The study of trigonometric functions began in Chapter 9 with radians and transformations of trigonometric functions. This chapter focuses on solving equations involving trigonometric functions by operating on the variable. This work will review the inverse trigonometric functions as well as introduce the reciprocal trigonometric functions.

See the Math Notes box in Lesson 12.2.1 for additional information.

Example 1

For what values of $\theta$ are the following equations true?

a. $\cos(\theta) = \frac{\sqrt{3}}{2}$

b. $2\sin(\theta) = \sqrt{2}$

c. $\cos(\theta) = 5$

a. The graph of $y = \cos(x)$ is a periodic function, and the graph of $y = \frac{\sqrt{3}}{2}$ is a horizontal line. By graphing both equations on the same set of axes, it can be seen that they intersect infinitely many times. How do you determine all solutions?

Solving by using inverse cosine gives one solution.

$\cos(\theta) = \frac{\sqrt{3}}{2}$

$\cos^{-1}(\cos(\theta)) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\theta = \frac{\pi}{6}$ radians

It helps to remember the unit circle. For what values of $\theta$ does $\cos(\theta) = \frac{\sqrt{3}}{2}$? Two points are easily found: $\frac{\pi}{6}$ and $-\frac{\pi}{6}$.

How do you determine the rest? On the unit circle, $\frac{\pi}{6}$ and $-\frac{\pi}{6}$ are revisited at each rotation of $2\pi$. Therefore, not only does $\frac{\pi}{6}$ make the equation true, but so do $\frac{\pi}{6} \pm 2\pi$, $\frac{\pi}{6} \pm 4\pi$, $\frac{\pi}{6} \pm 6\pi$, etc. Similarly, $-\frac{\pi}{6} \pm 2\pi$, $-\frac{\pi}{6} \pm 4\pi$, $-\frac{\pi}{6} \pm 6\pi$, … will also make the equation true. Consolidate this information as $\theta = \pm \frac{\pi}{6} \pm 2\pi n$, for all integers $n$. Note: There are other ways to write this solution that are equivalent to this expression.
b. Solving using inverse sine gives: 
\[ 2 \sin(\theta) = \sqrt{2} \]
\[ \sin(\theta) = \frac{\sqrt{2}}{2} \]
\[ \theta = \sin^{-1} \left( \frac{\sqrt{2}}{2} \right) \]
\[ \theta = \frac{\pi}{4} \]

There are usually two solutions within the unit circle, so in which two quadrants is sine positive? Since the value of sine depends on \( y \), sine is positive in Quadrants I and II. Therefore \( \theta = \frac{\pi}{4} \pm 2\pi n \) or \( \theta = \frac{5\pi}{4} \pm 2\pi n \).

c. Since the range of \( y = \cos(x) \) is \(-1 \leq y \leq 1\), this equation has no solution.

Example 2

Let \( f(x) = \sin(x) \), \( g(x) = \cos(x) \), and \( h(x) = \tan(x) \). Graph each of the following functions on separate axes.

\[ y = \frac{1}{f(x)} \quad y = \frac{1}{g(x)} \quad y = \frac{1}{h(x)} \]

Compare your graphs to the graphs of these functions:

\[ y = \sin^{-1}(x) \quad y = \cos^{-1}(x) \quad y = \tan^{-1}(x) \]

The first three functions are the reciprocal functions of sine, cosine and tangent. However, rather than writing them as reciprocals \( \left( \frac{1}{f(x)} \right) \), they are given new names:

\[ \frac{1}{\sin(x)} = \csc(x) \quad \frac{1}{\cos(x)} = \sec(x) \quad \frac{1}{\tan(x)} = \cot(x) \]

The abbreviation for cosecant is csc, for secant it is sec, and for cotangent it is cot. Their graphs are:

Since these are reciprocal functions, everywhere the first functions were zero, the corresponding reciprocal functions will be undefined. Check that this is the case.
In comparing these functions to the inverse trig functions, it is important to note that
\( \frac{1}{\sin(x)} \neq \sin^{-1}(x) \) (and similarly for the other corresponding functions). This is very clear by examining the graphs. The exponent of “\(-1\)” indicates that the function is the inverse, not the reciprocal.

**Problems**

For each of the following equations, determine all the solutions. You may use your calculator but remember that your calculator only gives one answer.

1. \( 2 \cos(x) = \sqrt{2} \)
2. \( 5\tan(x) - 5 = 0 \)
3. \( 4\cos^2(x) - 1 = 0 \)
4. \( 4\sin^2(x) = 3 \)
5. \( \sin(x) + 2 = 3\sin(x) \)
6. \( \tan^2(x) + \tan(x) = 0 \)

Graph each of the following equations on a separate set of axes. Label all the important points.

