In this chapter, students turn their attention back to logarithms. Using pattern recognition, and other problem solving strategies, students develop several properties of logarithms that enable them to solve equations that, until now, they have been unable to solve algebraically. These properties are listed in the Math Notes box in Lesson 7.1.4.

Example 1

Solve each of the following equations for $x$.

a. $5^x = 67$

b. $3(7^x) + 4 = 124$

Each of these equations has the variable as the exponent, which makes them different from other equations that students have been solving. The log property, $\log(b^x) = x \log(b)$, can be used to solve these equations. As with other equations, however, the variable must be isolated on one side of the equation.

a. $5^x = 67$

\[
\log(5^x) = \log(67) \\
x \log(5) = \log(67) \\
x = \frac{\log(67)}{\log(5)} \\
x \approx 2.613
\]

b. $3(7^x) + 4 = 124$

\[
3(7^x) = 120 \\
7^x = 40 \\
\log(7^x) = \log(40) \\
x \log(7) = \log(40) \\
x = \frac{\log(40)}{\log(7)} \\
x \approx 1.896
\]

Note: The decimal answer is an approximation.

The exact answer is the fraction $\frac{\log(67)}{\log(5)}$.

Example 2

Using the properties of logs to rewrite each expression.

a. $\log_3(16x)$

b. $\log_6(32) + \log_6(243)$

c. $\log_8\left(\frac{3x}{7}\right)$

d. $\log_{12}(276) - \log_{12}(23)$

The two properties to use are $\log_b(mn) = \log_b(m) + \log(b)$ and $\log_b\left(\frac{m}{n}\right) = \log_b(m) - \log_b(n)$.

a. $\log_3(16x) = \log_3(16) + \log_3(x)$

b. $\log_6(32) + \log_6(243) = \log_6(32 \cdot 243) = \log_6(7776) = 5$

c. $\log_8\left(\frac{3x}{7}\right) = \log_8(3x) - \log_8(7) = \log_8(3) + \log_8(x) - \log_8(7)$

d. $\log_{12}(276) - \log_{12}(23) = \log_{12}\left(\frac{276}{23}\right) = \log_{12}(12) = 1$
Example 3

Fall came early in Piney Orchard, and the community swimming pool was still full of water when the first frost occurred. The outside temperature hovered at 30° F. Maintenance quickly turned off the heat so that energy would not be wasted heating a pool that nobody would be swimming in for at least six months. As Tess walked by the pool each day on her way to school, she would peer through the fence at the slowly cooling pool. She could just make out the thermometer across the deck that displayed the water’s temperature. On the first day, she noted that the water temperature was 68° F. Four days later, the temperature reading was 58° F. Write an equation that models this situation. If the outside temperature remains at 30° F, and the pool is allowed to continue to cool, how long before it freezes (32° F)?

Heating and cooling problems are typical application problems that use exponential equations. The equation that will model this problem is an exponential equation of the form \( y = ab^x + k \). In the problem description, two data points are given: \((0, 68°)\) and \((4, 58°)\). There is one other piece of important information. The outside temperature is hovering at 30°. This is the temperature the water will approach, that is, \( y = 30 \) is the horizontal asymptote for this equation. Knowing this fact means that the equation \( y = ab^x + 30 \) can be written. To determine \( a \) and \( b \), substitute values into the equation and solve for \( a \) and \( b \).

\[
\begin{align*}
(0, 68) \Rightarrow y &= ab^x + 30 \Rightarrow 68 = ab^0 + 30 \\
(4, 58) \Rightarrow y &= ab^x + 30 \Rightarrow 58 = ab^4 + 30
\end{align*}
\]

Doing this gives two equations with two unknowns that can be solved. Simplifying first makes the work a lot easier. The first equation simplifies to 38 = a since \( b^0 = 1 \). Since \( a = 38 \) this value can be substituted into the second equation to determine \( b \) as shown at left.

\[
\begin{align*}
58 &= 38b^4 + 30 \\
28 &= 38b^4 \\
b^4 &= \frac{28}{38} = 0.7368 \\
b &= 0.9265
\end{align*}
\]

