $36C_8$. This is the number in the sample space, i.e., the denominator. How many ways can we pull out all blue? $12C_8$. Therefore the probability is $\frac{12C_8}{36C_8} \approx 0.0000164$.  
22. Same denominator. Now we want to choose 4 from the 12 blue, $12C_4$, and 4 from the 4 whites, $4C_4$. $\frac{12C_44C_4}{36C_8} \approx 0.0000164$, the same answer!  
23. Seven green: $7C_7$, one yellow: $5C_1$. $\frac{7C_75C_1}{36C_8} \approx 0.0000001652$  
24. Here we have to get at least one red: $8C_1$, and at least two yellow: $5C_2$, but the other five marbles can come from the rest of the pot: $33C_5$. Therefore, $\frac{8C_15C_233C_5}{36C_8} \approx 0.627$.  
25. To get no blue marbles means we want all eight from the other 24 non-blue marbles. $\frac{24C_8}{36C_8} \approx 0.0243$.  

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