Introduction to the Parent Guide with Extra Practice

Welcome to the Core Connections Integrated III Parent Guide with Extra Practice. The purpose of this guide is to assist you should your child need help with homework or the ideas in the course. We believe all students can be successful in mathematics as long as they are willing to work and ask for help when they need it. We encourage you to contact your child’s teacher if your student has additional questions that this guide or other resources do not answer.

This guide was written to address the major topics in each chapter of the textbook. Each section begins with a title bar and the lesson(s) in the book that it addresses. In many cases the explanation box at the beginning of the section refers you to one or more Math Notes boxes in the student text for additional information about the fundamentals of the idea. Detailed examples follow a summary of the concept or skill and include complete solutions. The examples are similar to the work your child has done in class. Additional problems, with answers, are provided for your child to practice.

There will be some topics that your child understands quickly and some concepts that may take longer to master. The big ideas of the course take time to learn. This means that students are not necessarily expected to master a concept when it is first introduced. When a topic is first introduced in the textbook, there will be several problems to do for practice. Subsequent lessons and homework assignments will continue to practice the concept or skill over weeks and months so that mastery will develop over time.

Practice and discussion are required to understand mathematics. When your child comes to you with a question about a homework problem, often you may simply need to ask your child to read the problem and then ask them what the problem is asking. Reading the problem aloud is often more effective than reading it silently. When you are working problems together, have your child talk about the problems. Then have your child practice on their own.

Below is a list of additional questions to use when working with your child. These questions do not refer to any particular concept or topic. Some questions may or may not be appropriate for some problems.

- What have you been doing in class or during this chapter that might be related to this problem? Let’s look at your notebook, class notes, and Learning Log. Do you have them?
- Were the other members of your team having difficulty with this as well? Can you call your study partner or someone from your study team?
- Have you checked the online homework help (homework.cpm.org)?
- What have you tried? What steps did you take?
- What did not work? Why did it not work?
- Which words are most important? Why? What does this word/phrase tell you?
- What do you know about this part of the problem?
- Explain what you know right now.
- What is unknown? What do you need to know to solve the problem?
- How did the members of your study team explain this problem in class?
- What important examples or ideas were highlighted by your teacher?
- How did you organize your information? Do you have a record of your work?
- Can you draw a diagram or sketch to help you?
- Have you tried making a list, looking for a pattern, etc.?
- What is your estimate/prediction?
- Is there a simpler, similar problem we can do first?
If your student has made a start at the problem, try these questions:

- What do you think comes next? Why?
- What is still left to be done?
- Is that the only possible answer?
- Is that answer reasonable? Are the units correct?
- How could you check your work and your answer?

If you do not seem to be making any progress, you might try these questions.

- Let’s look at your notebook and class notes. Do you have them?
- Were you listening to your team members and teacher in class? What did they say?
- Did you use the class time working on the assignment? Show me what you did.
- Were the other members of your team having difficulty with this as well? Can you call your study partner or someone from your study team?

This is certainly not a complete list; you will probably come up with some of your own questions as you work through the problems with your child. Ask any question at all, even if it seems too simple to you.

To be successful in mathematics, students need to develop the ability to reason mathematically. To do so, students need to think about what they already know and then connect this knowledge to the new ideas they are learning. Many students are not used to the idea that what they learned yesterday or last week will be connected to today’s lesson. Too often students do not have to do much thinking in school because they are usually just told what to do. When students understand that connecting prior learning to new ideas is a normal part of their education, they will be more successful in this mathematics course (and any other course, for that matter). The student’s responsibilities for learning mathematics include the following:

- Actively contributing in whole class and study team work and discussion.
- Completing (or at least attempting) all assigned problems and turning in assignments in a timely manner.
- Checking and correcting problems on assignments (usually with their study partner or study team) based on answers and solutions provided in class and online.
- Asking for help when needed from their study partner, study team, and/or teacher.
- Attempting to provide help when asked by other students.
- Taking notes and using his/her toolkit when recommended by the teacher or the text.
- Keeping a well-organized notebook.
- Not distracting other students from the opportunity to learn.

Assisting your child to understand and accept these responsibilities will help them to be successful in this course, develop mathematical reasoning, and form habits that will help them become a life-long learner.
ADDITIONAL SUPPORT

Consider these additional resources for assisting students with the CPM Educational Program:

- **This Core Connections Integrated III Parent Guide with Extra Practice**
  This booklet can be downloaded free of charge at cpm.org. It can also be purchased at shop.cpm.org.

- **CPM Homework Help website at homework.cpm.org**
  A variety of complete solutions, hints, and answers are provided. Some problems refer back to other similar problems. The homework help is designed to assist students to be able to do the problems but not necessarily do the problems for them.

- **Checkpoints**
  The student text has Checkpoint materials to assist students with skills they should master. The checkpoints are numbered to align with the chapter in the text. For example, the topics in Checkpoint 5A and Checkpoint 5B should be mastered while students complete Chapter 5.

- **Resource Pages**
  The resource pages referred to in the student text can be found at cpm.org.

- **Previous tests**
  Many teachers allow students to examine their own tests from previous chapters in the course. Even if they are not allowed to bring these tests home, a student can learn much by analyzing errors on past tests.

- **Math Notes in the student text**
  The Closure section at the end of each chapter has a list of the Math Notes in that chapter. Note that relevant Math Notes are sometimes found in other chapters than the one currently being studied.

- **Glossary and Index in the student text**

- **“Answers and Support” table**
  The “What Have I Learned” questions (in the Closure section a the end of each chapter) are followed by an “Answers and Support” table that indicates where students can get more help with the problems.

- **After-school assistance**
  Some schools have after-school or at-lunch support programs for students. Ask the teacher.

- **Other students**
  Consider asking your child to obtain the contact information for a couple other students in class.

- **Parent Guides with Extra Practice from previous courses**
  If your student needs help with the concepts from previous courses that are necessary preparation for this class, the Core Connections Integrated I and Core Connections Integrated II Parent Guides with Extra Practice are available for free download at cpm.org.

Many of these resources can also be found in the student eBook.
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INVESTIGATIONS AND FUNCTIONS  

This opening section introduces the big ideas of *Core Connections Integrated III*, as well as different ways of thinking and problem solving strategies. Not only are problems challenging and interesting, they also review topics from previous math courses such as graphing and solving equations. Functions and function notation is reviewed by using function machines and sketching graphs of several types of functions. For more information about functions and their graphs see the Math Notes boxes in Lessons 1.1.1 and 1.1.3.

Example 1

Talula’s function machine at right shows its inner “workings,” written in function notation. Note that \( y = 10 - x^2 \) is an equivalent form. What will the output be if:

a. \( 2 \) is dropped in?

b. \( -2 \) is dropped in?

c. \( \sqrt{10} \) is dropped in?

d. \( -3.45 \) is dropped in?

Solution: The number “dropped in,” that is, substituted for \( x \), takes the place of \( x \) in the equation in the machine. Follow the Order of Operations to simplify the expression to determine the value of \( f(x) \).

a. \[ f(2) = 10 - (2)^2 \]
   \[ = 10 - 4 \]
   \[ = 6 \]

b. \[ f(-2) = 10 - (-2)^2 \]
   \[ = 10 - 4 \]
   \[ = 6 \]

c. \[ f(\sqrt{10}) = 10 - (\sqrt{10})^2 \]
   \[ = 10 - 10 \]
   \[ = 0 \]

d. \[ f(-3.45) = 10 - (-3.45)^2 \]
   \[ = 10 - 11.9025 \]
   \[ = -1.9025 \]
Example 2

Graph and completely describe the function \( f(x) = \frac{3}{\sqrt{x+4}} \).

Solution: Graphs of functions can be described using these attributes:

- shape
- line of symmetry
- opens upward or downward
- asymptotes
- increasing or decreasing
- \( x \)- and \( y \)-intercepts
- domain and range
- endpoints
- maximum or minimum points
- continuous, discrete, or neither
- whether it is function

Graph this function on a graphing calculator.

This function has two domain restrictions. First, the denominator cannot be 0, so \( x \neq -4 \).

Second, the radical sign cannot have a negative quantity inside, \( \sqrt{x+4} \geq 0 \), or \( x \geq -4 \).

Combining these two restrictions, the domain of the function is \( x > -4 \).

Asymptotes occur when the graph approaches a value. Here, as the \( x \)-values become larger and larger, the \( f(x) \)-values move closer and closer to zero. This happens because the value in the denominator increases, making smaller and smaller fractions. Therefore \( y = 0 \) is a horizontal asymptote. Similarly, the line \( x = -4 \) is a vertical asymptote. You can convince yourself that the graph becomes very close to this line as you substitute values for \( x \) that are very close to \(-4\).

The \( x \)-intercepts occur when \( f(x) = 0 \). For a rational expression (a fraction) to equal 0, the numerator (the top) must have a value of 0.

Therefore, \( f(x) = 0 \rightarrow \frac{3}{\sqrt{x+4}} = 0 \rightarrow 3 = 0 \). Since \( 3 \neq 0 \), there are no \( x \)-intercepts.

To determine the \( y \)-intercept, substitute \( x = 0 \) into the equation. \( f(0) = \frac{3}{\sqrt{0+4}} = \frac{3}{2} \)

The function can be completely described as follows:

- The graph of this function is curved.
- There are no lines of symmetry.
- This is not a quadratic function, so there is no direction of opening.
- The horizontal asymptote is \( y = 0 \).
- The vertical asymptote is \( x = -4 \).
- The function is decreasing.
- There are no \( x \)-intercepts.
- The \( y \)-intercept is \((0, \frac{3}{2})\).
- The domain is \( x > -4 \).
- The range is \( y > 0 \).
Example 3

Consider the functions \( f(x) = \frac{\sqrt{x}}{3-x} \) and \( g(x) = (x + 5)^2 \).

a. What is \( f(4) \)?  
   b. What is \( g(4) \)?

c. What is the domain of \( f \)?  
   d. What is the domain of \( g \)?

e. What is the range of \( f \)?  
   f. What is the range of \( g \)?

Solution:

Substitute \( x = 4 \) in the functions for parts (a) and (b):

\[
\begin{align*}
\text{a. } f(4) &= \frac{\sqrt{4}}{3-4} \\
&= \frac{2}{-1} \\
&= -2 \\
\text{b. } g(4) &= (4 + 5)^2 \\
&= (9)^2 \\
&= 81
\end{align*}
\]

The domain of a function is the set of \( x \)-values that can be input and still obtain an output.

c. The function \( f \) has some domain restrictions. First, we cannot take the square root of a negative number, so \( x \) cannot be less than zero. Additionally, the denominator of a fraction cannot be zero, so \( x \neq 3 \). Therefore, the domain of \( f \) is \( x \geq 0 \) and \( x \neq 3 \).

d. For the function \( g \), there are no restrictions on adding five and squaring the result. The domain of \( g \) is all real numbers.

The range of a function is the set of all possible output values. The range of a function can be seen by sketching a graph of the function.

e. Try some possibilities first. Can this function ever equal zero? Yes, when the numerator is zero, so if \( x = 0 \), then \( f(x) = 0 \). Can the function ever equal a very large positive number? Yes, this happens when \( x < 3 \), but very close to 3. (For example, let \( x = 2.9999 \), then \( f(x) \) is approximately equal to 17,320.) Can \( f(x) \) become an extremely negative number? Yes, when \( x > 3 \), but very close to 3. (Here, try \( x = 3.0001 \), and then \( f(x) \) is approximately equal to \(-17,320\).) There does not seem to be any restrictions on the range of \( f \), therefore we can say that the range is all real numbers.

f. Since the function \( g \) squares the value in the final step, the output will always be positive. It can equal zero (when \( x = -5 \)), but it will never be negative. Therefore, the range of \( g \) is \( y \geq 0 \).

Use a graphing calculator to graph the functions and verify that your answers are correct.
Problems

If \( f(x) = 3x^2 - 6x \), calculate:
1. \( f(1) \)
2. \( f(-3) \)
3. \( f(2.75) \)

If \( g(x) = -0.3x + 6.3x^2 \), calculate:
4. \( g(-2) \)
5. \( g(0.4) \)
6. \( g(18) \)

Graph and completely describe each of the following functions. Be sure to label each graph carefully so that all key points are identified.
7. \( f(x) = x^2 - 2x - 3 \)
8. \( f(x) = \frac{2}{x+1} \)
9. \( f(x) = -0.1x + 3.2 \)
10. \( f(x) = (x + 2)^3 - 1 \)
11. \( f(x) = \sqrt{x+5} - 1 \)
12. \( f(x) = 2\left(\frac{3}{2}\right)^x \)

Answers
1. \(-3\)
2. \(45\)
3. \(6.1875\)

4. \(25.8\)
5. \(0.888\)
6. \(2035.8\)

7. [Graph of a parabola opening upward]
   - Line of symmetry: \( x = 1 \)
   - \( x \)-intercepts: \((-1, 0)\) and \((3, 0)\)
   - \( y \)-intercept: \((0, -3)\)
   - Domain: all real numbers
   - Range: \( y \geq -4 \)
   - Vertex: \((1, -4)\)

8. [Graph of a curved, decreasing function]
   - Curved, decreasing
   - Vertical asymptote: \( x = -1 \)
   - Horizontal asymptote: \( y = 0 \)
   - \( y \)-intercept: \((0, 2)\)
   - Domain: \( x \neq -1 \)
   - Range: \( y \neq 0 \)

9. [Graph of a line, decreasing]
   - Line, decreasing
   - \( x \)-intercept: \((32, 0)\)
   - \( y \)-intercept: \((0, 3.2)\)
   - Domain: all real numbers
   - Range: all real numbers

10. [Graph of a cubic, increasing function]
    - Cubic, increasing
    - \( x \)-intercept: \((-1, 0)\)
    - \( y \)-intercept: \((7, 0)\)
    - Domain: all real numbers
    - Range: all real numbers

11. [Graph of a square root, increasing function]
    - Square root, increasing
    - Endpoint: \((-5, -1)\)
    - \( x \)-intercept: \((-4, 0)\)
    - \( y \)-intercept: \((1.2, 0)\)
    - Domain: \( x \geq -5 \)
    - Range: \( y \geq -1 \)

12. [Graph of an exponential, increasing function]
    - Exponential, increasing
    - Horizontal asymptote: \( y = 0 \)
    - \( y \)-intercept: \((2, 0)\)
    - Domain: all real numbers
    - Range: \( y > 0 \)
DETERMINING POINTS OF INTERSECTION  1.2.1

Multiple representations (table, graph, equation, situation) can be used to determine where the graphs of two functions will intersect. In a table, find the entries with the same $x$- and $y$-values. On a graph, you can usually see the point(s) of intersection, although they might not be “nice” coordinates. The equations of two functions can be set equal to each other and solved to determine the exact points of intersection.

Example

Where do the graph of the functions $f(x) = 10x^2 - 5x - 3$ and $g(x) = -10x + 2$ intersect?

Solution: Create tables and a graph for the given functions.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$x$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>102</td>
<td>-3</td>
<td>32</td>
</tr>
<tr>
<td>-2</td>
<td>47</td>
<td>-2</td>
<td>22</td>
</tr>
<tr>
<td>-1</td>
<td>12</td>
<td>-1</td>
<td>12</td>
</tr>
<tr>
<td>0</td>
<td>-3</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>-8</td>
</tr>
<tr>
<td>2</td>
<td>27</td>
<td>2</td>
<td>-18</td>
</tr>
<tr>
<td>3</td>
<td>72</td>
<td>3</td>
<td>-28</td>
</tr>
</tbody>
</table>

Both table contain the point $(-1, 12)$, so that is one point of intersection. This point of intersection cannot be seen in a standard graphing window.