Therefore the equation is \( y = 38(0.9265)^x + 30 \). To determine when the pool will freeze, use the equation to determine when the water’s temperature reaches 32° F.

\[
\begin{align*}
32 &= 38(0.9265)^x + 30 \\
2 &= 38(0.9265)^x \\
\frac{2}{38} &= 0.9265^x \\
\log\left(\frac{2}{38}\right) &= \log(0.9265^x) \\
\log\left(\frac{2}{38}\right) &= x \log(0.9265) \\
x &= \frac{\log\left(\frac{2}{38}\right)}{\log(0.9265)} \approx 38.57
\end{align*}
\]

In approximately 39 days, the water in the pool will freeze if the outside temperature remains at 30° F for those days. In reality, the pool would be drained to prevent damage from freezing.
Problems

Solve each of the following equations.

1. \((2.3)^x = 7\)
2. \(12^x = 6\)
3. \(\log_7 49 = x\)
4. \(\log_3 x = 4\)
5. \(5(3.14)^x = 18\)
6. \(7x^8 = 294\)
7. \(\log_x 100 = 4\)
8. \(\log_5 45 = x\)
9. \(2(6.5)^x + 7 = 21\)
10. \(−\frac{1}{2} (14)^x + 6 = −9.1\)

Use the properties of logarithms to rewrite each log expression.

11. \(\log(23 \cdot 3)\)
12. \(\log \left( \frac{3\times}{8} \right)\)
13. \(\log_2 \left( \frac{60}{7} \right)\)
14. \(\log_8(12) − \log_8(2)\)
15. \(\log_5(25) + \log_5(25)\)
16. \(\log(10 \cdot 10)\)
17. \(\log_{13}(15x^2)\)
18. \(\log(123) + \log(456)\)
19. \(\log(10^8) − \log(10^3)\)
20. \(\log(5x − 4)\)

Simplify.

21. \(\log_2(64)\)
22. \(\log_{17}(17^{1/8})\)
23. \(8^{\log_8(1.3)}\)
24. \(2.3^{\log_{2.3}(1)}\)

25. Climbing Mt. Everest is not an easy task! Not only is it a difficult hike, but the Earth’s atmosphere decreases exponentially as you climb above the Earth’s surface, and this makes it harder to breathe. The air pressure at the Earth’s surface (sea level) is approximately 14.7 pounds per square inch (or 14.7 psi). In Denver, Colorado, elevation 5280 feet, the air pressure is approximately 12.15 psi. Write an equation to represent this situation expressing air pressure as a function of altitude. What is the air pressure in Mexico City, elevation 7300 feet? At the top of Mt. Everest, elevation 29,000 feet? (Note: You will need to carry out the decimal values several places to get an accurate equation and air pressures.)
Answers

1. \[ x = \frac{\log(7)}{\log(2.3)} \approx 2.336 \]

2. \[ x = \frac{\log(6)}{\log(12)} \approx 0.721 \]

3. \[ x = 2 \]

4. \[ x = 81 \]

5. \[ x = \frac{\log(3.6)}{\log(3.14)} \approx 1.119 \]

6. \[ x = 42^{1/8} \approx 1.596 \]

7. \[ x = 100^{1/4} \approx 3.162 \]

8. \[ x = \frac{\log(45)}{\log(5)} \approx 2.365 \]

9. \[ x = \frac{\log(7)}{\log(6.5)} \approx 1.040 \]

10. \[ x = \frac{\log(30.2)}{\log(14)} \approx 1.291 \]

11. \[ \log(23) + \log(3) \]

12. \[ \log(3x) - \log(8) \]

13. \[ \log_2(60) - \log_2(7) \]

14. \[ \log_8\left(\frac{12}{7}\right) = \log_8(6) \]

15. \[ \log_5(625) \]

16. \[ \log(10) + \log(10) = 2 \]

17. \[ \log_{13}(15) + \log_{13}(x^2) \]

18. \[ \log(56,088) \]

19. \[ \log\left(\frac{10^8}{10^3}\right) = \log 10^5 = 5 \]

20. cannot be rewritten

21. 6

22. \[ \frac{1}{8} \]

