Complex numbers arise naturally when trying to solve some equations such as $x^2 + 1 = 0$. The solutions to this equation are $x = \pm \sqrt{-1}$, or $x = \pm i$.

Sometimes polynomials have complex roots. Complex roots always come in pairs called complex conjugates. For example, if $x = 3 + 2i$ is a root, then $x = 3 - 2i$ is also a root.

**Example 1**

Simplify each of the following expressions.

a. $3 + \sqrt{-16}$

b. $(3 + 4i) + (-2 - 6i)$

c. $(4i)(-5i)$

d. $(8 - 3i)(8 + 3i)$

Remember that $i = \sqrt{-1}$, or $i^2 = -1$.

a. $3 + \sqrt{-16} = 3 + 4\sqrt{-1} = 3 + 4i$

b. Real parts can be combined with real parts, and imaginary parts with imaginary parts:

$(3 + 4i) + (-2 - 6i) = 1 - 2i$

c. $(4i)(-5i) = (4 \cdot -5)(i \cdot i) = -20i^2 = -20(-1) = 20$

d. The Distributive Property or an area model can be used to compute this product.

$$
(8 - 3i)(8 + 3i) = 8(8) + 8(3i) - 3i(8) - 3i(3i) = 64 + 24i - 24i + 9 = 73
$$

The two factors are called **complex conjugates**, and they are useful when working with complex numbers.

Multiplying a complex number by its conjugate produces a real number! This will always happen. Also, whenever a function with real coefficients has a complex root, it always has the conjugate as a root as well.
Example 2

Determine the roots of the function below using the Quadratic Formula. Explain what the roots tell you about the graph of the function.

\[ f(x) = 2x^2 - 20x + 53 \]

Quadratic Formula: If \( ax^2 + bx + c = 0 \) then \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \).

The roots of the function occur when \( f(x) = 0 \). Here, \( a = 2 \), \( b = -20 \), and \( c = 53 \). The solution is shown at right.

This creates an expression with a negative under the radical. This equation has no real solutions, but it does have complex solutions.

In mathematics, \( i = \sqrt{-1} \) is defined as an imaginary number. When an imaginary number is combined with a real number, then it is called a complex number. Complex numbers are written in the form \( a + bi \). Using \( i = \sqrt{-1} \), the answer above can be simplified.

\[
x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(2)(53)}}{2(2)}
\]

\[
x = \frac{20 \pm \sqrt{400 - 424}}{4}
\]

\[
x = \frac{20 \pm \sqrt{-24}}{4}
\]

\[
x = \frac{20 \pm 2i \sqrt{6}}{4}
\]

\[
x = \frac{2(10 \pm i \sqrt{6})}{4}
\]

\[
x = \frac{10 \pm i \sqrt{6}}{2}
\]

Therefore, the graph of the equation \( y = 2x^2 - 20x + 53 \) has no \( x \)-intercepts, but it does have two complex roots, \( x = \frac{10 \pm i \sqrt{6}}{2} \). Recall that the degree of a polynomial function indicates the maximum number of roots. In fact the degree indicates the exact number of roots; some (or all) which might be complex.
Example 3

Make a sketch of a graph of a polynomial function \( y = p(x) \) so that \( p(x) \) has exactly four real roots. Then change the graph so that \( p(x) \) has two real roots and two complex roots.

If \( p(x) \) has four real roots, then this will be a fourth degree polynomial that crosses the x-axis at exactly four different places. One such graph is shown at right.

In order for the graph to have only two real and two complex roots, it must be changed so one of the “dips” does not reach the x-axis. One example is shown at right.

Problems

Simplify the following expressions.

1. \((6 + 4i) - (2 - i)\)  
2. \(8i - \sqrt{-16}\)  
3. \((-3)(4i)(7i)\)

4. \((5 - 7i)(-2 + 3i)\)  
5. \((3 + 2i)(3 - 2i)\)  
6. \((\sqrt{3} - 5i)(\sqrt{3} + 5i)\)

Below are the graphs of some polynomial functions. Based on the shape and location of the graph, describe all the roots. Be sure to include information such as whether roots are double or triple, real or complex, etc.

7.

8.

9. Write the specific equation for the polynomial function passing through the point \((0, 5)\), and with roots \( x = 5, x = -2, \) and \( x = 3i \).
Answers

1. $4 + 5i$
2. $4i$
3. $84$
4. $11 + 29i$
5. $13$
6. $28$
7. A third degree polynomial with one real root at $x = 5$ and two complex roots.
8. A fifth degree polynomial with one real root at $x = -4$ and four complex roots.
9. $y = -\frac{1}{18} (x - 5)(x + 2)(x - 3i)(x + 3i) = -\frac{1}{18} (x^2 - 3x - 10)(x^2 + 9)$