By looking at the graph we can see that there is another point of intersection, but it does not have integer coordinates. Use the Equal Values Method (from Core Connections Integrated I) to solve for this point. Since the graphs of the functions intersect when $f(x) = g(x)$, start by solving the equation $10x^2 - 5x - 3 = -10x + 2$.

1. $10x^2 - 5x - 3 = -10x + 2$
2. $10x^2 + 5x - 5 = 0$
3. $5(2x^2 + x - 1) = 0$
4. $5(2x - 1)(x + 1) = 0$
5. $5 = 0$ or $2x - 1 = 0$ or $x + 1 = 0$
6. $5 \neq 0$ so only $x = \frac{1}{2}$ or $x = -1$

This is a quadratic equation, so start be setting it equal to 0. Move all terms to one side. Factor out the greatest common factor. Factor the quadratic expression. Apply the Zero Product Property. Solve each equation.

Solve for the corresponding $y$-values. It does not matter which equation you use.
1. If $x = \frac{1}{2}$, then $y = -10\left(\frac{1}{2}\right) + 2$ or $y = -3$. $\left(\frac{1}{2}, -3\right)$ is a point of intersection.
2. If $x = -1$, then $y = -10(-1) + 2$ or $y = 12$. $(-1, 12)$ is a point of intersection.
Problems

Determine where the graphs of each of the following pairs of functions intersect.

1. \( f(x) = -5x + 5 \)  
   \( g(x) = -5x - 6 \)
2. \( f(x) = -x + 4 \)  
   \( g(x) = -9x + 8 \)
3. \( f(x) = -4x + 10 \)  
   \( g(x) = x^2 + 9x - 6 \)
4. \( f(x) = 10x \)  
   \( g(x) = x^2 + 4x + 8 \)
5. \( f(x) = x^2 - 5x \)  
   \( g(x) = x^2 - 6x - 3 \)
6. \( f(x) = 4x^2 - x + 6 \)  
   \( g(x) = -x^2 - 5x + 7 \)

Answers

1. no points of intersection (the lines are parallel)  
   2. \( \left( \frac{1}{2}, \frac{7}{2} \right) \)
3. (1, 4) and (–14, 64)  
   4. (2, 20) and (4, 40)
4. (–3, 24)  
   5. (–1, 11) and (0.2, 5.96)
COMPARING AND REPRESENTING DATA

Data distributions can be represented graphically with histograms and boxplots. For assistance with box plots and histograms, see the Parent Guide with Extra Practice for CPM Core Connections Course 1, 2, and 3, which is available free for download at cpm.org. For more details on describing spread and data distributions see the Math Notes boxes in Lessons 1.2.2 and 1.2.3.

Two distributions of data can be compared by comparing their center, shape, spread, and outliers.

The center, or “typical” value, of a data distribution can be described by the median. If the distribution is symmetric and has no outliers, the mean can be used to describe the center.

The spread of a distribution can be described with the interquartile range (IQR) or the standard deviation, as described in the Math Notes box in Lesson 1.2.2. Since the standard deviation is based upon the mean, it should be used only to describe the spread of distributions that are symmetric and without outliers. Since gathering data for entire populations is often impractical, most of the data sets we analyze are samples. In this course, in general, calculate the sample standard deviation.

Example 1

University professors are complaining that the English Literature classes at community colleges are not demanding enough. Specifically, the university professors claim that community college literature courses are not assigning enough novels to read. A community college statistics student collected the following data from 42 universities and community college literature courses in the state. Compare the number of novels read in the two types of colleges.

Number of novels assigned in community college literature courses:
13, 10, 15, 12, 14, 9, 11, 15, 12, 14, 9, 10, 13, 15, 12, 9, 11, 15, 12, 10, 15, 14

Number of novels assigned in university literature courses:
11, 8, 14, 13, 25, 11, 7, 13, 8, 16, 11, 10, 20, 7, 8, 13, 14, 16, 18, 10

Solution:

Any analysis of data distributions should begin with a graphical representation of the data. A bin width of two was chosen for the histograms that follow. So that the distributions can be compared, both graphs have the same scale on the x-axis, and the boxplots are graphed above the histograms.
The checksum is used to verify that data has been entered into the graphing technology correctly. The sum of the data set, as determined by the statistical functions of the calculator, should match the given checksum value.

When comparing the distributions, the center, shape, spread, and outliers should be considered. Since neither of the distributions is nearly symmetric, and one of the distributions has an outlier, it would not be appropriate to use the mean nor standard deviations to compare. The five number summaries are shown below each graph.

**Center:** Both types of colleges assign the same median of 12 novels.

**Shape:** The distribution for community colleges is skewed, with a low of 8 to 9 novels and increasing to a peak at 14 to 15 novels. The distribution at universities is skewed in the other direction, with a peak at 10 to 11 novels.

**Spread:** The variability in the number of novels assigned at the community college level is much less than the variability between courses at the university level. The IQR for community colleges is 4 novels (14 – 10 = 4), while the university IQR of 6 (15 – 9 = 6) is one-and-a-half times as wide.

**Outliers:** One course at a university is an outlier; 25 books are assigned in that course. Twenty-five books is far away from the bulk of the university courses. The TI-83+/TI-84+ calculator can mark an outlier or a boxplot with dots.

**Conclusions:** The university professors claim that their courses are more demanding because they assign more novels. However that data does not bear this claim out. Twenty-five percent of university courses assign more novels than any of the community college courses (the right “whisker,” or the top 25% of the courses, for universities is beyond the entire boxplot for community colleges). But just as dramatically, 25% of the university classes assign fewer books than any of the community colleges (the left “whisker,” or lowest 25%, for universities is below the entire boxplot for community colleges). Furthermore, the median number of novels assigned at the two universities is the same (12 books). Community colleges are more consistent from course-to-course in the number of novels they assign (IQR is 4) than are the universities (IQR is 6).
Example 2

A rabbit breeder kept track of the number of offspring from five does (female rabbits) this year. The does had: 243, 215, 184, 280, and 148 kits (baby rabbits) respectively. Show how to calculate the mean and standard deviation number of kits per doe without using the statistical functions of a calculator.

The mean is \( \frac{243 + 215 + 184 + 280 + 148}{5} = 214 \) kits.

To calculate the sample standard deviation, first calculate the distance each data point is from the mean:

\[
\begin{align*}
243 - 214 &= 29 \\
215 - 214 &= 1 \\
184 - 214 &= -30 \\
280 - 214 &= 66 \\
148 - 214 &= -66
\end{align*}
\]

Then, calculate each of the distances squared:

\[
\begin{align*}
29^2 &= 841 \\
1^2 &= 1 \\
(-30)^2 &= 900 \\
66^2 &= 4356 \\
(-66)^2 &= 4356
\end{align*}
\]

Divide by one less than the number of data points:

\[
\frac{841 + 1 + 900 + 4356 + 4356}{5 - 1} = 2613.5
\]

Take the square root: \( \sqrt{2613.5} \approx 52.122 \)

Since the precision of the original measurements was an integer, the final result should also be an integer. The mean number of kits per doe is 214 with a standard deviation of 52 kits.

Problems

1. Different types of toads tend to lay different numbers of eggs. The following data was collected from two different species. Compare the number of eggs laid by American toads to the number laid by Fowler’s toads. Is it appropriate to summarize the distributions by using mean and standard deviation? Use a bin width of 250 eggs.

   American toads: 9100, 8700, 10300, 9500, 7800, 8900, 9200, 9300, 8900, 8300, 9400, 8000, 9000, 8400, 9700, 10000, 8600, 8900, 9900, 9300 \( \text{checksum 181,200} \)

   Fowler’s toads: 9500, 9100, 9400, 8800, 9000, 8400, 9200, 9200, 8900, 9100, 8600, 9200, 8700, 9800, 9300, 8800, 9200, 9300, 9000, 9100 \( \text{checksum 181,600} \)

2. Without using the statistical functions on your calculator, calculate the sample standard deviation of the number of eggs laid by each of the first five American toads.
3. A psychologist collected the following data for the age at which children started crawling:
low weight babies: 10, 12, 11, 11, 7, 13, 10, 12, 11, 13, 10, 11, 15, 11, 14, 10 months  
   checksum 181
average weight babies: 7, 6, 13, 9, 8, 7, 5, 7, 9, 8, 10, 8, 11, 7, 7, 10, 6, 8, 7, 6, 12, 8, 7 months  
   checksum 186

Do low birth weight babies start crawling at a later age than babies born at an average weight?

4. Compare the amount of time the flavor lasted for people chewing brand “10” chewing gum to the amount of time the flavor lasted in “Strident” chewing gum. See the graph at right. Estimate the mean for each type of gum.

Answers

1. American Toads

Mean and standard deviation are appropriate statistics because both distributions are fairly symmetric with no outliers.

Both types of toads lay a mean of between 9000 and 9100 eggs. Both distributions are single-peaked and symmetric with no apparent outliers. However there is much greater variability in the number of eggs that American toads lay. The sample standard deviation for American toads is about 654 while the sample standard deviation for Fowler’s toads is only about half as much, about 324 eggs.
2. \[ \sqrt{\frac{20^2 + (-380)^2 + 1220^2 + 420^2 + (-1280)^2}{5-1}} \approx 928 \text{ eggs} \]

3. The median and IQR will be used to compare statistics since mean and standard deviation are not appropriate—both distributions are skewed and one has an outlier.

The median age at which low-weight babies start crawling is 11 months, while the median age for average-weight babies is 8 months.

Both distributions are single-peaked and skewed. The low-weight babies appear to have an outlier at 7 months, although the calculator does not identify it as a true outlier. The average-weight babies have an outlier at 13 months.

The variability in the crawling age is roughly the same for low-weight babies (IQR is 2.5 months) as for average-weight babies (IQR is 2 months).

Low-weight babies have their development delayed by about 3 months. About 75% of low-weight babies have not even started crawling after 75% of average-weight babies are already crawling.

4. The median for both types of gum was about 18 minutes of flavor time. The times for “10” were skewed, while the times for Strident were symmetric. The lower half of the distributions for both gums was the same. But there was much more variability in the upper half of people chewing “10” than in the upper half of Strident. Indeed, more than 25% of “10” chewers reported flavor lasting longer than any of the Strident chewers. Neither gum had outliers in flavor time.

There was more variability in flavor time for “10”—the IQR was about 9 minutes (25 – 16 = 9). The IQR of 4 minutes (20 – 16 = 4) for Strident was less than half that of “10”. That variability is an advantage. If you chew “10,” you will probably be no worse off than chewing Strident, and you could have much longer flavor.

The mean for Strident is about the same as the median since the distribution is symmetric—about 18 minutes. But the mean for “10” is longer than 18 minutes due to the skew in the shape—maybe 22 minutes or so.
SAT Prep

These problems are very similar to actual SAT test questions. Use a calculator whenever you need one. On multiple choice questions, choose the best answer from the ones provided. When a picture has a diagram, assume the diagram is drawn accurately except when a problem says it is not. These questions address more topics than you have done in class so far.

1. If $x + 9$ is an even integer, then which of the following could be the value of $x$?
   a. 4  b. 2  c. 0  d. $-1$  e. $-2$

2. If $(m + 5)(11 - 7) = 24$, then $m =$?
   a. 1  b. 4  c. 8  d. 11  e. 17

3. The fractions $\frac{3}{d}$, $\frac{4}{d}$, and $\frac{5}{d}$ are in simplest reduced form. Which of the following could be the value of $d$?
   a. 20  b. 21  c. 22  d. 23  e. 24

4. A group of three numbers is called a “j-triple” for some number $j$, if $(\frac{3}{4}, j, \frac{5}{4})$. Which of the following is a j-triple?
   a. $(0, 4, 5)$  b. $\left(\frac{5}{4}, 6, 6\frac{1}{4}\right)$  c. $(6, 2, 10)$
   d. $(750, 1000, 1250)$  e. $(575, 600, 625)$

5. A ball is thrown straight up. The height of the ball can be modeled with the equation $h = 38t - 16t^2$ where $h$ is the height in feet and $t$ is the number of second since the ball was thrown. How high is the ball two seconds after it is thrown?
   a. 12  b. 16  c. 22  d. 32  e. 40

6. In the figure at right, $\overline{AC}$ is a line segment with a length of 4 units. What is the value of $k$?

7. Let the operation $\&$ be defined as $a \& b$ is the sum of all integers between $a$ and $b$. For example, $4 \& 10 = 5 + 6 + 7 + 8 + 9 = 35$. What is the value of $(130 \& 170) - (131 \& 169)$?
8. An isosceles triangle has a base of length 15. The length of each the other two equal sides is an integer. What is the shortest possible length of these other two sides?

9. Assume that $\frac{1}{4}$ quart of cranberry concentrate is mixed with $1\frac{3}{4}$ quarts of apple juice to make cranapple juice for four people. How many quarts of cranberry concentrate are needed to make a cranapple drink at the same strength for 15 people?

10. A stack of five cards is labeled with a different integer ranging from 0 to 4. If two cards are selected at random without replacement, what is the probability that the sum will be 2?

Answers

6. $\frac{1}{2}$  7. 300  8. 8  9. $\frac{15}{16}$  10. $\frac{1}{10}$
TRANSFORMATIONS OF \( f(x) = x^2 \) 2.1.1 – 2.1.2

Students investigate the general equation for a family of quadratic functions, discovering ways to shift and change the graphs. Additionally, they learn how to graph a quadratic function when it is written in graphing form.

Example 1

The graph of \( f(x) = x^2 \) is shown at right. For each function listed below, explain how its graph differs from the graph at right.

\[
\begin{align*}
g(x) &= 3x^2 \\
h(x) &= (x + 6)^2 \\
j(x) &= x^2 + 5 \\
k(x) &= -\frac{1}{2}x^2 \\
l(x) &= -2(x - 4)^2 + 1
\end{align*}
\]

Every function listed above has something in common: they are all quadratic functions, which means they will form a parabola when graphed. The only differences will be in the direction of opening (upward or downward), the shape (vertically compressed or vertically stretched), and/or the location of the vertex.

The “3” in \( g(x) = 3x^2 \) stretches the graph vertically, making it appear “skinnier”.

The graph of \( h(x) = (x + 6)^2 \) has the same shape as \( f(x) = x^2 \) and opens upward, but it has a new location: it moves to the left 6 units.

The graph of \( j(x) = x^2 + 5 \) has the same shape as \( f(x) = x^2 \), opens upward, and is shifted up 5 units.

The function \( k(x) = -\frac{1}{2}x^2 \) does not shift position, but the \( \frac{1}{4} \) compresses the parabola vertically, making it appear “wider” and the negative sign reflects the graph vertically across the x-axis.

The last function, \( l(x) = -2(x - 4)^2 + 1 \), combines all of these transformations. The “−2” stretches the graph vertically (appear “skinnier”) and reflects the graph so that it opens downward, the “−4” shifts the graph to the right 4 units, and the “+ 1” shifts the graph up 1 unit.
Example 2

For each of the quadratic functions below, where is the vertex of the parabola?

\[ f(x) = -(x + 5)^2 + 6 \quad \text{g}(x) = -4(x - 1)^2 \quad h(x) = 5x^2 + 3 \]

For a quadratic function, the vertex is called the **locator point**. This point gives a starting point for graphing the parabola. The vertex of a quadratic function in **graphing form**, \( f(x) = a(x - h)^2 + k \), is the point \((h, k)\).