23. 1.3

24. 6

25. The equation is \( y = 14.7(0.999964)^x \) where \( x \) is the elevation in feet, and \( y \) is the number of pounds per square inch (psi). The air pressure in Mexico City is approximately 11.3 psi, and at the top of Mt. Everest, the air pressure is approximately 5.175 psi.
Several tools can be used for calculating parts of right triangles, including the Pythagorean Theorem, the tangent ratio, the sine ratio, and the cosine ratio. These relationships only work, however, with right triangles. What if the triangle is not a right triangle? Can the lengths and angles of any triangle still be calculated with trigonometry from certain pieces of information? Yes, by using two laws, the Law of Sines and the Law of Cosines state that for any triangle:

**Law of Sines**

\[
\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}
\]

**Law of Cosines**

\[
c^2 = a^2 + b^2 - 2ab\cos(C)
\]

See the Math Notes boxes in Lessons 7.2.2 and 7.2.3.

**Example 1**

Use the Law of Sines to calculate the value of \(x\) in each triangle.

a. 

\[
\begin{align*}
35^\circ & \quad x & \quad 21 \\
65^\circ & \quad 65^\circ
\end{align*}
\]

Solution: Set up ratios that are equal according to the Law of Sines. The ratio compares the sine of the measure of an angle to the length of the side opposite that angle.

\[
\frac{\sin(35^\circ)}{21} = \frac{\sin(65^\circ)}{x}
\]

\[
x \sin(35^\circ) = 21 \sin(65^\circ)
\]

\[
x = \frac{21 \sin(65^\circ)}{\sin(35^\circ)}
\]

\[
x \approx 33.2
\]

b. 

\[
\begin{align*}
15 & \quad x & \quad 19 \\
13 & \quad 52^\circ
\end{align*}
\]

\[
\frac{\sin(x)}{13} = \frac{\sin(52^\circ)}{15}
\]

\[
15 \sin(x) = 13 \sin(52^\circ)
\]

\[
\sin(x) = \frac{13 \sin(52^\circ)}{15}
\]

\[
\sin^{-1}(\sin(x)) = \sin^{-1}\left(\frac{13 \sin(52^\circ)}{15}\right)
\]

\[
x \approx 43^\circ
\]
Example 2

Use the Law of Cosines to calculate the value of $x$ in each triangle.

a.

![Diagram of triangle with sides 6, 9, and $x$]

The Law of Cosines does not use ratios, as the Law of Sines does. Rather, it uses a formula somewhat similar to the Pythagorean Theorem.

$$x^2 = 6^2 + 9^2 - 2(6)(9)\cos(93^\circ)$$

$$x^2 \approx 36 + 81 - 108(-0.052)$$

$$x^2 \approx 117 + 5.612$$

$$x \approx 122.612$$

$$x \approx 11.07$$

b.

![Diagram of triangle with sides 7, 21, and $x$]

$$x^2 = 17^2 + 21^2 - 2(17)(21)\cos(x)$$

$$49 = 289 + 441 - 714\cos(x)$$

$$49 = 730 - 714\cos(x)$$

$$-681 = -714\cos(x)$$

$$\frac{-681}{-714} = \cos(x)$$

$$x = 17.5^\circ \text{ using } \cos^{-1}(x)$$

Example 3

Marisa’s, June’s, and Daniel’s houses form a triangle. The distance between June’s and Daniel’s houses is 1.2 km. Standing at June’s house, the angle formed by looking out to Daniel’s house and then to Marisa’s house is $63^\circ$. Standing at Daniel’s house, the angle formed by looking out to June’s house and then to Marisa’s house is $75^\circ$. What are the distances between each of the houses?