7. \( y = 3\csc(x) \)
8. \( y = 4 + \sec(x) \)
9. \( y = \cot(x - \pi) \)
Answers

1. $x = \pm \frac{\pi}{4} \pm 2\pi n$ for all integers $n$

2. $x = \frac{\pi}{2} \pm \pi n$ for all integers $n$

3. $x = \pm \frac{\pi}{4} n$ for all integers $n$

4. $x = \frac{\pi}{12} \pm \pi n$ or $x = \frac{2\pi}{3} \pm \pi n$ for all integers $n$

5. $x = \frac{\pi}{2} \pm 2\pi n$ for all integers $n$

6. $x = \pm \pi n$ or $x = \frac{3\pi}{4} \pm \pi n$ for all integers $n$

7. [Diagram]

8. [Diagram]

9. [Diagram]
Before the widespread availability of calculators, tables were used to look up the trig values of various angle measures. Knowing that \( \sin(2\theta) = 2\sin(\theta)\cos(\theta) \), for instance, meant that a trig table of values did not need to extend to 120° angles. \( \sin(120^\circ) \) can be written as \( 2\sin(60^\circ)\cos(60^\circ) \), and only the trig values for 60° need to be known.

Another practice common before the advent of calculators was the proving of trig identities. These proofs usually employ algebraic steps and previously proven identities to show that one side of the equation equals the other. Equivalent trigonometric expressions are known as trig identities. These identities allow trigonometric equations to be rewritten and/or solved.

See the Math Notes box in Lesson 12.2.3 for additional information.

**Example 1**

Graph the function \( f(x) = \frac{1}{\cos^2(x)} - \tan^2(x) \). Based on the graph, what can you conclude about the expression \( \frac{1}{\cos^2(x)} - \tan^2(x) \)? (That is, what trig identity can you write?) What substitution can you make in the identity so that you no longer have a fraction?

By graphing the given function above, it can readily be seen that the function is a constant, that is, a horizontal line. This graph is equivalent to the graph of \( y = 1 \). Because their graphs are equivalent for all values of \( x \), the expressions are also equivalent. Therefore you can write \( \frac{1}{\cos^2(x)} - \tan^2(x) = 1 \).

How can the given expression be rewritten so it no longer has a fraction?

Since \( \frac{1}{\cos(x)} = \sec(x) \), write \( \sec(x) - \tan^2(x) = 1 \).

This trig identity is more commonly written as \( 1 + \tan^2(x) = \sec^2(x) \).
Example 2

Prove the following trig identity:

\[
\frac{\sin(x)}{1-\cos(x)} + \frac{1-\cos(x)}{\sin(x)} = 2 \csc(x)
\]

For the identity above, start with the left side of the equation, get common denominators so that the fractions can be added, and see where this goes. It is also important to be aware of the right hand side of the equation, which is the goal. Remember that \( \csc(x) = \frac{1}{\sin(x)} \).

\[
\frac{\sin(x)}{1-\cos(x)} + \frac{1-\cos(x)}{\sin(x)} = 2 \csc(x)
\]

\[
\frac{\sin(x)}{\sin(x)} \left( \frac{\sin(x)}{1-\cos(x)} \right) + \frac{1-\cos(x)}{\sin(x)} \left( \frac{1-\cos(x)}{\sin(x)} \right) =
\]

\[
\frac{\sin^2(x)}{(\sin(x))(1-\cos(x))} + \frac{(1-\cos(x))^2}{(\sin(x))(1-\cos(x))} =
\]

\[
\frac{\sin^2(x) + (1-\cos(x))^2}{(\sin(x))(1-\cos(x))} =
\]

\[
\frac{\sin^2(x) + 1 - 2 \cos(x) + \cos^2(x)}{(\sin(x))(1-\cos(x))} =
\]

\[
\frac{\sin^2(x) + \cos^2(x) + 1 - 2 \cos(x)}{(\sin(x))(1-\cos(x))} =
\]

\[
\frac{1 + 1 - 2 \cos(x)}{(\sin(x))(1-\cos(x))} =
\]

\[
\frac{2 - 2 \cos(x)}{(\sin(x))(1-\cos(x))} =
\]

\[
\frac{2 (1-\cos(x))}{(\sin(x))(1-\cos(x))} =
\]

\[
\frac{2}{\sin(x)} = 2 \csc(x)
\]

This proves that this identity is true.

Problems

1. Show graphically that \( \sin(x + y) \) does not equal \( \sin(x) + \sin(y) \).
2. Graphically, determine what \( \cos(x + 90^\circ) \) equals.
3. Graphically, determine what \( \sin(180^\circ - x) \) equals.

Prove the following identities.

4. \[
\frac{\sin(2x)}{2 \sin^2(x)} = \cot(x)
\]

5. \[
\sin^2(x) - \cos^2(x) = \frac{\tan(x) - \cot(x)}{\tan(x) + \cot(x)}
\]

6. \[
\frac{\sin^2(x)}{1 + \cos(x)} = 1 - \frac{1}{\sec(x)}
\]

7. \[
\cos^4(x) - \sin^4(x) = 2 \cos^2(x) - 1
\]