For the function \( f(x) = -(x + 5)^2 + 6 \), \( h = -5 \) and \( k = 6 \), so the vertex is \((-5, 6)\).

Since \( g(x) = -4(x - 1)^2 \) can also be written \( g(x) = -4(x - 1)^2 + 0 \), the vertex of its graph is \((1, 0)\).

Rewrite \( h(x) = 5x^2 + 3 \) as \( h(x) = 5(x - 0)^2 + 3 \) to see that the vertex of its graph is \((0, 3)\).

Example 3

In a neighborhood water balloon battle, Dudley develops a winning strategy. His home base is situated 5 feet behind an 8-foot fence. Twenty-five feet away, on the other side of the fence, is his opponent’s camp. Dudley uses a water balloon launcher, and shoots his balloons so that they just miss the fence and land in his opponent’s camp. Write an equation to model the path of the water balloon.

Solution:

It is helpful to first draw a sketch of the situation as shown at right. The parabola shows the path the balloon will take, starting 5 feet away from the fence (point A) and landing 25 feet past the fence (point B).

There are different ways to set up axes for this problem. The location of the axes will determine the equation. Note that there are multiple correct equations.

Here, the y-axis will be at the fence. With the axes in place, label the known coordinates to show all of the information from the problem description. Find the coordinates of the vertex (highest point) of this parabola and then write its equation in graphing form.

Since parabolas are symmetric, the vertex is halfway between the two x-intercepts. The average of the x-intercepts is 10, so the vertex is located at \((10, y)\). This means the equation will be in the form \( y = a(x - 10)^2 + k \). Since the parabola opens down, \( a < 0 \). Also, \( k > 8 \) since the vertex is higher than the y-intercept of \((0, 8)\).
The parabola passes through the points \((0, 8), (–5, 0), \) and \((25, 0)\). Substitute the point \((0, 8)\) into the equation and write:

\[ 8 = a(0 - 10)^2 + k \]

or

\[ 8 = 100a + k \]

This equation has two variables, which means that another (different) equation with \(a\) and \(k\) is needed to be able to solve for them. Substituting the point \((–5, 0)\) into the original equation yields:

\[ 0 = a(-5 - 10)^2 + k \]

or

\[ 0 = 225a + k \]

Begin solving the system of equations by subtracting the second equation from the first:

\[
\begin{align*}
8 &= 100a + k \\
- (0) &= -225a + k \\
\hline
8 &= -125a \\
\therefore \quad a &= \frac{-8}{125}
\end{align*}
\]

Substitute this \(a\)-value back into one of the two equations above to determine \(k\).

\[
\begin{align*}
8 &= 100\left(-\frac{8}{125}\right) + k \\
8 &= -\frac{32}{5} + k \\
\therefore \quad k &= \frac{72}{5}
\end{align*}
\]

An equation for the path of a water balloon is \( y = -\frac{8}{125}(x - 10)^2 + \frac{72}{5} \). 
Problems

For each quadratic function below, describe the transformation, sketch the graph, and state the vertex.

1. $y = -2(x - 5)^2 + 4$

2. $y = (x - 2)^2 - 5$

3. $y = (x + 3)^2 - 2$

4. $y = \frac{1}{2}(x - 6)^2 + 2$

For each situation, write an equation that will model the situation.

5. Twinkle Toes Tony kicked a football, and it landed 100 feet from where he kicked it. It also reached a maximum height of 125 feet. Write an equation that models the path of the ball while it was in the air.

6. When some software companies develop software, they do it with “planned obsolescence” in mind. This means that they plan on the sale of the software to rise, hit a point of maximum sales, then drop and eventually stop when they release a newer version of the software. Suppose a graph of the curve showing the number of sales over time can be modeled with a parabola and that the company plans on the “life span” of its product to be 6 months, with maximum sales reaching 1.5 million units. Write an equation that models this situation.

7. A new skateboarder’s ramp just arrived at Bungey’s Family Fun Center. A cross-sectional view shows that the shape is parabolic. The sides are 12 feet high and 15 feet apart. Write an equation that models the cross section of this ramp.

Answers

5. Placing the start of the kick at the origin gives an equation of $y = -0.05x(x - 100)$.

6. Let the $x$-axis be the number of months, and the $y$-axis be the number of sales in millions. Placing the origin at the beginning of sales, gives an equation of $y = -\frac{1}{6}(x - 3)^2 + 1.5$.

7. Placing the lowest point of the ramp at the origin gives $y = \frac{48}{225}x^2$. 
Students generalize what they have learned about transforming the graph of \( f(x) = x^2 \) to transform the graphs of several other functions. The students start with the simplest form of each function’s graph, which is called the “parent graph.” Students use \( y = x^3 \), \( y = \frac{1}{x} \), \( y = \sqrt{x} \), \( y = |x| \), and \( y = b^x \) as the equations for parent graphs. They also apply their knowledge to non-functions. For additional information see the Math Notes box in Lesson 2.2.5.

Example 1

For each of the following functions, state the parent equation, and use it to graph each equation as a transformation of its parent equation.

a. \( y = (x + 4)^3 - 1 \)  
b. \( y = -\frac{1}{x} \)  
c. \( y = 3\sqrt{x} - 2 \)  
d. \( y = 3^x - 6 \)

Solutions:

For each function, graph both the function its parent on the same set of axes to help visualize the transformation.

a. This is a cubic function (the name given to a polynomial with 3 as the highest power of \( x \)), thus its parent graph is \( y = x^3 \) as shown by the lighter curve at right. The graph of the given function will have the same shape as \( y = x^3 \), but it will be shifted to the left 4 units (from the “+ 4” within the parentheses), and down 1 unit (from the “–1”). The transformed graph is the darker curve shown on the graph at right. Notice that the point \((0, 0)\) on the original graph has shifted left 4 units, and down 1 unit, so it is at \((-4, -1)\). This point is known as a locator point. It is a key point of the graph, and knowing its position helps to graph the rest of the curve.

b. \( y = -\frac{1}{x} \), has had only one change from the parent function \( y = \frac{1}{x} \): the negative sign. The negative sign here “flips” each part of the parent graph vertically, or across the \( x \)-axis. The lighter curve shown at right is the parent \( y = \frac{1}{x} \), and the darker curve is \( y = -\frac{1}{x} \).
c. The graph of $y = 3\sqrt{x} - 2$ is shifted to the right 2 units from the parent graph of $y = \sqrt{x}$ because of the “–2” under the radical sign. The transformed graph will be vertically stretched by a factor of 3 because the radical is multiplied by 3. The transformed graph is the darker curve on the graph shown at right. Notice that the point (0, 0) on the parent graph (the locator point) has shifted right 2 units.

d. $y = 3^x - 6$ is an exponential function with a parent graph of $y = b^x$, which changes in steepness as $b$ changes. The greater the $b$-value, the steeper the graph. The graph of $y = 3^x - 6$ is a bit steeper than the graph of $y = 2^x$, which is often thought of as the simplest exponential function. The transformed graph is also shifted down 6 units. In the graph at right, the lighter curve is $y = 2^x$, while the darker graph is $y = 3^x - 6$.

Example 2

A graph of $y = f(x)$ is shown at right. Use the graph to sketch:

a. $y = f(x) + 3$

b. $y = f(x) - 4$

c. $y = f(x - 2)$

d. $y = f(x + 1)$

e. $y = 3f(x)$

Solution: Even though the equation of the function is not given, the graph can still be transformed on the coordinate grid.

a. Remember that $f(x)$ represents the $y$-values of the original function. Therefore $y = f(x) + 3$ can be thought of as saying “the $y$-values, plus 3.” Adding 3 to the $y$-values will shift the graph 3 units up. This is shown at right. Notice that the shape of the graph is identical to the original graph.

b. If $y = f(x) + 3$ shifts the graph up three units, then $y = f(x) - 4$ will shift the graph down 4 units. This graph is shown at right.

c. The graph of $y = f(x - 2)$ is shifted to the right 2 units. This graph is shown at right.
d. \( y = f(x + 1) \) is shifted to the left 1 unit. This graph is shown at right.

e. \( y = 3f(x) \) stretches the graph vertically by a factor of 3. Notice that the point \((-3, -1)\) on the original graph is \((-3, -3)\) on the transformed graph. The point \((3, 0)\) remains \((3, 0)\) and the point \((4, 1)\) becomes \((4, 3)\).

Example 3

Apply your knowledge of parent graphs and transformations to graph the following two non-functions.

a. \( x = y^2 + 3 \)

b. \((x - 2)^2 + (y + 3)^2 = 36\)

Solution: Not every equation represents a function, and the two parent equation of the two non-functions in this problem are \(x = y^2\) and \(x^2 + y^2 = r^2\). The first is the equation of a “sleeping parabola,” or a parabola that opens towards the positive \(x\)-axis. The second equation is the equation of a circle with center \((0, 0)\), and radius of length \(r\).

a. The “+ 3” tells us the graph will shift 3 units, but is it up, down, left, or right? Rewriting the equation as \(y = \pm \sqrt{x - 3}\) helps us see that this graph is shifted to the right 3 units. At right, the gray curve is the graph of \(x = y^2\), and the darker curve is the graph of \(x = y^2 + 3\).

b. A circle with center \((h, k)\) and radius \(r\) has the equation \((x - h)^2 + (y + k)^2 = r^2\). Therefore the graph of this equation is a circle with center \((2, -3)\), and a radius of 6. The graph of the circle is shown at right.
Problems

Sketch the graph each of the following equations. State the parent equation and include any key and/or locator points.

1. \( y = (x - 5)^2 \)
2. \( y = -\frac{1}{3} (x + 4)^2 + 7 \)
3. \( (x - 2)^2 + (y + 1)^2 = 9 \)
4. \( y = |x + 5| - 2 \)
5. \( y = \frac{1}{x+1} + 10 \)
6. \( y = 2^x - 8 \)
7. \( y = -(x - 2)^3 + 1 \)
8. \( y = \sqrt{x + 7} \)
9. \( y = 3|x - 5| \)
10. \( x = y^2 + 9 \)

For each of the following problems, state whether or not it is a function. If it is not a function, explain why not.

11. \( y = 7 \pm \sqrt{9 - x^2} \)
12. \( y = 3(x - 4)^2 \)
Answers

1. parent graph $f(x) = x^2$
   vertex $(5, 0)$

2. parent graph $f(x) = x^2$
   vertex $(-4, 7)$

3. parent graph $(x - h)^2 + (y - k)^2 = r^2$
   center $(2, -1)$, radius 3

4. parent graph $f(x) = |x|$
   vertex $(-5, -2)$

5. parent graph $f(x) = \frac{1}{x}$
   asymptotes $x = -1, y = 10$

6. parent graph $f(x) = 2^x$
   asymptote $x = -8$

7. parent graph $f(x) = x^3$
   locator point $(2, 1)$

8. parent graph $f(x) = \sqrt{x}$
   vertex $(-7, 0)$

9. parent graph $f(x) = |x|$
   vertex $(5, 0)$

10. parent graph $x = y^2$
    vertex $(9, 0)$

11. Yes.

12. No, on the left part of the graph, for each $x$-value there are two possible $y$-values. This can be seen by drawing a vertical line through the graph. If a vertical line passes through the graph more than once, it is not a function.

13. No, because the equation has “±,” for each value substituted for $x$, there will be two $y$-values produced. A function can have only one output for each input.

14. Yes.
Although students can determine the vertex of a parabola by averaging the $x$-intercepts, they also can use the algebraic method known as completing the square. Completing the square is also used when the equation of a circle is written in general form. The process of completing the square was introduced using algebra tiles in CPM Core Connections Integrated I.

Example 1

The function $f(x) = x^2 + 6x + 3$ is written in standard form. Complete the square to write it in graphing form. Then state the vertex of the parabola and sketch the graph.

The equation of a parabola in graphing form is $f(x) = a(x - h)^2 + k$, where $(h, k)$ is the vertex. The original equation needs to be rewritten in this form.

$f(x) = x^2 + 6x + 3$
$f(x) - 3 = x^2 + 6x$  Move the constant term to the other side.
$f(x) - 3 + \_\_ = x^2 + 6x + \_\_ $ What needs to be added to make a perfect square trinomial?
$f(x) - 3 + 9 = x^2 + 6x + 9$  Add 9 (because it is half of 6, squared).
$f(x) + 6 = (x + 3)^2$  Factor the perfect square trinomial.
$f(x) = (x + 3)^2 - 6$  Move the constant term back to the right side of the equation.

With the equation written in graphing form, it is easy to identify that the vertex is at $(-3, -6)$. The graph of the parabola is shown at right.
Note: From the original equation we know that the parabola has a $y$-intercept of $(0, 3)$. 
Example 2

The equation \( x^2 - 8x + y^2 + 16y = 41 \) is the equation of a circle. Complete the square to determine the coordinates of its center and its radius.

As with the last example, fill in the blanks to create perfect squares. This needs to be done twice: for \( x \) and for \( y \).

\[
x^2 - 8x + y^2 + 16y = 41
x^2 - 8x + 16 + y^2 + 64 = 41 + 16 + 64
(x - 4)^2 + (y + 8)^2 = 121
\]

This is a circle with center \((4, -8)\) and a radius of \(\sqrt{121} = 11\).

Problems

Write each of the following equations in graphing form. Then state the vertex of the parabola.

1. \( f(x) = x^2 - 8x + 18 \)
2. \( f(x) = -x^2 - 2x - 7 \)
3. \( f(x) = 3x^2 - 24x + 42 \)
4. \( f(x) = 2x^2 - 6 \)
5. \( f(x) = \frac{1}{2} x^2 - 3x + \frac{1}{2} \)
6. \( f(x) = x^2 + 18x + 97 \)

Identify the center and radius of each circle.

7. \( (x + 2)^2 + (y + 7)^2 = 25 \)
8. \( 3(x - 9)^2 + 3(y + 1)^2 = 12 \)
9. \( x^2 + 6x + y^2 = 91 \)
10. \( x^2 - 10x + y^2 + 14y = -58 \)
11. \( x^2 + 50x + y^2 - 2y = -602 \)
12. \( x^2 + y^2 - 8x - 16y = 496 \)

Answers

1. \( f(x) = (x - 4)^2 + 2 \), vertex: \((4, 2)\)
2. \( f(x) = -(x + 1)^2 - 6 \), vertex: \((-1, -6)\)
3. \( f(x) = 3(x - 4)^2 - 6 \), vertex: \((4, -6)\)
4. \( f(x) = 2(x - 0)^2 - 6 \), vertex: \((0, -6)\)
5. \( f(x) = \frac{1}{2} (x - 3)^2 - 4 \), vertex: \((3, -4)\)
6. \( f(x) = (x + 9)^2 + 16 \), vertex: \((-9, -16)\)
7. center: \((-2, -7)\), radius: 5
8. center: \((9, -1)\), radius: 2
9. center: \((-3, 0)\), radius: 10
10. center: \((5, -7)\), radius: 4
11. center: \((-25, 1)\), radius: \(\sqrt{24} = 2\sqrt{6}\)
12. center: \((4, 8)\), radius: 24
SAT Prep

1. If \(m\) is an integer, which of the following could not equal \(m^3\)?
   a. 27  b. 0  c. 1  d. 16  e. 64

2. If \(n\) is divided by 7 the remainder is 3. What is the remainder if \(3n\) is divided by 7?
   a. 2  b. 3  c. 4  d. 5  e. 6

3. What is the slope of the line passing through the point \((-3, -1)\) and the origin?
   a. \(-3\)  b. \(-\frac{1}{3}\)  c. 0  d. \(\frac{1}{3}\)  e. 3

4. If \(x = 2y + 3\) and \(3x = 7 – 4y\), what does \(x\) equal?
   a. \(-5\)  b. \(-\frac{1}{3}\)  c. \(\frac{13}{5}\)  d. \(\frac{2}{3}\)  e. 15

5. A bag contains a number of marbles of which 35 are blue, 16 are red and the rest are yellow. If the probability of selecting a yellow marble from the bag at random is \(\frac{1}{4}\), how many yellow marbles are in the bag?
   a. 4  b. 17  c. 19  d. 41  e. 204

6. If \(n > 0\) and \(16x^2 + kx + 25 = (4x + n)^2\) for all values of \(x\), what does \(k – n\) equal?
   a. 0  b. 5  c. 35  d. 40  e. 80

7. A rectangular solid has two faces congruent to the figure labeled I at right and four faces congruent to the figure labeled II at right. What is the volume of the solid?