As with any application, it is helpful to draw a picture of the situation. One distance is already known: the distance from June’s house to Daniel’s house. Label the length of the side from $D$ to $J$ as 1.2. The situation also indicates that the $m\angle J = 63^\circ$ and $m\angle D = 75^\circ$. From this it can be determined that the $m\angle M = 42^\circ$. The needed information is the lengths of $DM$ and $MJ$. To do this, use the Law of Sines.

$$\frac{\sin(75^\circ)}{MJ} = \frac{\sin(42^\circ)}{1.2}$$

$$1.2\sin(75^\circ) = (MJ)\sin(42^\circ)$$

$$\frac{1.2\sin(75^\circ)}{\sin(42^\circ)} = MJ$$

$$MJ \approx 1.7 \text{ km}$$

$$\frac{\sin(63^\circ)}{DM} = \frac{\sin(42^\circ)}{1.2}$$

$$1.2\sin(63^\circ) = (DM)\sin(42^\circ)$$

$$\frac{1.2\sin(63^\circ)}{\sin(42^\circ)} = DM$$

$$DM \approx 1.6 \text{ km}$$

Therefore the distances between the homes are: From Marisa’s to Daniel’s: 1.6 km, from Marisa’s to June’s: 1.7 km, and from Daniel’s to June’s: 1.2 km.
Problems

Solve for $x$, $y$, and/or $\theta$ in each diagram. Round all answers to the nearest tenth.

1. \[
\begin{array}{c}
7 \\
42^\circ \\
x
\end{array} \quad \begin{array}{c}
y \\
9.1 \\
31^\circ \\
\end{array}
\]

2. \[
\begin{array}{c}
6 \\
76^\circ \\
x
\end{array} \quad \begin{array}{c}
y \\
17
\end{array}
\]

3. \[
\begin{array}{c}
9.4 \\
20^\circ \\
x
\end{array} \quad \begin{array}{c}
10
\end{array}
\]

4. \[
\begin{array}{c}
y \\
94^\circ \\
x
\end{array} \quad \begin{array}{c}
15 \\
52^\circ
\end{array}
\]

5. \[
\begin{array}{c}
\theta \\
30 \\
18^\circ \\
27
\end{array}
\]

6. \[
\begin{array}{c}
21 \\
13 \\
\theta \\
14
\end{array}
\]

7. \[
\begin{array}{c}
\theta \\
8 \\
59^\circ \\
5
\end{array}
\]

8. \[
\begin{array}{c}
x \\
8.2 \\
43^\circ \\
38^\circ
\end{array}
\]

9. \[
\begin{array}{c}
6.25 \\
6.25 \\
1.75 \\
\theta
\end{array}
\]

10. \[
\begin{array}{c}
6.2 \\
93^\circ \\
x \\
8.2
\end{array}
\]
Draw and label a triangle for each part below. Then use the given information to determine the required part(s).

17. $m\angle A = 40^\circ, m\angle B = 88^\circ, a = 15$
   Calculate $b$.

18. $m\angle B = 75^\circ, a = 13, c = 14$
   Calculate $b$.

19. $m\angle B = 50^\circ, m\angle C = 60^\circ, b = 9$
   Calculate $a$.

20. $m\angle A = 62^\circ, m\angle C = 28^\circ, c = 24$
   Calculate $a$.

21. $m\angle A = 51^\circ, c = 8, b = 12$
   Calculate $a$.

22. $m\angle B = 34^\circ, m\angle C = 98^\circ, b = 3$
   Calculate $a$.

23. $a = 9, b = 12, c = 15$
   Calculate $m\angle B$.

24. $m\angle B = 96^\circ, m\angle A = 32^\circ, a = 6$
   Calculate $c$.

25. $m\angle C = 18^\circ, m\angle B = 54^\circ, b = 18$
   Calculate $c$.

26. $a = 15, b = 12, c = 14$
   Calculate $m\angle C$.

27. $m\angle C = 76^\circ, a = 39, b = 19$
   Calculate $c$.

28. $m\angle A = 30^\circ, m\angle C = 60^\circ, a = 8$
   Calculate $b$.

29. $a = 34, b = 38, c = 31$
   Calculate $m\angle B$.

30. $a = 8, b = 16, c = 7$
   Calculate $m\angle C$.

31. $m\angle C = 84^\circ, m\angle B = 23^\circ, c = 11$
   Calculate $a$.

32. $m\angle A = 36^\circ, m\angle B = 68^\circ, b = 8$
   Calculate $a$ and $c$.

33. $m\angle B = 40^\circ, b = 4$, and $c = 6$
   Calculate $a$, $m\angle A$, and $m\angle C$.

34. $a = 2, b = 3, c = 4$
   Calculate $m\angle A$, $m\angle B$, and $m\angle C$.

