After exploring patterns in Pascal’s Triangle, students see how the numbers in the triangle relate to the counting principles. The connection to the Binomial Theorem allows students to quickly expand binomials raised to various powers as well as determine probabilities in situations involving two outcomes.

See the Math Notes boxes in Lessons 10.1.1 and 10.3.3 for additional information.

Example 1

Use Pascal’s Triangle to expand \((a + b)^4\), then apply the same pattern to expand \((2x – 3y)^4\).

The fourth line of Pascal’s Triangle gives the coefficients of the terms in the expansion of this binomial since its degree is 4. The fourth line is 1 4 6 4 1. Therefore the expansion is:

\[
(a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.
\]

Notice that the powers of \(a\) decrease while the powers of \(b\) increase, but the sum of the two exponents in each term is always 4.

Apply this pattern to the second binomial, replacing each \(a\) with \(2x\) and each \(b\) with \(-3y\):

\[
(2x – 3y)^4 = 1(2x)^4 + 4(2x)^3(-3y) + 6(2x)^2(-3y)^2 + 4(2x)(-3y)^3 + (-3y)^4
\]
\[
= 16x^4 + 32x^3(-3)y + 24x^2(9)y^2 + 8x(-27)y^3 + 81y^4
\]
\[
= 16x^4 – 96x^3y + 216x^2y^2 – 216xy^3 + 81y^4
\]

Example 2

Use the Binomial Theorem to determine the eighth term in the expansion of \((p + 3q)^{14}\).

The Math Notes box in Lesson 10.1.1 provides the formula for using combinations to generate the numbers in Pascal’s Triangle. The Math Notes box in Lesson 10.3.3 shows how to use the Binomial Theorem in combinations form to determine the expansion of any binomial raised to any positive integer value. In particular, the \(k^{th}\) term of the expansion \((a + b)^n\) is given by \(\binom{n}{k}a^{n-k}b^k\). In this case the term is:

\[
\binom{14}{14 - (8 - 1)}a^{14 - (8 - 1)}b^{8 - 1} = \binom{14}{7}a^7b^7
\]
\[
= \frac{14!}{7!7!}a^7b^7
\]
\[
= 3432a^7b^7
\]
Problems

Expand each of the following binomials.

1. \((5x - 6y)^6\)
2. \((8 + 3q)^5\)
3. Determine the fifth term in the expansion of \((x + y)^{12}\).
4. Determine the seventh term in the expansion of \((x + y)^{12}\).
5. Determine the 15th term in the expansion of \((3p - 2q)^{32}\).
6. Eight coins are tossed. What is the probability that exactly five are heads?

Answers

1. \(15625x^6 - 112500x^5y + 337500x^4y^2 - 540000x^3y^3 + 486000x^2y^4 - 233280xy^5 + 46656y^6\)
2. \(32768 + 61440q + 46080q^2 + 17280q^3 + 3240q^4 + 243q^5\)
3. \(495x^8y^4\)
4. \(924x^6y^6\)
5. \(\binom{32}{18}(3p)^{18}(-2q)^{14} = 471435600(387420489)p^{18}(16384)q^4\)
6. \(\binom{8}{5} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^3 = \frac{7}{32}\)