8. In the figure at right, \(PQ = QR\). What is the \(x\)-coordinate of point \(Q\)?

9. The time \(t\), in hours, needed to produce \(u\) units of a product is given by the formula \(t = ku + c\), where \(k\) and \(c\) are constants. If it takes 430 hours to produce 100 units and 840 hours to produce 200 units, what is the value of \(c\)?

10. In the figure at right, a square is inscribed in a circle. If the sides of the square measure \(\sqrt{3}\) and the area of the circle is \(c\pi\), what is the exact value of \(c\)?
Answers

1. D
2. A
3. D
4. C
5. B
6. C
7. 250
8. 5
9. 20
10. 1.5
SOLVING SYSTEMS OF EQUATIONS  

In this course, one focus is on what a solution means, both algebraically and graphically. By understanding the nature of solutions, students are able to solve equations in new and different ways. This understanding also provides opportunities to solve some challenging application problems. In Chapter 11 this knowledge is extended to solve equations and systems of equations with three variables.

Example 1

The graph of \( y = (x - 5)^2 - 4 \) is shown at right. Use the graph to solve each of the following equations.

a. \( (x - 5)^2 - 4 = 12 \)

b. \( (x - 5)^2 - 4 = -3 \)

c. \( (x - 5)^2 = 4 \)

Solutions:

a. Add the graph of \( y = 12 \) which is a horizontal line to the graph of \( y = (x - 5)^2 - 4 \). These two graphs intersect at two points, \((1, 12)\) and \((9, 12)\). The \( x \)-coordinates of these points are the solutions to the original equation. Notice that there is no “\( y \)” in the equation in part (a). Therefore the solutions to the equation are \( x = 1 \) and \( x = 9 \).

b. Add the graph of \( y = -3 \) to see that the graphs intersect at \((4, -3)\) and \((6, -3)\). Therefore the solutions to the equation are \( x = 4 \) and \( x = 6 \).

c. The equation might look as if it cannot be solved with the graph, but it can. By recognizing the equation is equivalent to \( (x - 5)^2 - 4 = 0 \) (subtract 4 from both sides), then the graph can be used to find where the parabola crosses the line \( y = 0 \) (the \( x \)-axis). The graph tells us the solutions are \( x = 7 \) and \( x = 3 \).
Example 2

Solve the equation $\sqrt{x + 2} = 2x + 1$ using at least two different methods. Explain your methods and the implications of the solution(s).

Solution:

One method is to use algebra to solve this equation. This involves squaring both sides and solving a quadratic equation as shown at right.

A problem arises, however, if the solutions are not checked. When each $x$-value is substituted back into the original equation, only one $x$-value checks: $x = \frac{1}{4}$. This is the only solution.

$\sqrt{\frac{1}{4} + 2} = 2\left(\frac{1}{4}\right) + 1$
$\sqrt{\frac{9}{4}} = \frac{3}{2} + 1$
$\frac{3}{2} = \frac{3}{2} \checkmark$

$\sqrt{-1 + 2} = 2(-1) + 1$
$\sqrt{1} = -2 + 1$
$1 \neq -1$

To see why the other solution does not work use a graph to solve the equation. The graphs of $y = \sqrt{x + 2}$ and $y = 2x + 1$ are shown at right. Notice that the graphs only intersect at one point, namely $x = \frac{1}{4}$. There is only solution to the equation; the other solution is called an extraneous solution.

Remember that a solution makes the equation true. In the original equation, this means that both sides of the equation will be equal for certain values of $x$. Using the graphs, the solution is the $x$-value that has the same $y$-value for both functions, or the $x$-coordinate(s) of the point(s) at which the graphs intersect.
Example 3

Algebraically solve each system of equations below. For each system, explain what the solution (or lack thereof) tells about the graph of the system.

a. \( y = -\frac{2}{5} x + 3 \)
   \( y = \frac{3}{5} x - 2 \)

b. \( y = -2(x - 2)^2 + 35 \)
   \( y = -2x + 15 \)

c. \( y = \frac{1}{6} x^2 - \frac{17}{3} \)
   \( x^2 + y^2 = 25 \)

Solutions:

a. The two equations are written in “\( y = \)” form, which makes substitution the most efficient method for solving. Set the expressions on the right side of each equation equal to each other and solve for \( x \).

\[ y = -\frac{2}{5} x + 3 \]
\[ y = \frac{3}{5} x - 2 \]

Then substitute this value for \( x \) back into either one of the original equations to determine the value of \( y \). Finally, check that the solution satisfies both equations.

\[ y = -\frac{2}{5} x + 3, \quad x = 5 \]
\[ y = -\frac{2}{5} (5) + 3 \]
\[ y = -2 + 3 \]
\[ y = 1 \]

Solution to check: (5, 1)

The solution is the point (5, 1), which means that the graphs of these two equations intersect at one point, the point (5, 1).

b. The two equations are written in “\( y = \)” form, which means that substitution can again be used. This is shown at right.

\[ y = -2(x - 2)^2 + 35 \]
\[ y = -2x + 15 \]

Now substitute each \( x \)-value into either equation to calculate the corresponding \( y \)-value.

\[ x = 6, \quad y = -2x + 15 \]
\[ y = -2(6) + 15 \]
\[ y = -12 + 15 \]
\[ y = 3 \]
Solution: (6, 3)

\[ x = -1, \quad y = -2x + 15 \]
\[ y = -2(-1) + 15 \]
\[ y = 2 + 15 \]
\[ y = 17 \]
Solution: (-1, 17)

Solution continues on next page →
Lastly, check each point in both equations to make sure there are not any extraneous solutions.

\[
\begin{align*}
(6, 3): & \quad y = -2(x - 2)^2 + 35 \\
& \quad 3 = -2(6 - 2)^2 + 35 \\
& \quad 3 = -2(16) + 35 \checkmark
\end{align*}
\]

\[
\begin{align*}
(6, 3): & \quad y = -2x + 15 \\
& \quad 3 = -2(6) + 15 \checkmark
\end{align*}
\]

\[
\begin{align*}
(-1, 17): & \quad y = -2(x - 2)^2 + 35 \\
& \quad 17 = -2(-1 - 2)^2 + 35 \\
& \quad 17 = -2(9) + 35 \checkmark
\end{align*}
\]

\[
\begin{align*}
(-1, 17): & \quad y = -2x + 15 \\
& \quad 17 = -2(-1) + 15 \checkmark
\end{align*}
\]

In solving these two equations with two unknowns, two solutions were found, both of which checked in the original equations. This means that the graphs of the equations, a parabola and a line, intersect in two distinct points.

c. This system requires substitution to solve. One option is to replace \( y \) in the second equation with the right hand side of the first equation, but that would require solving an equation of degree four (an exponent of 4). Instead, rewrite the first equation without fractions in order to simplify. This is done by multiplying both sides of the equation by 6, as shown at right.

\[
\begin{align*}
y & = \frac{1}{6} x^2 - \frac{17}{3} \\
6y & = x^2 - 34
\end{align*}
\]

Now, instead, replace the \( x^2 \) in the second equation with \( 6y + 34 \). Then solve.

\[
\begin{align*}
x^2 + y^2 & = 25 \\
(6y + 34) + y^2 & = 25 \\
y^2 + 6y + 34 & = 25 \\
y^2 + 6y + 9 & = 0 \\
(y + 3)(y + 3) & = 0 \\
y & = -3
\end{align*}
\]

Next, substitute this value back into either equation to find the corresponding \( x \)-value.

\[
\begin{align*}
y = -3: & \quad 6y + 34 = x^2 \\
& \quad 6(-3) + 34 = x^2 \\
& \quad 16 = x^2 \\
& \quad x = \pm 4
\end{align*}
\]

\[
\begin{align*}
(4, -3): y & = \frac{1}{6} x^2 - \frac{34}{6}, \quad -3 = \frac{1}{6} (4)^2 - \frac{34}{6} - \frac{16}{6} - \frac{34}{6} = -\frac{18}{6} \checkmark \\
(4, -3): x^2 + y^2 & = 25, \quad (4)^2 + (-3)^2 = 16 + 9 = 25 \checkmark \\
(-4, -3): y & = \frac{1}{6} x^2 - \frac{34}{6}, \quad -3 = \frac{1}{6} (-4)^2 - \frac{34}{6} - \frac{16}{6} - \frac{34}{6} = -\frac{18}{6} \checkmark \\
(-4, -3): x^2 + y^2 & = 25, \quad (-4)^2 + (-3)^2 = 16 + 9 = 25 \checkmark 
\end{align*}
\]

Since there are two points that make this system true, the graphs of this parabola and this circle intersect in only two points, \((4, -3)\) and \((-4, -3)\).
Example 4

Jo has small containers of lemonade and lime soda. She once mixed one container of lemonade with three containers of lime soda to make 17 ounces of a tasty drink. Another time, she combined five containers of lemonade with six containers of lime soda to produce 58 ounces of another splendid beverage. Given this information, how many ounces are in each small container of lemonade and lime soda?

Solution:

Solve this problem by using a system of equations. To start, let $x =$ the number of ounces of lemonade in each small container, and let $y =$ the number of ounces of lime soda in each of its small containers. Write an equation that describes each mixture Jo created.

The first mixture used one container ($1x$ ounces) of lemonade and three containers ($3y$ ounces) of lime soda for a total of 17 ounces. This can be represented as $1x + 3y = 17$.

The second mixture used five containers ($5x$ ounces) of lemonade and six containers ($6y$ ounces) of lime soda for a total of 58 ounces. This can be represented by the equation $5x + 6y = 58$.

Solve this system to determine the values of $x$ and $y$.

\[
\begin{align*}
x + 3y &= 17 \\
5x + 6y &= 58
\end{align*}
\]

\[
\begin{align*}
x + 3y &= 17 \\
\times(-5) &
\rightarrow & -5x - 15y &= -85
\end{align*}
\]

If $y = 3$, then:

\[
\begin{align*}
x + 3(3) &= 17 \\
x + 9 &= 17
\end{align*}
\]

\[
x = 8
\]

\[
y = 3
\]

(Note: Check these values!)

Therefore each container of lemonade has 8 ounces, and each container of lime soda has 3 ounces.
Problems

Solve each of the following systems of equations. Then explain what the solution(s) tells you about the graphs of the equations. Be sure to check your work.

1. \[ x + y = 11 \]
   \[ 3x - y = 5 \]

2. \[ 2x - 3y = -19 \]
   \[ -5x + 2y = 20 \]

3. \[ 15x + 10y = 21 \]
   \[ 6x + 4y = 11 \]

4. \[ 8x + 2y = 18 \]
   \[ -6x + y = 14 \]

5. \[ 12x - 16y = 24 \]
   \[ y = \frac{3}{4}x - \frac{3}{2} \]

6. \[ \frac{1}{2}x - 7y = -15 \]
   \[ 3x - 4y = 24 \]

The graph of \[ y = \frac{1}{2} (x - 4)^2 + 3 \] is shown at right. Use the graph to solve each of the following equations. Explain how you get your answers.

7. \[ \frac{1}{2} (x - 4)^2 + 3 = 3 \]
8. \[ \frac{1}{2} (x - 4)^2 + 3 = 5 \]
9. \[ \frac{1}{2} (x - 4)^2 + 3 = 1 \]
10. \[ \frac{1}{2} (x - 4)^2 = 8 \]

Solve each equation below.

11. \[ 3(x - 4)^2 + 6 = 33 \]
12. \[ \frac{x}{4} + \frac{x}{5} = \frac{9x-4}{20} \]
13. \[ 3 + \left( \frac{10-3x}{2} \right) = 5 \]
14. \[ -3\sqrt{2x-1} + 4 = 10 \]

Solve each of the following systems of equations algebraically. What does the solution tell you about the graph of the system?

15. \[ y = -\frac{2}{3} x + 7 \]
   \[ 4x + 6y = 42 \]

16. \[ y = (x + 1)^2 + 3 \]
   \[ y = 2x + 4 \]

17. \[ y = -3(x - 4)^2 - 2 \]
   \[ y = -\frac{4}{7} x + 4 \]

18. \[ x + y = 0 \]
   \[ y = (x - 4)^2 - 6 \]

19. Adult tickets for the Mr. Moose’s Fantasy Show on Ice are $6.50 while a child’s ticket is only $2.50. At Tuesday night’s performance, 435 people were in attendance. The box office brought in $1667.50 for that evening. How many of each type of ticket were sold?

20. The next math test will contain 50 questions. Some will be worth three points while the rest will be worth six points. If the test is worth 195 points, how many three-point questions are there, and how many six-point questions are there?
21. Dudley’s water balloons follow the path described by the equation \( y = -\frac{8}{125} (x - 10)^2 + \frac{72}{5} \). Suppose Dudley’s nemesis, in a mad dash to save his base from total water balloon bombardment, ran to the wall and set up his launcher at its base. Dudley’s nemesis launches his balloons to follow the path \( y = -x \left( x - \frac{100}{25} \right) \) in an effort to knock Dudley’s water bombs out of the air. Is Dudley’s nemesis successful? Explain.

**Answers**

1. \((4, 7)\)  
2. \((-2, 5)\)  
3. no solution  

4. \((-\frac{1}{2}, 11)\)  
5. all real numbers  
6. \((12, 3)\)  

7. \(x = 4\)  
The horizontal line \( y = 3 \) crosses the parabola at one point, at the vertex.

8. \(x = 2 \text{ or } x = 6\)  
The horizontal line \( y = 5 \) crosses the parabola at two points.

9. no real solution  
The horizontal line \( y = 1 \) does not cross the parabola.  
(Solving algebraically yields \( x = 4 \pm 2i \).)

10. \(x = 0 \text{ or } x = 8\)  
Add three to both sides to rewrite the equation as \( \frac{1}{2} (x - 4)^2 + 3 = 11 \). The horizontal line \( y = 11 \) crosses the parabola at two points.

11. \(x = 7 \text{ or } x = 1\)  

12. no solution  

13. \(x = 2\)  

14. no solution  

15. all real numbers  
When graphed, these equations give the same line.

16. \((0, 4)\)  
The parabola and the line intersect at one point.

17. no solution  
This parabola and this line do not intersect.

18. \((2, -2) \text{ and } (5, -5)\)  
The line and the parabola intersect twice.

19. 145 adult tickets were sold, while 290 child tickets were sold.