35. Marco wants to cut a sheet of plywood to fit over the top of his triangular sandbox. One angle measures $38^\circ$, and it is between sides with lengths 14 feet and 18 feet. What is the length of the third side?

36. From the planet Xentar, Dweeble can see the stars Quazam and Plibit. The angle between these two stars is $22^\circ$. Dweeble knows that Quazam and Plibit are $93,000,000$ miles apart. He also knows that when standing on Plibit, the angle made from Quazam to Xentar is $39^\circ$. How far is Xentar from Quazam?
Answers

1. \( x \approx 13.0, y = 107^\circ \)
2. \( x \approx 16.6 \)
3. \( x \approx 3.4 \)
4. \( x \approx 8.4, y = 34^\circ \)
5. \( \theta \approx 16.2^\circ \)
6. \( \theta \approx 37.3^\circ \)
7. \( \theta \approx 32.4^\circ \)
8. \( x \approx 9.1 \)
9. \( \theta = 90^\circ \)
10. \( x \approx 10.5 \)
11. \( x \approx 24.5 \)
12. \( x \approx 22.6 \)
13. \( x \approx 4.0 \)
14. \( \theta \approx 83.3^\circ \)
15. \( x \approx 17.2 \)
16. \( x \approx 11.3 \)
17. \( b \approx 23.3 \)
18. \( b \approx 16.5 \)
19. \( a \approx 11.0 \)
20. \( a \approx 45.1 \)
21. \( a \approx 9.3 \)
22. \( a \approx 4.0 \)
23. \( m\angle B \approx 53.1^\circ \)
24. \( c \approx 8.9 \)
25. \( c \approx 6.9 \)
26. \( m\angle C \approx 61.3^\circ \)
27. \( c \approx 39.0 \)
28. \( b = 16 \)
29. \( m\angle B \approx 71.4^\circ \)
30. no possible triangle (Triangle Inequality)
31. \( b \approx 4.3 \)
32. \( a \approx 5.0, c \approx 8.4 \)
33. \( a \approx 5.7, m\angle A \approx 65.4^\circ, m\angle C \approx 74.6^\circ \)
34. \( m\angle A \approx 29.0^\circ, m\angle B \approx 46.6^\circ, m\angle C \approx 104.5^\circ \)
35. \( \approx 11.1 \) feet
36. \( \approx 156,235,361 \) miles
Sometimes the information known about the sides and angles of a triangle is not enough to determine one unique triangle. Sometimes a triangle may not even exist, as demonstrated by the Triangle Inequality. When a triangle formed is not unique (that is, more than one triangle can be made with the given conditions) this is called the ambiguous case. This can happen when two sides and an angle not between the two sides are given, known as SSA.

Example

In $\triangle ABC$, $m \angle A = 50^\circ$, $AB = 12$, and $BC = 10$. Can you make a unique triangle? If so, calculate all the angle measures and side lengths for $\triangle ABC$. If not, show more than one triangle that meets these conditions.

As with many problems, it is helpful to first make a sketch of what the problem is describing. Once the diagram is labeled, it can be seen that the information displays the SSA pattern.

Use the Law of Sines to determine $m \angle C$.

\[
\frac{\sin(50^\circ)}{10} = \frac{\sin(C)}{12}
\]

\[
12 \cdot \frac{\sin(50^\circ)}{10} = \sin(C)
\]

\[
0.919 = \sin(C)
\]

\[
66.8^\circ = m \angle C \quad \Rightarrow \quad m \angle B \approx 63.2^\circ
\]

Because this triangle displays the SSA pattern, check to see if there is another possibility for $m \angle C$. See the diagram at right. The triangle formed with the two possible arrangements (the light grey triangle) is isosceles. From that it can be concluded that the two possibilities for $\angle C$ are supplementary.

\[
180^\circ - 66.8^\circ = 113.2^\circ, \text{ so another possibility is } m \angle C \approx 113.2^\circ.
\]

In this case, $m \angle B \approx 16.8^\circ$.

