Once the meaning of a solution is understood, it can be applied to understanding solutions of inequalities and systems of inequalities. Inequalities typically have infinitely many solutions, and students learn ways to represent such solutions. For additional information see the Math Notes boxes in Lessons 3.2.2 and 3.2.4.

Example 1

Solve each equation or inequality below. Explain what the solution for each one represents. Then explain how the equation and inequalities are related to each other.

\[ x^2 - 4x - 5 = 0 \]

\[ x^2 - 4x - 5 < 0 \]

\[ y \geq x^2 - 4x - 5 \]

Solution:

There are many ways to solve the equation, including graphing, factoring and using the Zero Product Property, or using the Quadratic Formula. Factoring and use the Zero Product Property to solve is shown below.

\[
\begin{align*}
    x^2 - 4x - 5 &= 0 \\
    (x - 5)(x + 1) &= 0 \\
    x &= 5, \quad x = -1
\end{align*}
\]

Check:

If \( x = 5 \):

\[
(5)^2 - 4(5) - 5 = 25 - 20 - 5 = 0 \quad \checkmark
\]

If \( x = -1 \):

\[
(-1)^2 - 4(-1) - 5 = 1 + 4 - 5 = 0 \quad \checkmark
\]

The second quadratic is an inequality. To solve this inequality, utilize a number line to emphasize what the solution represents.

Start by solving the related quadratic equation (which was done above).

\[ x^2 - 4x - 5 = 0 \]

The solutions to the equatoin are \( x = 5 \) and \( x = -1 \).

By placing these two points on a number line, they act as boundary points, dividing the number line into three regions. Since the original inequality is strictly “less than”, use open circles for the boundary points.

Choose any number in each of the regions to see if the number will make the original inequality true or false. Solutions will make the inequality true. Note: You only need to check one point in each region.

Solution continues on next page →
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Choosing a point in each region and substituting in into the inequality gives:

<table>
<thead>
<tr>
<th>Point</th>
<th>Expression</th>
<th>Left Side Value</th>
<th>Right Side Value</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>x = -2</td>
<td>(( -2)^2 - 4(-2) - 5 )</td>
<td>?</td>
<td>&lt; 0</td>
<td>false</td>
</tr>
<tr>
<td></td>
<td></td>
<td>?</td>
<td>0 - 0 - 5</td>
<td>&lt; 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>?</td>
<td>7</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>x = 0</td>
<td>(( 0)^2 - 4(0) - 5 )</td>
<td>?</td>
<td>&lt; 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>?</td>
<td>0 - 0 - 5</td>
<td>&lt; 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>?</td>
<td>-5</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>x = 7</td>
<td>(( 7)^2 - 4(7) - 5 )</td>
<td>?</td>
<td>&lt; 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>?</td>
<td>49 - 28 - 5</td>
<td>&lt; 0</td>
</tr>
</tbody>
</table>

Highlight the region on the number line that makes the inequality true, as shown at right. This solution can also be represented algebraically as \(-1 < x < 5\).

The last inequality in the example has a \(y\). Having both \(x\) and \(y\) means an \(xy\)-coordinate graph needs to be used to show the solutions. Graph the parabola using a solid line because the original inequality is “greater than or equal to”. The graph of the parabola at right divides the plane into two regions: the part within the “bowl” of the parabola—the interior—and the region outside the parabola. The points on the parabola represent where \(y = x^2 - 4x - 5\).

Test a point from one of the regions to check whether it will make the inequality true or false. As before, we are looking for the “true” region.

The point \((0, 0)\) is an easy point to use.

\[
\begin{align*}
0 & \geq (0)^2 - 4(0) - 5 \\
0 & \geq 0 - 0 - 5 \\
0 & \geq -5
\end{align*}
\]

True! Therefore the region containing the point \((0, 0)\) is the solution. This means any point chosen in this region, the “bowl” of the parabola, will make the inequality true.