20. There are 35 three-point questions and 15 six-point questions on the test.

21. By graphing we see that the nemesis’ balloon when launched at the base of the wall (the y-axis), hits the path of the Dudley’s water balloon. Therefore, if timed correctly, the nemesis is successful.
Once the meaning of a solution is understood, it can be applied to understanding solutions of inequalities and systems of inequalities. Inequalities typically have infinitely many solutions, and students learn ways to represent such solutions. For additional information see the Math Notes boxes in Lessons 3.2.2 and 3.2.4.

Example 1

Solve each equation or inequality below. Explain what the solution for each one represents. Then explain how the equation and inequalities are related to each other.

\[ x^2 - 4x - 5 = 0 \quad x^2 - 4x - 5 < 0 \quad y \geq x^2 - 4x - 5 \]

Solution:

There are many ways to solve the equation, including graphing, factoring and using the Zero Product Property, or using the Quadratic Formula. Factoring and use the Zero Product Property to solve is shown below.

\[ x^2 - 4x - 5 = 0 \]
\[ (x - 5)(x + 1) = 0 \]
\[ x = 5, \ x = -1 \]

Check:

If \( x = 5 \): \( (5)^2 - 4(5) - 5 = 25 - 20 - 5 = 0 \) \( \checkmark \)

If \( x = -1 \): \( (-1)^2 - 4(-1) - 5 = 1 + 4 - 5 = 0 \) \( \checkmark \)

The second quadratic is an inequality. To solve this inequality, utilize a number line to emphasize what the solution represents.

Start by solving the related quadratic equation (which was done above). \( x^2 - 4x - 5 = 0 \)

The solutions to the equation are \( x = 5 \) and \( x = -1 \).

By placing these two points on a number line, they act as boundary points, dividing the number line into three regions. Since the original inequality is strictly “less than,” use open circles for the boundary points.

Choose any number in each of the regions to see if the number will make the original inequality true or false. Solutions will make the inequality true. Note: You only need to check one point in each region.

Solution continues on next page →
Choosing a point in each region and substituting into the inequality gives:

<table>
<thead>
<tr>
<th></th>
<th>$x = -2$</th>
<th>$x = 0$</th>
<th>$x = 7$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$(-2)^2 - 4(-2) - 5 &lt; 0$</td>
<td>$(0)^2 - 4(0) - 5 &lt; 0$</td>
<td>$(7)^2 - 4(7) - 5 &lt; 0$</td>
</tr>
<tr>
<td></td>
<td>$4 + 8 - 5 &lt; 0$</td>
<td>$0 - 0 - 5 &lt; 0$</td>
<td>$49 - 28 - 5 &lt; 0$</td>
</tr>
<tr>
<td></td>
<td>$7 &lt; 0$</td>
<td>$-5 &lt; 0$</td>
<td>$16 &lt; 0$</td>
</tr>
<tr>
<td></td>
<td>false</td>
<td>true</td>
<td>false</td>
</tr>
</tbody>
</table>

Highlight the region on the number line that makes the inequality true, as shown at right. This solution can also be represented algebraically as \(-1 < x < 5\).

The last inequality in the example has a $y$. Having both $x$ and $y$ means an $xy$-coordinate graph needs to be used to show the solutions. Graph the parabola using a solid line because the original inequality is “greater then or equal to”. The graph of the parabola at right divides the plane into two regions: the part within the “bowl” of the parabola—the interior—and the region outside the parabola. The points on the parabola represent where $y = x^2 - 4x - 5$.

Test a point from one of the regions to check whether it will make the inequality true or false. As before, we are looking for the “true” region.

The point $(0, 0)$ is an easy point to use.

<table>
<thead>
<tr>
<th></th>
<th>$0 \geq (0)^2 - 4(0) - 5$</th>
<th>$0 \geq 0 - 0 - 5$</th>
<th>$0 \geq -5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>True! Therefore the region containing the point $(0, 0)$ is the solution. This means any point chosen in this region, the “bowl” of the parabola, will make the inequality true.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

To illustrate that this region is the solution, we shade this region of the graph.

To see how these equations and inequalities are related, examine the graph of the parabola. Where are the zeros? Where are the $y$-values of the parabola negative? The zeros are the $x$-intercepts on the graph, or $x = -1$ and $x = 5$, which you determined by solving the equation. The graph is negative when it dips below the $x$-axis, and this happens when $x$ is between $-1$ and $5$. Solving the first inequality answered this question as well. Therefore, the graph could have answered the first two parts quickly.
Example 2

Han and Lea have been building jet roamers and pod racers. Each jet roamer requires one jet pack and three crystallic fuel tanks, while each pod racer requires two jet packs and four crystallic fuel tanks. Han and Lea’s suppliers can only produce 100 jet packs and 270 fuel tanks each week, and due to manufacturing conditions, Han and Lea can build no more than 30 pod racers each week. Each jet roamer makes a profit of 1 tig (their form of currency) while each pod racer makes a profit of 4 tigs.

a. If Han and Lea receive an order for 12 jet roamers and 22 pod racers, how many of each part will they need to fill this order? If they can fill this order, how many tigs will they make?

b. Write a list of constraints, an inequality for each constraint, and sketch a graph showing all inequalities with the points of intersection labeled. How many jet roamers and pod racers should Han and Lea build to maximize their profits?

Solution:

This problem is an example of a linear programming problem, and although the name might conjure up images of computer programming, these problems are not done on a computer. This problem can be solved by creating a system of inequalities that, when graphed, creates a feasibility region. This region contains the solution for the number of jet roamers and pod racers Han and Lea should make to maximize their profit.

a. Jet packs: \(1(12) + 2(22) = 12 + 44 = 56\)
   Fuel tanks: \(3(12) + 4(22) = 36 + 88 = 124\)
   Each result is within the constraints, so it is possible for Han and Lea to fill this order.
   If they do, they will make \(1(12) + 4(22) = 12 + 88 = 100\) tigs.

b. Begin by defining the variables. Let \(x\) = the number of jet roamers Han and Lea will make, and \(y\) = the number of pod racers.

   \(x \geq 0,\) and \(y \geq 0\) because a negative number of items cannot be produced.
   A jet roamer requires one jet pack while a pod racer requires two. There are only 100 jet packs available each week, so \(x + 2y \leq 100.\)
   Each jet roamer requires three crystallic fuel tanks and each pod racer requires four. This translates into the inequality \(3x + 4y \leq 270\) since only 270 fuel tanks are available each week. Lastly, since Han and Lea cannot make more than 30 pod racers, we can write \(y \leq 30.\)

   These inequalities are all shown on the graph at right. The region common to all constraints is shaded. This is the feasibility region because choosing a point in this shaded area gives you a combination of jet roamers and pod racers that Han and Lea can produce under the given restraints.

Solution continues on next page →
Solution continued from previous page.

The equation used to calculate the profit is \( P = 1x + 4y \).

To maximize profits, test all the vertices of the feasibility region in the profit equation. These points are (0, 0), (0, 30), (40, 30), (70, 15), and (90, 0).

(0, 0): \( P = 1(0) + 4(0) = 0 \)

(0, 30): \( P = 1(0) + 4(30) = 120 \)

(40, 30): \( P = 1(40) + 4(30) = 160 \)

(70, 15): \( P = 1(70) + 4(15) = 130 \)

(90, 0): \( P = 1(90) + 4(0) = 90 \)

The greatest profit is 160 tigs when Han and Lea build 40 jet roamers and 30 pod racers.

Problems

Graph the following system of inequalities.

1. \( y < \frac{1}{2} x + 6 \)
   \( y > -\frac{1}{2} x + 6 \)
   \( x < 12 \)

2. \( x + y < 10 \)
   \( x + y > 4 \)
   \( y < 2x \)
   \( y > 0 \)

3. \( y \leq 3x + 4 \)
   \( y \geq -\frac{1}{4} x + 8 \)
   \( y \geq -\frac{1}{3} x + 4 \)
   \( y \geq 5x - 6 \)

4. \( 3x + 4y < 12 \)
   \( y > (x + 1)^2 - 4 \)

5. \( y < -\frac{3}{4} (x - 1)^2 + 6 \)
   \( y > x - 7 \)

6. \( y < (x + 2)^3 \)
   \( y > x^2 + 3x \)

For each of the following problems, write a system of inequalities that when graphed will produce the shaded region.

7. 

8. 

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9. Ramon and Thea are considering opening their own business. They wish to make and sell alien dolls they call *Hauteans* and *Zotions*. Each *Hautean* sells for $1.00 while each *Zotion* sells for $1.50. They can make up to 20 *Hauteans* and 40 *Zotions*, but no more than 50 dolls total. When Ramon and Thea go to city hall to get a business license, they find there are a few more restrictions on their production. The number of *Zotions* (the more expensive item) can be no more than three times the number of *Hauteans* (the cheaper item). How many of each doll should Ramon and Thea make to maximize their profit? What will the profit be?

10. Sam and Emma are plant managers for the *Sticky Chewy Candy Company* that specializes in delectable gourmet candies. Their two most popular candies are *Chocolate Chews* and *Peanut and Jelly Jimmies*. Each batch of *Chocolate Chews* takes 1 teaspoon of vanilla while each batch of the *Peanut and Jelly Jimmies* uses two teaspoons of vanilla. They have at most 20 teaspoons of vanilla on hand as they use only the freshest of ingredients. The *Chocolate Chews* use two teaspoons of baking soda while the *Peanut and Jelly Jimmies* use three teaspoons of baking soda. They only have 36 teaspoons of baking soda on hand. Because of production restrictions, they can make no more than 15 batches of *Chocolate Chews* and no more than 7 batches of *Peanut and Jelly Jimmies*. Sam and Emma have been given the task of determining how many batches of each candy they should produce if they make $3.00 profit for each batch of *Chocolate Chews* and $2.00 for each batch of *Peanut and Jelly Jimmies*. Help them out by writing the inequalities described here, graphing the feasibility region, and determining their maximum profit.

### Answers

1. ![Graph 1](image)

2. ![Graph 2](image)

3. ![Graph 3](image)

4. ![Graph 4](image)

5. ![Graph 5](image)

6. ![Graph 6](image)
7. \[ y \leq \frac{1}{3} x + 4 \]
   \[ y \leq -x + 8 \]
   \[ y \geq -\frac{1}{2} x + 4 \]

8. \[ y \geq (x - 6)^2 - 5 \]
   \[ y \leq 0 \]

9. The graph of the feasibility region is shown at right. The inequalities are \( x \geq 0, y \geq 0, x + y \leq 50, x \leq 20, y \leq 40, \) and \( y \leq 3x, \) where \( x = \) number of Hauteans and \( y = \) number of Zotions. The profit is given by \( P = x + 1.5y. \) Maximum profit seems to occur at point A \((12.5, 37.5), \) but there is a problem with this point. Ramon and Thea cannot make a half of a doll (or at least that does not seem possible). Try these nearby points: \((12, 37), (12, 38), (13, 37), \) and \((13, 38).\) The point that gives maximum profit and is still in the feasibility region is \((13, 37).\) They should make 13 Hautean and 37 Zotion dolls for a profit of $68.50.

10. The graph of the feasibility region is shown at right. The inequalities are \( x \geq 0, y \geq 0, y \leq 7, x \leq 15, x + 2y \leq 20, \) and \( 2x + 3y \leq 36, \) where \( x = \) number of Chocolate Chews and \( y = \) number of Peanut and Jelly Jimmies. The profit is given by \( P = 3x + 2y. \) The point that seems to give the maximum profit is \((15, 2.5)\) but this only works if half batches can be made. Instead, choose the point \((15, 2)\) which means Sam and Emma should make 15 batches of Chocolate Chews and 2 batches of Peanut and Jelly Jimmies. Their profit will be $49.00.
1. If \( \frac{x+4}{12} = \frac{4}{3} \), then \( x \) equals:
   a. 3     b. 6     c. 8     d. 10     e. 12

2. What is the least of three consecutive integers whose sum is 21?
   a. 5     b. 6     c. 7     d. 8     e. 9

3. Juanita has stocks, bonds, and t-bills for investments. The number of t-bills she has is one more than the number of stocks, and the number of bonds is three times the number of t-bills. Which of the following could be the total number of investments?
   a. 16     b. 17     c. 18     d. 19     e. 20

4. Through how many degrees would the minute hand of a clock turn from 5:20 p.m. to 5:35 p.m. the same day?
   a. 15°     b. 30°     c. 45°     d. 60°     e. 90°

5. The length of a rectangle is six times its width. If the perimeter of the rectangle is 56, what is the width of the rectangle?
   a. 4     b. 7     c. 8.5     d. 18     e. 24

6. If \( m > 1 \) and \( m^n m^5 = m^{15} \), then what does \( n \) equal?

7. In the triangle at right, what is the value of \( a + b + c + d \)?

8. If \( x \) and \( y \) are positive integers, \( x + y < 12 \), and \( x > 4 \), what is the greatest possible value for \( x - y \)?

9. If \( (2x^2 + 5x + 3)(2x + 4) = ax^3 + bx^2 + cx + d \) for all values of \( x \) what does \( c \) equal?

10. Four lines intersect in one point creating eight congruent adjacent angles. What is the measure of one of these angles?
Answers

1. E
2. B
3. D
4. E
5. A
6. 10
7. 280°
8. 9
9. 26
10. 45°
In many situations, data from samples is required to estimate characteristics (parameters) of large populations. The reason for using a sample may be that the population is too large to permit data to be gathered from every subject. For example, a national opinion poll about a political issue would require data from samples. Another reason is that sometimes the act of collecting the data ruins the item being studied, as when crash-testing automobiles.

Understandably, to get reliable results, one must minimize the sources of bias. Random selection of subjects is used to reduce bias in statistical studies. Problem 4-3 explains types of bias found within survey questions.

For additional information, see the Math Notes box in Lesson 4.1.1.

Examples

Determine whether each situation describes a sample or census. If the situation describes a sample, discuss the sampling technique and potential sources of bias.

a. The Chief of Police calls in his five newest officers to get their opinion on new requirements for department-wide promotions.
   Answer: The Chief is using a sample, which is not likely representative of the population affected by the new requirements. The newest officers would likely want to please the Chief and would echo his support or dislike of the new requirements.

b. A manager compares the annual sales totals for 12 store locations to rank their performance.
   Answer: This is a census. The manager has information from all 12 of the stores he is interested in comparing.

c. Shoppers are invited to fill out a questionnaire about their shopping experience at the cash register. The store manager uses the results in a report to corporate headquarters to demonstrate the level of customer satisfaction at her store.
   Answer: The manager is using a voluntary response sample (or “convenience sample”), which would not represent all customers very well. A sample like this tends to over-represent those with the strongest opinions.

d. The student council is trying to determine how much space they will need for the prom. Carmine, the Junior Class President, walks around during lunchtime at her school with a clipboard, asking, “Are you planning on going to the best prom ever?”
   Answer: Carmine is not taking a random sample. She is more likely to talk to students she knows, and they, as a group, are likely to be biased about the prom. There may also be a strong desire to please the interviewer in this situation. The wording of the question itself is biased, using “best prom ever” to describe the dance.
Problems

Determine whether each situation describes a sample or census. If it is a sample, discuss the sampling technique and potential sources of bias.

1. A math teacher wants to determine whether playing classical music during testing benefits high school math students. He plays classical music to half of his classes while they test, and compares their scores to the other half of his classes, who tested without the music.