Use the Law of Cosines to determine the length of side $AC$. In this case a quadratic equation needs to be solved, so use the Quadratic Formula.

\[
x^2 - 15.43x + 44 \approx 0
\]

Both of these answers are positive numbers, and could be lengths of sides of a triangle. In one arrangement, $m \angle C$ is fairly small and $AC$ is longer, while in the other arrangement, $m \angle C$ is large and $AC$ is shorter.

\[
x = \frac{15.43 \pm \sqrt{15.43^2 - 4(1)(44)}}{2(1)}
\]

\[
x = \frac{15.43 \pm 16.28}{2}
\]

\[
x \approx 11.65 \text{ or } x \approx 3.78
\]
Problems

Solve for the remaining parts of each triangle described below, explain why a triangle does not exist, or explain why there is more than one possible triangle.

1. In ΔABC, ∠A = 32°, AB = 20, and BC = 12.

2. In ΔXYZ, ∠Z = 84°, XZ = 6, and YZ = 9.

3. In ΔABC, m∠A = m∠B = 45°, and AB = 7.


5. In ΔXYZ, ∠X = 59°, XY = 18, and YZ = 10.


Answers

1. Two triangles: AC ≈ 22.6, m∠B ≈ 85.3°, m∠C ≈ 62.7° or AC ≈ 11.4, m∠B ≈ 30.0°, m∠C ≈ 118.0°

2. One triangle: XY ≈ 10.3, m∠X ≈ 60.6°, m∠Y ≈ 35.5°

3. One triangle: m∠C ≈ 90°, BC = AC = \(\frac{7\sqrt{2}}{2}\) ≈ 4.9

4. Two triangles: QR ≈ 30.7, m∠Q ≈ 46.0°, m∠P ≈ 106.0°, or QR ≈ 9.90, m∠Q ≈ 134.0°, m∠P ≈ 18.0°

5. No triangle exists.

6. One triangle: m∠Q = 90°, QR ≈ 8.3, PR ≈ 10.2
1. If $-1 < t < 0$, which of the following statements must be true?
   a. $t^3 < t < t^2$
   b. $t^2 < t^3 < t$
   c. $t^2 < t < t^3$
   d. $t < t^3 < t^2$
   e. $t < t^2 < t^3$

2. Without taking a single break, Mercedes hiked for 10 hours, up a mountain and back down by the same path. While hiking, she averaged 2 miles per hour on the way up and 3 miles per hour on the way down. How many miles was it from the base of the mountain to the top?
   a. 4
   b. 6
   c. 9
   d. 12
   e. 18

3. When a certain rectangle is folded in half, it forms two squares. If the perimeter of one of these squares is 28, what is the perimeter of the original rectangle?
   a. 30
   b. 42
   c. 49
   d. 56
   e. Cannot be determined from the information given.

4. A class of 50 girls and 60 boys sponsored a road rally race. If 60% of the girls and 50% of the boys participated in the road rally, what percent of the class participated in the road rally?
   a. 54.5%
   b. 55%
   c. 57.5%
   d. 88%
   e. 110%

5. The sum of four consecutive integers is $s$. In terms of $s$, which of the following is the smallest of these four integers?
   a. $\frac{s-6}{4}$
   b. $\frac{s-4}{4}$
   c. $\frac{s-3}{4}$
   d. $\frac{s-2}{4}$
   e. $\frac{s}{4}$

6. On a certain map, 30 miles is represented by one-half inch. On the same map, how many miles are represented by 2.5 inches?

7. How many of the first one hundred positive integers contain the digit 9?

8. The sum of $n$ and $n + 1$ is greater than five but less than 15. If $n$ is an integer, what is one possible value of $n$?
9. In the figure at right, \( \triangle ABC \) is a right triangle and
\[
\frac{y}{6} = \frac{6}{10}
\]. What is the value of \( y \)?

10. For three numbers \( a, b, \) and \( c \), the average (arithmetic mean) is twice the median.
If \( a < b < c, a = 0, \) and \( c = kb \), what is the value of \( k \)?

**Answers**

1. D
2. D
3. B
4. A
5. A
6. 150 miles
7. 19 integers
8. \( n \) can be 3, 4, 5, or 6
9. \( y = 3.6 \)
10. \( k = 5 \)