To illustrate that this region is the solution, we shade this region of the graph.

To see how these equations and inequalities are related, examine the graph of the parabola. Where are the zeros? Where are the \(y\)-values of the parabola negative? The zeros are the \(x\)-intercepts on the graph, or \(x = -1\) and \(x = 5\), which you determined by solving the equation. The graph is negative when it dips below the \(x\)-axis, and this happens when \(x\) is between \(-1\) and \(5\). Solving the first inequality answered this question as well. Therefore, the graph could have answered the first two parts quickly.
Example 2

Han and Lea have been building jet roamers and pod racers. Each jet roamer requires one jet pack and three crystallic fuel tanks, while each pod racer requires two jet packs and four crystallic fuel tanks. Han and Lea’s suppliers can only produce 100 jet packs and 270 fuel tanks each week, and due to manufacturing conditions, Han and Lea can build no more than 30 pod racers each week. Each jet roamer makes a profit of 1 tig (their form of currency) while each pod racer makes a profit of 4 tigs.

a. If Han and Lea receive an order for 12 jet roamers and 22 pod racers, how many of each part will they need to fill this order? If they can fill this order, how many tigs will they make?

b. Write a list of constraints, an inequality for each constraint, and sketch a graph showing all inequalities with the points of intersection labeled. How many jet roamers and pod racers should Han and Lea build to maximize their profits?

Solution:

This problem is an example of a linear programming problem, and although the name might conjure up images of computer programming, these problems are not done on a computer. This problem can be solved by creating a system of inequalities that, when graphed, creates a feasibility region. This region contains the solution for the number of jet roamers and pod racers Han and Lea should make to maximize their profit.

a. Jet packs: 1(12) + 2(22) = 12 + 44 = 56
   Fuel tanks: 3(12) + 4(22) = 36 + 88 = 124
   Each result is within the constraints, so it is possible for Han and Lea to fill this order. If they do, they will make 1(12) + 4(22) = 12 + 88 = 100 tigs.

b. Begin by defining the variables. Let \( x \) = the number of jet roamers Han and Lea will make, and \( y \) = the number of pod racers.

   \( x \geq 0 \), and \( y \geq 0 \) because a negative number of items cannot be produced.
   A jet roamer requires one jet pack while a pod racer requires two. There are only 100 jet packs available each week, so \( x + 2y \leq 100 \).
   Each jet roamer requires three crystallic fuel tanks and each pod racer requires four. This translates into the inequality \( 3x + 4y \leq 270 \) since only 270 fuel tanks are available each week. Lastly, since Han and Lea cannot make more than 30 pod racers, we can write \( y \leq 30 \).

These inequalities are all shown on the graph at right. The region common to all constraints is shaded. This is the feasibility region because choosing a point in this shaded area gives you a combination of jet roamers and pod racers that Han and Lea can produce under the given restraints.

Solution continues on next page →
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The equation used to calculate the profit is \( P = 1x + 4y \).

To maximize profits, test all the vertices of the feasibility region in the profit equation. These points are (0, 0), (0, 30), (40, 30), (70, 15), and (90, 0).

(0, 0): \( P = 1(0) + 4(0) = 0 \)
(0, 30): \( P = 1(0) + 4(30) = 120 \)
(40, 30): \( P = 1(40) + 4(30) = 160 \)
(70, 15): \( P = 1(70) + 4(15) = 130 \)
(90, 0): \( P = 1(90) + 4(0) = 90 \)

The greatest profit is 160 tigs when Han and Lea build 40 jet roamers and 30 pod racers.

Problems

Graph the following system of inequalities.