2. A sports news reporter wants to know the win-loss record of the local high school girls lacrosse team, so he looks at the league website and sees that they have 8 wins and 5 losses.

3. A battery manufacturer wants to monitor the durability and peak voltage of its products. A machine is programmed to select every 1000th battery from the production line and subject it to a series of tests.

4. Scott is doing a report on the risk factors for cancer. He asks all of the members of his PE class: “Do you have a family history of cancer?” and “Do you eat five servings of fruits and vegetables each day?”

Answers

1. Because the math teacher is using his data to make a statement about all high school math students (the population), his classes represent a sample of those students. His students would have many things in common, like where they live and how they were taught, which other high school math students do not have. They would be a poor representation of the population. The teacher may also be biased in grading the exams to obtain the results he favors.

2. This is a census. The reporter has access to every game outcome for the team he is researching and is not using the information to draw conclusions about other teams or the sport of lacrosse in general.

3. The batteries tested represent a sample of those being made. They might be a reasonable representation of the population of batteries. However, if the 1000th battery is usually made at the same time every day, or corresponds to some other variable like periodic machine maintenance, it might be subject to bias.

4. Scott’s PE class represents a sample of people or students. However this is a convenience sample and cannot be considered a reliable representation of the population of people or students. The question order may cause bias in the results as people being reminded of a disease like cancer would likely want to appear concerned enough to be eating a healthy diet with lots of fruits and vegetables. Members of a PE class may be more inclined to be thinking about health issues such as diet.
Two ways to gather information for statistical studies are observational studies and experiments. Observational studies seek to collect data about subjects without changing the subjects, while experiments impose treatments (variables of interest) upon the subjects. Observational studies are generally easier to perform but cannot demonstrate cause-and-effect relationships because it is not possible to separate the many effects of lurking variables (those outside of what is being observed directly).

For additional information, see the Math Notes box in Lesson 4.2.1.

Examples

For each question below:

- Does a census or a sample make more sense in this situation?
- Should an observational study or an experiment be carried out to answer the question?
- If an observational study is suitable and asks for an association between variables, discuss the effects of possible lurking variables.
- If an experiment would be suitable, outline an experimental design.

a. Is there a relationship between drinking green tea and longevity?
   Answer: Because it would be impossible to find every green tea consumer, this study would need to rely on sampling. Green tea would not be a difficult treatment to administer in an experiment, but because life-spans are being measured, an experiment would be impractical. An observational study may determine a relationship between these two variables but would be unable to establish cause and effect because of the many possible lurking variables. Perhaps people who choose to drink green tea also choose to be more health conscious in other ways such as smoke less than the general population. They may also tend to be of higher income and have better access to health care. Hence their longevity causes could really be smoking less and having better health care, not from drinking green tea.

b. What is the mean verbal SAT score in the state of Colorado for a given year?
   Answer: This information is easily found by a census.

c. What percentage of razorback clams show signs of stress from acidic ocean water?
   Answer: One could not find every clam for study and the stress testing of the clams may harm them, so a sample would be appropriate. It would be important to gather razorback clams from a variety of locations as environmental conditions may vary considerably.

d. Would brighter classroom lighting increase exam scores?
   Answer: This would require using a sample of students in a controlled experiment. One could take student volunteers and give them the same instruction on a topic, then randomly assign them to classrooms with different lighting levels for the same exam. The mean test scores from each group could be compared.
Problems

For each question:

• Does a census or a sample make more sense in this situation?
• Should an observational study or an experiment be carried out to answer the question?
• If an observational study is suitable and asks for an association between variables, discuss the effects of possible lurking variables.
• If an experiment would be suitable, outline an experimental design.

1. Is there a relationship between the temperature of water and its electrical resistance?

2. What is the mean distance students commute to school in your county?

3. What percentage of car crashes causing injury occurred when at least one driver was impaired by drugs or alcohol?

4. Is there an association between baldness and being colorblind?

Answers

1. One would have to work with samples of water in this situation. This question would best be answered as an experiment. Begin with water from the same source so its other variables like impurities remain constant and divide it into several like containers. Randomly assign different temperatures to the water containers and measure the electrical resistance between two wires inserted an equal distance apart in each container, then compare results.

2. For most counties, surveying every student about commute distances is not possible, so a random sample of students could be used to represent the population. This is clearly an observational study because the question seeks to determine how students commute without imposing any changes upon them.

3. This kind of information is routinely collected by law enforcement, so census data is likely available.

4. This study would require samples from the population. An observational study is the only reasonable alternative because an experiment would require imposing baldness or colorblindness upon subjects. A lurking variable here is gender. Men are more often colorblind and more often bald than women, so if the study included both genders then it would likely determine an association.
RELATIVE FREQUENCY HISTOGRAMS

When the frequencies (numbers or values) listed in a frequency table are divided by the total number of data points, the frequencies become ratios, and the frequency table becomes a relative frequency table. Histograms display the information from frequency tables and relative frequency histograms display information from relative frequency tables.

Example 1

Forty applicants interviewed for an administrative assistant position. Each applicant was given a typing test and earned a score in words per minute (wpm). Their results are shown in the table below.

<table>
<thead>
<tr>
<th>Typing Score (wpm)</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>30 to 39</td>
<td>1</td>
</tr>
<tr>
<td>40 to 49</td>
<td>6</td>
</tr>
<tr>
<td>50 to 59</td>
<td>11</td>
</tr>
<tr>
<td>60 to 69</td>
<td>16</td>
</tr>
<tr>
<td>70 to 79</td>
<td>5</td>
</tr>
<tr>
<td>80 to 89</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>40</td>
</tr>
</tbody>
</table>

a. Create a frequency table, histogram, relative frequency table, and relative frequency histogram.
b. What percentage of applicants typed 40 to 49 wpm?
c. What percentage of applicants typed at least 60 wpm?
d. If 50 wpm is the minimum job qualification, what percentage of applicants did not qualify?

Answers:

a. |
<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Typing Score (wpm)</td>
</tr>
<tr>
<td>---------------------</td>
</tr>
<tr>
<td>30 to 39</td>
</tr>
<tr>
<td>40 to 49</td>
</tr>
<tr>
<td>50 to 59</td>
</tr>
<tr>
<td>60 to 69</td>
</tr>
<tr>
<td>70 to 79</td>
</tr>
<tr>
<td>80 to 89</td>
</tr>
<tr>
<td>Total</td>
</tr>
</tbody>
</table>

b. 15% (from the relative frequency table or histogram)
c. 0.40 + 0.125 + 0.025 = 0.55 = 55%
d. 0.025 + 0.15 = 0.175 = 17.5%
Example 2

A random sample of 35 cars was taken from a car rental agency, and the highway mileage of each car was measured in miles per gallon (mpg).

<table>
<thead>
<tr>
<th>Highway Mileage (mpg)</th>
<th>Frequency</th>
<th>Relative Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>24 up to 27</td>
<td>2</td>
<td>$\frac{2}{35} \approx 0.0571$</td>
</tr>
<tr>
<td>27 up to 30</td>
<td>3</td>
<td>$\frac{3}{35} \approx 0.0857$</td>
</tr>
<tr>
<td>30 up to 33</td>
<td>1</td>
<td>$\frac{1}{35} \approx 0.0286$</td>
</tr>
<tr>
<td>33 up to 36</td>
<td>5</td>
<td>$\frac{5}{35} \approx 0.1429$</td>
</tr>
<tr>
<td>36 up to 39</td>
<td>13</td>
<td>$\frac{13}{35} \approx 0.3714$</td>
</tr>
<tr>
<td>39 up to 42</td>
<td>9</td>
<td>$\frac{9}{35} \approx 0.2571$</td>
</tr>
<tr>
<td>42 up to 45</td>
<td>2</td>
<td>$\frac{2}{35} \approx 0.0571$</td>
</tr>
<tr>
<td>total</td>
<td>35</td>
<td>1.00</td>
</tr>
</tbody>
</table>

c. $0.0571 + 0.0857 = 0.1428 = 14.28\%$

d. Counting the cars from either end of the mileage distribution, would show that the 18th car would be in the middle. The 18th car would also be in the 36 up to 39 mpg group.

e. $0.2571 + 0.0571 = 0.3142 = 31.42\%$
There are many situations when a relative frequency histogram will closely resemble a bell-shaped curve. Sometimes an equation called the normal probability density function is fitted to the bell-shaped relative frequency histogram. This function is commonly called the normal distribution and can be used to determine percentages and probabilities associated with what is being measured. The normal probability density function is also used to calculate percentiles. Percentiles identify the measurement below which a specified percentage of data points are found. For example, if the 90\textsuperscript{th} percentile for men’s heights were 72 inches, 90\% of men would be shorter than 72 inches.

For additional information, see the Math Notes box in Lesson 4.3.3.

Example 1

Half liter soda bottles are filled with cola at a bottling plant. The equipment used to fill the bottles is not capable of putting exactly 500 ml in each bottle, so it is adjusted such that the amount of cola placed in the bottles follows a normal distribution with a mean of 502 ml and a standard deviation of 1.8 ml.

a. What proportion of bottles will have more than 502 ml of cola in them?

b. What proportion of bottles will have between 500 and 502 ml in them?

c. What proportion of bottles are under-filled (hold less than 500 ml)?

d. What percentile corresponds to 503 ml?

e. What percentile corresponds to 505 ml?

Answers:

a. \texttt{normalcdf}(502, 10^{99}, 502, 1.8) = 0.500
   One may recognize that 502 ml is the mean and half (0.500) of the probability in a normal distribution is always above and below the mean.

b. \texttt{normalcdf}(500, 502, 502, 1.8) ≈ 0.367

c. \texttt{normalcdf}(–10^{99}, 500, 502, 1.8) ≈ 0.133

d. \texttt{normalcdf}(–10^{99}, 503, 502, 1.8) ≈ 0.711 or the 71\textsuperscript{st} percentile

e. \texttt{normalcdf}(–10^{99}, 505, 502, 1.8) ≈ 0.952 or the 95\textsuperscript{th} percentile
Example 2

Commercial cherry trees produce fruit according to a normal distribution with an average of 90 kg of cherries per tree and a standard deviation of 11 kg per tree.

a. What is the probability that a tree produces at least 95 kg of cherries?

b. What is the probability that a tree yields between 85 and 105 kg of cherries?

c. What percentile would a harvest of 80 kg be for a single tree?

Answers:

a. normalcdf(95, 10^{99}, 90, 11) \approx 0.325

b. normalcdf(85, 105, 90, 11) \approx 0.589

c. normalcdf(-10^{99}, 80, 90, 11) \approx 0.182 or the 18^{th} percentile

Problems

1. Concrete is loaded into trucks for a large construction project according to a normal distribution. The mean amount loaded is 230 cubic feet with a standard deviation of 7 cubic feet. Concrete trucks are rated by the number of cubic yards they can carry, and 9 cubic yards is a common maximum load. Hence the expression “the whole nine yards.”

   a. What proportion of the trucks will be loaded with between 220 and 240 cubic feet?

   b. What proportion of trucks are loaded with more concrete than the 9 cubic yard maximum. (3 feet = 1 yard)

   c. 235 cubic feet is what percentile load?

   d. If a particular job requires at least 220 cubic yards of concrete, what is the probability there will not be enough concrete in a single truck?

2. Assume the annual snowfall in the village of Thalhammer follows a normal distribution with a mean of 185 cm and standard deviation of 36 cm.

   a. Less than 100 cm of snow in a year is considered drought conditions in Thalhammer. What is the chance that a particular year will have drought conditions?

   b. The Rivera Valley lies below Thalhammer. Residents of Rivera Valley are placed on a flood-warning from snowmelt in the spring if annual snowfall in Thalhammer is greater than 250 cm. What is the probability the Rivera Valley residents will be placed on a flood-warning this spring?

   c. What percentile corresponds to 160 cm of annual snowfall in Thalhammer?
Answers

1. a. \( \text{normalcdf}(220, 240, 230, 7) \approx 0.847 \)
   b. \((3 \text{ ft})(3 \text{ ft})(3 \text{ ft}) = 27 \text{ ft}^3\), so 27 cubic feet per cubic yard. \((9)(27) = 243 \text{ cubic feet}\)
   \( \text{normalcdf}(243, 10^{99}, 230, 7) \approx 0.032 \)
   c. \( \text{normalcdf}(-10^{99}, 235, 230, 7) \approx 0.762 \) or the 76th percentile
   d. \( \text{normalcdf}(-10^{99}, 220, 230, 7) \approx 0.077 \)

2. a. \( \text{normalcdf}(-10^{99}, 100, 185, 36) \approx 0.0091 \)
   b. \( \text{normalcdf}(250, 10^{99}, 185, 36) \approx 0.035 \)
   c. \( \text{normalcdf}(-10^{99}, 160, 185, 36) \approx 0.244 \) or the 24th percentile
1. In the rectangle $ABCD$ at right, the area of the shaded region is given by $\frac{\pi lw}{6}$. If the area of the shaded region is $7\pi$, what is the total area, to the nearest whole number, of the unshaded regions of the rectangle $ABCD$?
   a. 14   b. 15   c. 20   d. 22   e. 25

2. Consider the following equations:
   \[ a = p^3 - 0.61 \]
   \[ b = p^2 - 0.61 \]
   \[ c = (p - 0.61)^2 \]

   If $p$ is a negative integer, what is the ordering of $a$, $b$, and $c$ from least to greatest?
   a. $c < a < b$   b. $a < c < b$   c. $b < a < c$   d. $a < b < c$   e. $c < b < a$

3. The figure at right represents six offices that will be assigned randomly to six different employees, one employee per office. If Maryanne and Ginger are two of the six employees, what is the probability that they will be assigned an office indicated with an *?
   a. $\frac{1}{6}$   b. $\frac{1}{8}$   c. $\frac{1}{15}$   d. $\frac{2}{15}$   e. $\frac{1}{30}$

4. Raul needed wire pieces 7 inches long. He cut as many as he possibly could from a wire 6 feet long. What is the total length of the wire that is left over?
   a. 2 inches   b. 3 inches   c. 4 inches   d. 5 inches   e. 8 inches

5. The $n^{th}$ term of a sequence is defined to be $5n + 2$. The $35^{th}$ term is how much greater than the $30^{th}$ term?
   a. 5   b. 18   c. 25   d. 36   e. 40

6. Matilda remembers only the first four digits of a seven-digit phone number. She is certain that none of the last three digits is zero. If she dials the first four digits, then dials the last three digits randomly from the non-zero digits, what is the probability that she will dial the correct number?

7. Let $a \triangle b$ be defined as $\frac{1}{a} + b$ where $a \neq 0$. What is the value of $6 \triangle 7$?
8. If \(4xy + 1 = 1\), what is the value of \(xy\)?

9. Eight consecutive integers are arranged in order from smallest to the largest. If the sum of the first four integers is 206, what is the sum of the last four integers?

10. If the points \(A(4, 1), B(4, 8),\) and \(C(3, 8)\) form the vertices of a triangle, what is the area of the triangle?

**Answers**

1. C
2. D
3. C
4. A
5. C
6. \(\frac{1}{729}\)
7. \(\frac{43}{6}\)
8. 0
9. 222
10. 24.5
In this section students explore inverse functions, that is, functions that “undo” the actions of original functions. The outputs of the original function are the inputs of the inverse function and vice versa. Multiple representations are used to verify that two functions are inverses of each other. Graphing is used to determine any domain restrictions necessary to ensure that two functions are inverses of each other. An inverse function has the notation \( f^{-1} \) (read as \( f \) inverse). Note that the \(-1\) is not a negative exponent. It is the mathematical symbol that indicates the inverse function of \( f \). The line of symmetry \( y = x \) is used to graph the inverse of a function and write equations for inverses.

