The chapter explores polynomial functions in greater depth. Students will learn how to sketch polynomial functions without using their graphing tool by using the factored form of the polynomial. In addition, they learn the reverse process: determining the polynomial equation from the graph.

For additional information see the Math Notes box in Lesson 8.1.1.

Example 1

State whether or not each of the following expressions is a polynomial. If it is not, explain why not. If it is a polynomial, state the degree of the polynomial.

a. \(-7x^4 + \frac{2}{3}x^3 + x^2 - 4.1x - 6\)

b. \(8 + 3.2x^2 - \pi x^5 - 61x^{10}\)

c. \(9x^3 + 4x^2 - 6x^{-1} + 7^x\)

d. \(x(x^3 + 2)(x^4 - 4)\)

A polynomial in one variable is an expression that can be written as the sum or difference of terms. The terms are in the form \(ax^n\) where \(a\) is any number called the coefficient of \(x\), and \(n\), the exponent, must be a whole number.

a. This is a polynomial. A coefficient that is a fraction \(\left(\frac{2}{3}\right)\) is acceptable. The degree of the polynomial is the largest exponent on the variable, so in this case the degree is 4.

b. This is also a polynomial, and its degree is 10.

c. This expression is not a polynomial for two reasons. First, the \(x^{-1}\) is not allowed because the exponents of the variable cannot be negative. The second reason is because of the \(7^x\). The variable cannot be a power in a polynomial.

d. Although the expression is not the sum or difference of terms, it can be written as the sum or difference of terms by multiplying the expression and simplifying. Doing so gives \(x^8 + 2x^5 - 4x^4 - 8x\), which is a polynomial of degree 8.
Example 2

Without using your graphing tool, make a sketch of each of the following polynomial functions by using the leading coefficient, the roots, and the degree.

a. \( f(x) = (x + 1)(x - 3)(x - 4) \)  
b. \( y = (x - 2)^2(x + 3) \)  
c. \( p(x) = x(x + 1)^2(x - 4)^2 \)  
d. \( f(x) = -(x + 1)^3(x - 1)^2 \)

The roots of the polynomial are the \( x \)-intercepts, which are easily determined when the polynomial is in factored form, as are all the polynomials above, by using the Zero Product Property. The degree of the polynomial and leading coefficient can be determined by multiplying the first terms of each factor. Note that some factors are repeated. The graphs shown below are possible sketches.

a. The leading term will be \( x^3 \), so this graph will be a cubic function. The roots of this polynomial are \( x = -1, 3, \) and 4. To help sketch the graph, if \( x = 0 \), then \( f(0) = 12 \), so the \( y \)-intercept is \((0, 12)\).

b. The leading term will be \( x^3 \), so this graph will be a cubic function. The distinct roots of this polynomial are \( x = -3 \) and 2. \( x = 2 \) is called a double root, since the expression \((x - 2)\) is squared and is thus equivalent to \((x - 2)(x - 2)\). The graph will just touch the \( x \)-axis at \( x = 2 \), and “bounce” off. The \( y \)-intercept is \((0, 12)\).

c. This fifth degree polynomial with a leading term of \( x^5 \) has three distinct roots, \( x = 0, -1, \) and 4. Both \( x = -1 \) and \( x = 4 \) are double roots. The \( y \)-intercept is \((0, 0)\).

d. This fifth degree polynomial with a leading term of \(-x^5\) has two distinct roots, \( x = -1 \) and 1. \( x = 1 \) is a double root, and \( x = -1 \) is a triple root. The leading coefficient in negative, so this graph will be a vertical reflection of a typical fifth degree polynomial. The \( y \)-intercept is \((0, -1)\).
Example 3

Write the equation of the graph shown at right.

From the graph, a general equation can be written based on the roots and y-intercept of the polynomial. Since the x-intercepts (roots) are \( x = -3, 3, \) and 8, then \( (x + 3), (x - 3), \) and \( (x - 8) \) are factors. Also, since the graph touches and bounces off at \( x = -3, \) \((x + 3)\) is a double root. Thus this function can be written as \( f(x) = a(x + 3)^2(x - 3)(x - 8) \). The value of \( a \) needs to be determined.

Using the fact that the graph passes through the point \( (0, -2) \), write:

\[
-2 = a(0 + 3)^2(0 - 3)(0 - 8)
-2 = a(9)(-3)(-8)
-2 = 216a
a = \frac{2}{216} = -\frac{1}{108}
\]

Therefore the equation is \( f(x) = -\frac{1}{108} (x + 3)^2(x - 3)(x - 8) \).

Problems

State whether or not each of the following functions is a polynomial function. If it is, give the degree. If it is not, explain why not.

1. \[ y = \frac{1}{8} x^7 + 4.23x^6 - x^4 - \pi x^2 + \sqrt{2}x - 0.1 \]
2. \[ f(x) = 45x^3 - 0.75x^2 - \frac{3}{100} x + \frac{5}{x} + 15 \]
3. \[ y = x(x + 2)\left(6 + \frac{1}{x}\right) \]

Sketch the graph of each of the following polynomial functions.

4. \[ y = (x + 5)(x - 1)^2(x - 7) \]
5. \[ y = -(x + 3)(x^2 + 2)(x + 5)^2 \]
6. \[ f(x) = -x(x + 8)(x + 1) \]
7. \[ y = x(x + 4)(x^2 - 1)(x - 4) \]
Below are the complete graphs of some polynomial functions. Based on the shape and location of the graph, describe all the roots of the polynomial function and state its minimum degree. Be sure to include information such as whether or not a root is a double or triple root.

8. 
9. 
10. 

Using the graphs below and the given information, write the specific equation for each polynomial function.

11. y-intercept: (0, 12) 
12. y-intercept: (0, -15) 
13. y-intercept: (0, 3)
Answers

1. Yes, degree 7.

2. No. You cannot have \( x \) in the denominator.

3. No. When multiplied out, there is an \( x \) in the denominator.

4. The roots are \( x = -5, 1, \) and 7, with \( x = 1 \) being a double root.

5. The roots are \( x = -3 \) and \( x = -5 \), which is a double root. The \((x^2 + 2)\) factor does not produce any real roots since this expression cannot equal zero. The graph crosses the \( y \)-axis at \( y = -150 \).

6. This graph has roots \( x = -8, -1, \) and 0.

7. \( x^2 - 1 \) gives us two roots. Since it factors to \((x + 1)(x - 1)\), the five roots are \( x = -4, -1, 0, 1, \) and 4.

8. A third degree polynomial (cubic) with one root at \( x = 0 \), and one double root at \( x = -4 \).

9. A fourth degree polynomial with real roots at \( x = -5 \) and \( -3 \), and a double root at \( x = 5 \).

10. A fifth degree polynomial with five real roots at \( x = -5, -1, 2, 4, \) and 6.

11. \( y = (x + 3)(x - 1)(x - 4) \)

12. \( y = -0.1(x + 5)(x + 2)(x - 3)(x - 5) \)

13. \( y = \frac{1}{12} (x + 3)^2(x - 1)(x - 4) \)