1. \( y < \frac{1}{2}x + 6 \)
   \( y > -\frac{1}{2}x + 6 \)
   \( x < 12 \)

2. \( x + y < 10 \)
   \( x + y > 4 \)
   \( y < 2x \)
   \( y > 0 \)

3. \( y \leq 3x + 4 \)
   \( y \geq -\frac{1}{3}x + 8 \)
   \( y \geq -\frac{1}{3}x + 4 \)
   \( y \geq 5x - 6 \)

4. \( 3x + 4y < 12 \)
   \( y > (x + 1)^2 - 4 \)

5. \( y < -\frac{3}{4}(x - 1)^2 + 6 \)
   \( y > x - 7 \)

6. \( y < (x + 2)^3 \)
   \( y > x^2 + 3x \)

For each of the following problems, write a system of inequalities that when graphed will produce the shaded region.

7. 

8. 

\[ \text{Graphical representations of shaded regions.} \]
9. Ramon and Thea are considering opening their own business. They wish to make and sell alien dolls they call Hauteans and Zotions. Each Hautean sells for $1.00 while each Zotion sells for $1.50. They can make up to 20 Hauteans and 40 Zotions, but no more than 50 dolls total. When Ramon and Thea go to city hall to get a business license, they find there are a few more restrictions on their production. The number of Zotions (the more expensive item) can be no more than three times the number of Hauteans (the cheaper item). How many of each doll should Ramon and Thea make to maximize their profit? What will the profit be?

10. Sam and Emma are plant managers for the Sticky Chewy Candy Company that specializes in delectable gourmet candies. Their two most popular candies are Chocolate Chews and Peanut and Jelly Jimmies. Each batch of Chocolate Chews takes 1 teaspoon of vanilla while each batch of the Peanut and Jelly Jimmies uses two teaspoons of vanilla. They have at most 20 teaspoons of vanilla on hand as they use only the freshest of ingredients. The Chocolate Chews use two teaspoons of baking soda while the Peanut and Jelly Jimmies use three teaspoons of baking soda. They only have 36 teaspoons of baking soda on hand. Because of production restrictions, they can make no more than 15 batches of Chocolate Chews and no more than 7 batches of Peanut and Jelly Jimmies. Sam and Emma have been given the task of determining how many batches of each candy they should produce if they make $3.00 profit for each batch of Chocolate Chews and $2.00 for each batch of Peanut and Jelly Jimmies. Help them out by writing the inequalities described here, graphing the feasibility region, and determining their maximum profit.

Answers

1.  

2.  

3.  

4.  

5.  

6.  

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7.  \[ y \leq \frac{1}{3}x + 4 \]  
7.  \[ y \leq -x + 8 \]  
7.  \[ y \geq -\frac{1}{2}x + 4 \] 

8.  \[ y \geq (x - 6)^2 - 5 \]  
8.  \[ y \leq 0 \] 

9. The graph of the feasibility region is shown at right. The inequalities are \( x \geq 0, y \geq 0, x + y \leq 50, x \leq 20, y \leq 40, \) and \( y \leq 3x, \) where \( x = \text{number of Hauteans} \) and \( y = \text{number of Zotions}. \) The profit is given by \( P = x + 1.5y. \) Maximum profit seems to occur at point A (12.5, 37.5), but there is a problem with this point. Ramon and Thea cannot make a half of a doll (or at least that does not seem possible). Try these nearby points: (12, 37), (12, 38), (13, 37), and (13, 38). The point that gives maximum profit and is still in the feasibility region is (13, 37). They should make 13 Hautean and 37 Zotion dolls for a profit of $68.50.

10. The graph of the feasibility region is shown at right. The inequalities are \( x \geq 0, y \geq 0, y \leq 7, x \leq 15, x + 2y \leq 20, \) and \( 2x + 3y \leq 36, \) where \( x = \text{number of Chocolate Chews} \) and \( y = \text{number of Peanut and Jelly Jimmies}. \) The profit is given by \( P = 3x + 2y. \) The point that seems to give the maximum profit is (15, 2.5) but this only works if half batches can be made. Instead, choose the point (15, 2) which means Sam and Emma should make 15 batches of Chocolate Chews and 2 batches of Peanut and Jelly Jimmies. Their profit will be $49.00.