For additional information see the Math Notes box in Lesson 5.1.3.

**Example 1**

The graph of \( f(x) = 0.2x^3 - 2.4x^2 + 6.4x \) is shown at right.
Graph the inverse of this function.

The graph of a function and its inverse have a special property: they are symmetric about the line \( y = x \).

If the line \( y = x \) is added to the graph, the inverse is the reflection across this line. Fold the paper along the line \( y = x \), and trace the result to create the reflection. The result is shown at right.
Example 2

Write the equation of the inverse for the functions below. Use function notation and state any domain restrictions necessary to make each pair of functions inverses of each other. Verify your answers by graphing.

a. \( f(x) = \frac{x-6}{3} \)

b. \( g(x) = (x + 4)^2 + 1 \)

The function in part (a) subtracts 6 from the input then divides by 3. The inverse function reverses this process. Therefore, the inverse function first multiplies by 3 then adds 6. Therefore the inverse function, called \( f^{-1} \), is \( f^{-1}(x) = 3x + 6 \). No domain restrictions are necessary since both functions have a domain of all real numbers. In the graph at right, the original function is dark gray and the inverse function is black.

The function in part (b) adds 4 to the input, squares that value, then adds 1. The inverse will first subtract 1, take the square root, then subtract 4. Therefore \( g^{-1}(x) = \sqrt{x - 1} - 4 \). The domain of the inverse function is \( x \geq 1 \) and the range is \( y \geq 4 \). Therefore the domain of the original function must be restricted to \( x \geq -4 \) and the corresponding range is \( y \geq 1 \). This can be see in the graph at right, where the original function is dark gray and the inverse function is black.

Problems

Write the equation of the inverse for each of the following functions. State any domain restrictions necessary to make each pair of functions inverses of each other.

1. \( f(x) = -5(x - 4) \)
2. \( h(x) = \frac{3}{8}x - 5 \)
3. \( k(x) = \frac{2}{x} + 3 \)
4. \( f(x) = x^2 + 6 \)
5. \( f(x) = \frac{3}{x} + 6 \)
6. \( g(x) = \frac{5}{x-8} \)
7. \( g(x) = (x + 1)^2 - 3 \)
8. \( f(x) = (x + 2)^3 \)
9. \( m(x) = 3 + \sqrt{x - 4} \)
10. \( g(x) = 3x + 6 \)

Sketch the graph of each function and its inverse function. Restrict domains as necessary.

11. \( h(x) = \frac{x}{6} + 2 \)
12. \( f(x) = 2x^2 - 1 \)
13. \( g(x) = \sqrt{x} - 4 \)
14. \( n(x) = \frac{1}{x-5} \)
15. \( y \)
Answers

1. \( f^{-1}(x) = -\frac{x}{2} + 4 \)

2. \( h^{-1}(x) = \frac{8}{3} x + 8 \)

3. \( k^{-1}(x) = (x - 3)^3 - 2 \)

4. \( f^{-1}(x) = \sqrt{x - 6} \)
   domain of \( f \) needs to be \( x \geq 0 \)

5. \( f^{-1}(x) = \frac{3}{x-6} \)

6. \( g^{-1}(x) = \frac{5}{x} + 8 \)

7. \( g^{-1}(x) = \sqrt{x} + 3 - 1 \)
   domain of \( g \) needs to be \( x \geq -1 \)

8. \( f^{-1}(x) = \frac{3}{\sqrt{x}} - 2 \)

9. \( m^{-1}(x) = (x - 3)^2 + 4, \text{ for } x \geq 3 \)

10. \( g^{-1}(x) = \frac{x-6}{3} = \frac{1}{3} x - 2 \)

The original functions are graphed in dark gray. The inverse functions are graphed in black.

11. [Graph of the original function and its inverse]

12. [Graph of the original function and its inverse]
   The domain of \( f \) needs to be restricted to \( x \geq 0 \).

13. [Graph of the original function and its inverse]

14. [Graph of the original function and its inverse]
   The domain of \( f^{-1} \) needs to be restricted to \( x \geq -5 \).

15. [Graph of the original function and its inverse]
In this section students explore the inverse of an exponential function. Although the graph of the inverse of an exponential function can be created by reflecting the graph across the line $y = x$, students cannot yet write the equation of this inverse function. Writing the equation requires the introduction of a new function, the logarithm.

For additional information see the Math Notes box in Lesson 5.2.4.

Example 1

Determine each of the missing values below and then justify your answer by writing the equation in its equivalent exponential form.

a. $\log_5(25) = ?$

b. $\log_7(?) = 3$

c. $\log_2\left(\frac{1}{8}\right) = ?$

In part (a), $\log_5(25)$, is asking “What exponent is needed to raise the base 5 to, to get 25?” This question can be translated into an equation $5^x = 25$. By phrasing it this way, the answer is more apparent: 2. This is true because $5^2 = 25$.

Part (b) can be rephrased as $7^3 = ?$. The answer is 343.

Part (c) asks “2 to what exponent gives $\frac{1}{8}$?” or $2^x = \frac{1}{8}$. The answer is –3 because $2^{-3} = \frac{1}{8}$.
Example 2

The graph of \( y = \log(x) \) is shown at right. Use this “parent graph” to graph each of the following equations. Describe how each graph is transformed from the parent graph. Note: When a logarithm is written without a base, as in \( y = \log(x) \) and the log key used on a calculator, the base is 10.

\[ y = \log(x - 4) \quad y = 6\log(x) + 3 \quad y = -\log(x) \]

The logarithm function follows the same rules for transforming its graphs as other functions. The parent graph \( y = \log(x) \) is shown in gray at right.

\( y = \log(x - 4) \) shifts the parent graph 4 units to the right.
\( y = 6\log(x) + 3 \) shifts the parent graph up 3 units, but it is also vertically stretched by a factor of 6.
\( y = -\log(x) \) is reflected vertically or across the \( x \)-axis.

Problems

Rewrite each logarithmic equation as an exponential equation and vice versa.

1. \( 2 = \log_5(x) \)
2. \( 3 = \log_2(x) \)
3. \( x = \log_5(30) \)
4. \( 4^x = 80 \)
5. \( \left( \frac{1}{2} \right)^x = 64 \)
6. \( x^3 = 343 \)
7. \( 5^x = \frac{1}{125} \)
8. \( \log(32) = x \)
9. \( 11^3 = x \)
10. \( -4 = \log_x \left( \frac{1}{16} \right) \)

What is the value of \( x \) in each equation below? If necessary, rewrite the expression in the equivalent exponential equation to verify your answer.

11. \( 4 = \log_5(x) \)
12. \( 2 = \log_9(x) \)
13. \( 6 = \log(x) \)
14. \( 81 = 9^x \)
15. \( \left( \frac{1}{3} \right)^x = 243 \)
16. \( 6^x = 7776 \)
17. \( 7^x = \frac{1}{49} \)
18. \( \log_2(32) = x \)
19. \( \log_{11}(x) = 3 \)
20. \( \log_5 \left( \frac{1}{125} \right) = x \)

Graph each of the following equations.

21. \( y = \log(x + 2) \)
22. \( y = 3\log(x - 7) + 5 \)
23. \( y = -\log(x - 4) \)
24. \( y = \log(x) - 5 \)
Answers

1. $4^2 = x$
2. $2^3 = x$
3. $5^x = 30$
4. $\log_4(80) = x$
5. $\log_{1/2}(64) = x$
6. $\log_x(343) = 3$
7. $\log_5\left(\frac{1}{125}\right) = x$
8. $10^x = 32$
9. $\log_{11}(x) = 3$
10. $x^{-4} = \frac{1}{16}$
11. $x = 625$
12. $x = 81$
13. $x = 1,000,000$
14. $x = 2$
15. $x = -5$
16. $x = 5$
17. $x = -2$
18. $x = 5$
19. $x = 1,331$
20. $x = -3$

21.

22.

23.

24.
1. In the figure at right, \( \triangle DAI \) is isosceles with \( DI = 13 \) and base 24. If \( \triangle VAI \cong \triangle DAI \) what is the area of quadrilateral \( DAVI \)?
   a. 60  b. 75  c. 120  d. 156  e. 240

2. An experimental jet flies at a speed of 5280 miles per hour. How many miles can this jet cover in 10 seconds?
   a. 1.467  b. 8.802  c. 11.237  d. 14.667  e. 88.022

3. If the angle (not shown) where \( a \) and \( b \) intersect is three times as large as the angle (not shown) where \( e \) and \( b \) intersect, what is the value of \( p \)?
   a. 70°  b. 85°  c. 140°  d. 160°  e. Cannot be determined

4. Let \( \zeta x \) be defined for all positive integer values of \( x \) as the product of all even factors of \( 4x \). For example, \( \zeta 3 \zeta = 12 \times 6 \times 4 \times 2 = 576 \). What is the value of \( \zeta 5 \zeta \)?
   a. 1600  b. 6400  c. 7200  d. 8000  e. 9600

5. The table at right shows the distribution of topics covered in a particular business text, in chapters and pages per chapter. According to the table, how many total pages are in this text?

<table>
<thead>
<tr>
<th>Topic</th>
<th>No. of Chapters</th>
<th>No. of pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Development</td>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>Marketing</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>Public Relations</td>
<td>1</td>
<td>11</td>
</tr>
</tbody>
</table>

   a. 31  b. 39  c. 48  d. 65  e. 79

6. In the figure at right, what is the sum of \( x \) and \( y \)? Note: The figure is not drawn to scale.

7. If \( 2^q = 8^{q-1} \), then \( q = ? \)
8. If \( a \) is 40 percent of 300, \( b \) is 40 percent of \( a \), and \( c \) is 25 percent of \( b \), what is \( a + b + c \)?

9. If \( \frac{x}{4} = \frac{11}{20} \), what is the value of \( x \)?

10. If \( \frac{3}{5} \) of \( \frac{1}{3} \) is added to 5, what is the result?

**Answers**

1. C
2. D
3. C
4. A
5. E
6. 220°
7. \( q = \frac{3}{2} \)
8. 180
9. \( x = 2.2 \)
10. 5 \( \frac{1}{5} \)
When theoretical probability calculations become too complex, statisticians often use simulations instead. Simulations can also be used to check statistical computations, or they can be used in place of a study which is too expensive, time-consuming, or unethical. Simulations require random numbers or outcomes. In this chapter the simulations are simple enough that students can use coins, dice, tables of random digits, or their calculators to generate required random numbers or outcomes.

**Example 1**

Dana is looking for clear wood (without knots) to make furniture. The chair he has designed requires four 3-foot boards of clear wood. If there is a 20% chance of a knot being in a particular foot of wood, on average, how many boards will Dana have to sort through to get enough clear wood for one chair?

Solution: To simulate each 3-foot board, you can use a calculator or random number table to produce random digits in sets of three from 0 to 9. Consider each digit to be a foot of wood within the “board” where 0s and 1s to represent knots (20% of the digits). Simulated boards would look like \{6\,0\,3\}, \{9\,9\,5\}, \{1\,4\,7\}, etc., with each “foot” having a 20% chance of having a “knot”.

Continue generating your simulated boards, counting the number created in order to get four sets without a 0 or 1. 

Repeat this process enough times to get a meaningful average. 

For example: 6 boards, 8 boards, 5 boards, 9 boards, 10 boards, 11 boards 

*The theoretical mean is approximately 7.81 boards.*

**Example 2**

Would it be easier for Dana to find a single 12-foot board of clear wood rather than many 3-foot pieces? Use a simulation to estimate the average number of 12-foot boards he would have to consider.

Solution: The mean will be about fifteen boards (theoretical mean approximately 14.55). This is considerably more than the mean boards examined when four 3-foot boards were used, so finding four 3-foot clear boards would be easier.
Problems

1. Jack is planning flights to Fort Lauderdale to catch a cruise ship. The least expensive route will require three separate flights to arrive in Miami and then a shuttle bus connection to Fort Lauderdale. Jack believes that if more than one of the legs is delayed he will miss his ship. Jack researches and finds the airline and bus company both have an 80% on-time rating. If Jack buys the least expensive ticket, what is the probability that Jack misses his ship?

2. Jack looks for an alternative and finds a more expensive flight to Fort Lauderdale with another airline, eliminating the need for the shuttle bus. To afford this ticket he will need to avoid airport parking fees and have his friend Clara drive him to the airport. This airline has an 85% on-time rating, but he estimates Clara’s on-time percentage at around 65%.

   a. Use simulation to estimate the probability that more than one leg of this journey is delayed.

   b. Which travel plan would you recommend to Jack, the one with the shuttle bus or the one involving Clara? Why?
**Answers**

1. To simulate each trip you could use a calculator or random number table to produce random digits in groups of four from 0 to 9. `randInt(0, 9, 4)`. Consider each digit to be a leg of his journey with zeros and ones to represent delays. Simulated journeys would look like \{5 8 1 1\}, \{4 9 6 0\}, \{7 6 7 3\}, etc., with each “leg” having a 20% chance of a “delay”. Continue generating simulated trips (about 30 to 50) until you feel you have an accurate proportion of missed cruise ships out of total trip attempts.

   In this example, 5 missed cruise ships out of 28 trips is $\frac{5}{28} \approx 0.179$. *The theoretical probability is approximately 0.181.*

2. a. To simulate each trip you could use a calculator or random number table to produce random numbers in groups of 3 from 0 to 99. `randInt(0, 99, 3)`. Consider each number to be a leg of Jack’s journey with 0 to 34 to represent delays in the first leg with Clara and 0 to 14 to represent delays in the flights. Simulated journeys would now look like \{75 81 41\}, \{49 60 76\}, \{72 37 4\}, etc. Continue generating simulated trips (about 30 to 50) until you feel you have an accurate proportion of missed cruise ships out of total trip attempts.

   In this example, 3 missed ships out of 28 trips is $\frac{3}{28} \approx 0.107$. *The theoretical probability is approximately 0.112.*

   b. Option 2 looks better with a lower chance of missing the ship. The advantages of eliminating one leg of the trip and choosing a more reliable airline make up for Clara’s lower on-time rating.
Sampling is performed to learn about populations when it is too time-consuming, expensive, or impractical to study the whole. Samples are rarely exact representations of their populations. In fact, two samples from the same population are rarely the same. However, sample-to-sample variability is measurable and predictable. Students investigate and predict this natural variation via repeated sampling from populations and simulations of repeated sampling. For more information see the Math Notes boxes in Lessons 6.2.2, 6.2.3, and 6.2.4.

Example 1

A dental health insurer wants to know the percentage of dental patients who opt to have root canal procedures performed by a general dentist rather than by an endodontist (root canal specialist). They examine the dental records for a sample of 50 patient records selected at random, all indicating at least one root canal. The sample proportion was found to be 0.20.

To get an idea of how much variation one could expect in samples of size 50, a simulation of the samples was performed 100 times. The simulated proportions are sorted and shown below.