8. \[
\frac{1}{1-\sin(x)} + \frac{1}{1+\sin(x)} = 2 \sec^2(x)
\]
Answers

1. The graphs are not the same.
2. \( \cos(x + 90^\circ) = -\sin(x) \)
3. \( \sin(180^\circ - x) = \sin(x) \)
4. \[
\frac{\sin(2x)}{2 \sin^2(x)} = \cot(x)
\]
\[
\frac{2 \sin(x) \cos(x)}{2 \sin(x) \sin(x)} =
\]
\[
\frac{2 \sin(x) \cos(x)}{2 \sin(x) \sin(x)} =
\]
\[
\frac{\cos(x)}{\sin(x)} = \cot(x)
\]
5. \[
\sin^2(x) - \cos^2(x) = \frac{\tan(x) - \cot(x)}{\tan(x) + \cot(x)}
\]
\[
\sin(x) \cos(x)
\]
\[
\sin(x) \sin(x)
\]
\[
\cos(x) \cos(x)
\]
\[
\cos(x) \sin(x)
\]
\[
\sin(x) - \cos^2(x)
\]
\[
\sin(x) + \cos^2(x)
\]
\[
\sin^2(x) - \cos^2(x)
\]
\[
\sin^2(x) + \cos^2(x)
\]
\[
\sin^2(x) - \cos^2(x)
\]
6. \[
\frac{\sin^2(x)}{1 + \cos(x)} = 1 - \frac{1}{\sec(x)}
\]
\[
= 1 - \cos(x)
\]
\[
= (1 - \cos(x))(\frac{1 + \cos(x)}{1 + \cos(x)})
\]
\[
= \frac{(1 - \cos(x))(1 + \cos(x))}{1 + \cos(x)}
\]
\[
= \frac{1 - \cos^2(x)}{1 + \cos(x)}
\]
\[
= \frac{\sin^2(x)}{1 + \cos(x)}
\]
7. \[
\cos^4(x) - \sin^4(x) = 2\cos^2(x) - 1
\]
\[
= (\cos^2(x) + \sin^2(x)) \cdot (\cos^2(x) - \sin^2(x))
\]
\[
= 1 \cdot (\cos^2(x) - \sin^2(x))
\]
\[
= \cos^2(x) - (1 - \cos^2(x))
\]
\[
= \cos^2(x) - 1 + \cos^2(x)
\]
\[
= 2\cos^2(x) - 1
\]
8. \[
\frac{1}{1 - \sin(x)} + \frac{1}{1 + \sin(x)} = 2 \sec^2(x)
\]
\[
\left(\frac{1 + \sin(x)}{1 + \sin(x)}\right) \cdot \left(\frac{1}{1 - \sin(x)}\right) + \left(\frac{1 - \sin(x)}{1 - \sin(x)}\right) \cdot \left(\frac{1}{1 + \sin(x)}\right) =
\]
\[
\frac{1 + \sin(x) + 1 - \sin(x)}{(1 + \sin(x))(1 - \sin(x))} =
\]
\[
\frac{2}{1 - \sin^2(x)} =
\]
\[
\frac{2}{\cos^2(x)} = 2 \sec^2(x)
\]
SAT Prep

1. If $7x < 2y$ and $2y < 9z$, which of the following statements is true?
   
   a. $7x < 9z$  
   b. $9z < 7x$  
   c. $z < x$  
   d. $7x = 9z$  
   e. $7x + 1 = 9z$

2. If $f(t) = 5t - 15$, then at what value of $t$ does the graph of $y = f(t)$ cross the $x$-axis?
   
   a. $-15$  
   b. $-5$  
   c. $0$  
   d. $2$  
   e. $3$

3. If $p^5 + 3 = p^5 + w$, then $w = ?$
   
   a. $-3$  
   b. $-\sqrt[5]{3}$  
   c. $\sqrt[5]{3}$  
   d. $3$  
   e. $3^5$

4. For all positive numbers $j$ and $k$, let $j \triangledown k$ be defined as $\frac{j + 4k}{j - 4k}$. What is the value of $1018 \triangledown 4.5$?
   
   a. $1.036$  
   b. $10.36$  
   c. $103.6$  
   d. $1036$  
   e. $10360$

5. If a number is rounded to 26.7, which of the following values could have been the original number?
   
   a. $26$  
   b. $26.605$  
   c. $26.655$  
   d. $26.776$  
   e. $27$

6. On a coordinate plane, the center of a circle is at $(9, -2)$. If the circle touches the $y$-axis in only one point, what is the radius of the circle?

7. The figure at right shows three squares with sides of length 6, 8, and $k$, respectively. If points $A$, $B$, and $C$ lie on line $l$, what is the value of $k$?

8. Exactly 875 out of 7000 seniors at college are majoring in mathematics. What percent of seniors are NOT majoring in mathematics?

9. Five SnookerBars cost as much as 2 Sodiepop Swirls. If the cost of one Sodiepop Swirl and one SnookerBar is $1.75, what is the cost in dollars of 1 Sodiepop Swirl?

10. The highest score possible on Professor Snape’s test is 100 and the lowest is 0. Harry, Ron, Hermione, and Neville’s tests had an average of 86. If Neville got the lowest score, what is the lowest possible score he could have gotten?

Answers

1. A  
2. E  
3. D  
4. A  
5. C  
6. 9  
7. $\frac{32}{3}$  
8. 87.5%  
9. $1.25$  
10. 4