<table>
<thead>
<tr>
<th>Simulated proportions of sample size 50</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.06 0.16 0.18 0.20 0.24</td>
</tr>
<tr>
<td>0.08 0.16 0.18 0.20 0.24</td>
</tr>
<tr>
<td>0.08 0.16 0.18 0.20 0.26</td>
</tr>
<tr>
<td>0.10 0.16 0.18 0.22 0.26</td>
</tr>
<tr>
<td>0.10 0.16 0.18 0.22 0.26</td>
</tr>
<tr>
<td>0.12 0.16 0.18 0.22 0.26</td>
</tr>
<tr>
<td>0.12 0.16 0.18 0.22 0.26</td>
</tr>
<tr>
<td>0.12 0.16 0.18 0.22 0.26</td>
</tr>
<tr>
<td>0.14 0.16 0.20 0.22 0.26</td>
</tr>
<tr>
<td>0.14 0.16 0.20 0.22 0.28</td>
</tr>
<tr>
<td>0.14 0.16 0.20 0.24 0.28</td>
</tr>
<tr>
<td>0.14 0.16 0.20 0.24 0.28</td>
</tr>
<tr>
<td>0.14 0.18 0.20 0.24 0.30</td>
</tr>
<tr>
<td>0.14 0.18 0.20 0.24 0.32</td>
</tr>
<tr>
<td>0.14 0.18 0.20 0.24 0.32</td>
</tr>
<tr>
<td>0.14 0.18 0.20 0.24 0.34</td>
</tr>
<tr>
<td>0.16 0.18 0.20 0.24 0.34</td>
</tr>
</tbody>
</table>

checksum 19.62

Example continues on next page →
Example continued from previous page.

a. What do you predict the population proportion is?

b. There is variation in the simulated samples of 50. The simulated sample proportions have a range of 0.28. Determine the boundaries that mark the middle 90% of the simulated samples.

c. What is the margin of error for samples of size 50?

d. State a reasonable conclusion for the population, taking into account that it is based on sample information, which varies from sample-to-sample.

e. Based on this study, could the dental insurance company conclude that less than one quarter of people getting root canals opt for the procedure to be performed by a general dentist? Explain.

f. If the dental insurance company has the hypothesis that fewer than one third of people getting root canals opt for the procedure to be performed by a general dentist, is it supported by their study? Explain.

Answers:

a. From the sample of 50 patient records the sample proportion was found to be 0.20.

b. There are 100 data points so an estimate of the middle 90% is from 0.11 to 0.29.

c. The margin of error is half the range of the middle 90% of the data $\frac{0.29-0.11}{2} = 0.09$.

d. I am 90% confident that the proportion of root canal patients opting for their surgery to be performed by a general dentist is 0.20 with a margin of error of 0.09. Or, with 90% confidence the proportion of root canal patients opting for their surgery to be performed by a general dentist is between 0.11 and 0.29.

e. No. Even though the sample mean seems to be substantially less than 0.25 or one quarter, 0.25 is within one margin of error from the sample mean, so it is still plausible the population mean is equal to or even more than 0.25.

f. Yes. One third, (0.3) is beyond one margin of error from the sample mean so the insurance company has significant evidence that the population mean is less than 0.3.
Problem

Capture and recapture programs are important ways to monitor fish populations. A known number of fish are marked or tagged and released into the wild. Then biologists or other anglers report the number of tagged fish they catch along with the total number of the same species they caught. The proportion of tagged fish captured can be used to estimate the total population of the species. Assume 200 rainbow trout were tagged and released into Paulina Lake. Last week anglers at the lake reported a catch of 121 rainbow trout out of which 19 were tagged.

a. What do you estimate the population proportion of tagged rainbow trout to be?

b. Estimate the rainbow trout population in the lake.

c. Amanda, a biologist for the state, realizes that the results in parts (a) and (b) are from a sample and are subject to variation. To get an idea of how much variation she could expect in samples of size 121, Amanda ran a simulation of the sample 100 times. Her simulated proportions are sorted and shown below.

```
Simulated proportions of sample size 121

| 0.066 | 0.116 | 0.140 | 0.149 | 0.157 | 0.179 | 0.190 |
| 0.074 | 0.116 | 0.140 | 0.149 | 0.157 | 0.179 | 0.190 |
| 0.083 | 0.116 | 0.140 | 0.157 | 0.165 | 0.179 | 0.198 |
| 0.091 | 0.124 | 0.140 | 0.157 | 0.165 | 0.179 | 0.198 |
| 0.099 | 0.124 | 0.140 | 0.157 | 0.165 | 0.182 | 0.215 |
| 0.099 | 0.124 | 0.140 | 0.157 | 0.165 | 0.182 | 0.223 |
| 0.099 | 0.132 | 0.140 | 0.157 | 0.165 | 0.182 | 0.223 |
| 0.099 | 0.132 | 0.140 | 0.157 | 0.165 | 0.182 | 0.231 |
| 0.099 | 0.132 | 0.140 | 0.157 | 0.179 | 0.182 | 0.240 |
| 0.099 | 0.132 | 0.140 | 0.157 | 0.179 | 0.182 | 0.240 |
| 0.099 | 0.132 | 0.140 | 0.157 | 0.179 | 0.182 | 0.240 |
| 0.107 | 0.140 | 0.149 | 0.157 | 0.179 | 0.182 | 0.182 |
| 0.107 | 0.140 | 0.149 | 0.157 | 0.179 | 0.182 | 0.182 |
| 0.107 | 0.140 | 0.149 | 0.157 | 0.179 | 0.190 | 1.90 |
| 0.116 | 0.140 | 0.149 | 0.157 | 0.179 | 0.190 | 1.90 |
```

As expected there is sample-to-sample variation in the simulated samples of 121. Determine the boundaries that mark the middle 90% of the simulated samples.

d. What is the margin of error for samples of size 121?

e. State a reasonable conclusion for the population proportion of tagged rainbow trout, taking into account that it is based on sample information.

f. State a reasonable conclusion for the total number of rainbow trout in the lake, taking into account that it is based on sample information.

g. Based on this study, could Amanda conclude that there are at least 1000 rainbow trout in the lake? Explain.
Answers:

a. The sample proportion from last week’s catch is the best estimate: \( \frac{19}{121} \approx 0.157 \)

b. Assuming the proportion of tagged fish in the sample is the same as the population and \( x \) represents the total number of rainbow trout in the lake:
\[
\frac{19}{121} = \frac{200}{x} \implies x = 121 \cdot \frac{200}{19} = 1274 \text{ rainbow trout in the lake}
\]

c. There are 100 data points so an estimate of the middle 90% is between 0.099 and 0.219.

d. The margin of error is half the range of the middle 90% of the data.
\[
\frac{0.219 - 0.099}{2} = 0.06
\]

e. I am 90% confident that the proportion of tagged rainbow trout in the lake is 0.157 with a margin of error of 0.06. Or, with 90% confidence the proportion of tagged rainbow trout in the lake is between 0.097 and 0.217.

f. Using ratios as in part (b) with the proportions from part (e).
\[
0.097 = \frac{200}{x_{(\text{high})}} \implies x_{(\text{high})} = 2062; \quad 0.217 = \frac{200}{x_{(\text{low})}} \implies x_{(\text{low})} = 922
\]

With 90% confidence the population of rainbow trout in the lake is between 922 and 2062 fish.

g. No. Even though the sample indicates a population of 1274 fish, 1000 is within the confidence interval (922, 2062), so it is plausible the population number of rainbow trout is equal to or even less than 1000.
To visualize these situations think about false alarms. Nearly all alarms do an excellent job of warning if there is a specific threat. Car alarms protect cars, smoke alarms protect against fires, and burglar alarms protect homes and businesses, with very high probabilities of detecting the problems they were designed to guard against. However, the frequency of these kinds of threats is thankfully very low, so alarm systems sound more often in response to more common and less exciting events (false alarms). This makes for statements that sound similar but have very different probabilities.

### Examples

<table>
<thead>
<tr>
<th>Very High Probability</th>
<th>Much Lower Probability</th>
<th>Some False Alarm Causes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Given that a car has been broken into, its alarm system would detect it.</td>
<td>Given that you hear a car alarm go off, the car has been broken into.</td>
<td>Loud noises or music, a large truck driving by, a cat climbs on the hood.</td>
</tr>
<tr>
<td>Given that there is a fire the smoke/fire detector will sound.</td>
<td>Given you hear a smoke/fire alarm, there is a fire.</td>
<td>Steamy showers, burning toast, inexperienced chemistry teacher doing a lab, scheduled fire drills.</td>
</tr>
<tr>
<td>Given that there is an unwelcome intruder, the security system will call the police.</td>
<td>Given that a security system has called police, there is an unwelcome intruder.</td>
<td>Pets wandering inside the home, someone watering your plants when you are on vacation, a cousin pays a surprise visit.</td>
</tr>
</tbody>
</table>

### Example 1

People of Madelinton are security-conscious and have sophisticated car alarms installed on their cars. Over the course of a year these alarms correctly distinguish between a break-in attempt and other harmless events at a 99% rate. Of the 820,600 cars in Madelinton about 100 are broken into each year. If a citizen of Madelinton hears a car alarm, what is the probability the car is being broken into?

Solution:

Possible diagram (multiply row by column to get numbers for events A, B, C, & D):

<table>
<thead>
<tr>
<th></th>
<th>100 are broken into</th>
<th>820,500 are not broken into</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alarm is correct 0.99</td>
<td>(A) 99</td>
<td>(B) 812,295</td>
</tr>
<tr>
<td>Alarm is not correct 0.01</td>
<td>(C) 1</td>
<td>(D) 8205 false alarms</td>
</tr>
</tbody>
</table>

Car alarms will sound in events (A) or (D) from the diagram. That would be $99 + 8205 = 8304$ alarms per year. Of those alarms only the 99 from event (A) are actual break-ins. If you hear a car alarm in Madelinton the probability the car is being broken into is just $\frac{99}{8304} \approx 0.0119$. 

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Example 2

Imagine there is a rare but serious disease that is carried by three people per ten thousand. However, there is a reliable test that will identify infected individuals for treatment. Given that a person is infected or not, the test will correctly identify them 99.9% of the time. Now assume the test has just identified a randomly chosen person as having the disease (a “positive” result in testing terms). What is the probability this person actually has the disease?

Solution:

Possible diagram:

<table>
<thead>
<tr>
<th></th>
<th>Person has disease 0.0003</th>
<th>Person does not have disease 0.9997</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test is correct 0.999</td>
<td>(A) 0.0002997</td>
<td>(B) 0.9987003</td>
</tr>
<tr>
<td>Test is not correct 0.001</td>
<td>(C) 0.0000003</td>
<td>(D) 0.0009997 false positives</td>
</tr>
</tbody>
</table>

The test will indicate a person has the disease (positive result) in events (A) or (D) from the diagram, making the associated probability $0.0002997 + 0.0009997 = 0.0012994$. Of the probability of a positive test result, only in event (A) does the person have the disease. So if a person has a positive test result the probability they have the disease is only $\frac{0.000299799}{0.0012994} \approx 0.231$.

Problems

1. The police chief of Madelinton is concerned about the number of false alarms from home security systems his officers respond to. The alarm manufacturers say that they are not to blame because during any given year their alarm systems correctly distinguish between criminal activity and other harmless events 99.5% of the time. Of the 319,200 residences in Madelinton only 200 were broken into last year, but the police chief says his officers responded to hundreds more false alarm calls. How many calls are the police likely to receive from home alarm systems in a year? If the Madelinton police department receives a call from a home alarm system, what is the probability it is the result of criminal activity?

2. The Madelinton fire chief is also concerned about false alarms for fires. Not just because they require a response from his fire units, but because as people grow tired of false alarms they do unwise things like disabling their required smoke detectors or ignoring them. If the required smoke detectors in Madelinton are accurate to 99% during a year, and typically 0.2% of residences catch fire each year, what is the chance of a false alarm when a residential smoke alarm sounds in Madelinton? Even with a high probability of a false alarm, why does the chief insist citizens continue paying attention to home smoke alarms?
Answers

1. Possible diagram:

<table>
<thead>
<tr>
<th></th>
<th>200 are broken into</th>
<th>319,000 are not broken into</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alarm is correct 0.995</td>
<td>(A) 199</td>
<td>(B) 317,405</td>
</tr>
<tr>
<td>Alarm is not correct 0.005</td>
<td>(C) 1</td>
<td>(D) 1595 false alarms</td>
</tr>
</tbody>
</table>

Home alarm systems will call police in events (A) or (D) from the diagram. That would be 199 + 1595 = 1794 calls per year. Of those alarms only the 199 from event (A) are actual break-ins. When the police dispatcher receives a call from a home alarm system there is only a \( \frac{199}{1794} \approx 0.111 \) chance it is due to criminal activity.

2. Possible diagram:

<table>
<thead>
<tr>
<th></th>
<th>0.002 have a fire</th>
<th>0.998 do not have a fire</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alarm is correct 0.99</td>
<td>(A) 0.00198</td>
<td>(B) 0.98802</td>
</tr>
<tr>
<td>Alarm is not correct 0.01</td>
<td>(C) 0.00002</td>
<td>(D) 0.00998 false alarms</td>
</tr>
</tbody>
</table>

Home smoke detectors sound in events (A) or (D) from the diagram. That would be a probability of 0.00198 + 0.00998 = 0.01196. Of the probability of a smoke alarm sounding, event (D) is a false alarm. So if a residential smoke alarm sounds in Madelinton it has a \( \frac{0.00998}{0.01196} \approx 0.834 \) chance of being a false alarm.

If the fire is real, the probability of substantial damage, injury or even death far out-weighs the inconvenience of responding to false alarms.
1. If \( k = 5t \) and \( t = 6 \), what is the value of \( 5k \)?
   a. 5  b. 6  c. 30  d. 36  e. 150

2. **Some integers in set U are negative.**
   If the statement above is true, which of the following must also be true?
   a. All integers in set U are negative.
   b. All integers in set U are positive.
   c. If an integer is negative, then it is in set U.
   d. If an integer is positive, then it is in set U.
   e. Not all integers in set U are positive.

3. Squaring the product of \( x \) and 4 gives the same result as squaring the difference of \( x \) and 4. Which of the following equations could be used to calculate all the possible values of \( x \)?
   a. \( 4x^2 = x^2 + 4^2 \)
   b. \( (4x^2) = (x - 4)^2 \)
   c. \( 4^2x = x^2 - 4^2 \)
   d. \( (4x)^2 = x^2 + 4^2 \)
   e. \( 4x^2 = (x + 4)^2 \)

4. To make Tam’s tantalizingly tasty toffee cookies, flour, sugar, and salt are mixed by weight in the ratio 7:3:1, respectively. In order to make 11 pounds of the dough for this cookie, what weight of sugar, in pounds, is needed?
   a. 7  b. 3  c. 1  d. \( \frac{1}{3} \)  e. \( \frac{1}{7} \)

5. The local donut shop donated some donuts to Professor Galactic’s astronomy class. If each student takes two donuts, there will be 16 donuts left. If four students do not take any donuts and the rest of the students take six donuts, there will be no donuts left. How many donuts were donated to the class?
   a. 24  b. 30  c. 36  d. 40  e. 48
6. Tran can mow the lawn in three hours while it takes Collin four hours to mow the same lawn. How long will it take if they mow the lawn together? (They start at opposite sides and work toward each other.)

   a. 12 hours  b. 7 hours  c. \( \approx 1.7 \) hours  d. \( \approx 42 \) minutes  e. \( \approx 9 \) minutes

7. On a certain map, 15 miles are represented by one-half inch. On the same map, how many miles are represented by 3.75 inches?

8. How many of the first one hundred positive integers contain the digit 1?

9. The sum of two consecutive integers is greater than three but less than 13. What is one possible integer fitting these conditions?

10. Give three values for \( k \) for which the trinomial \( 3x^2 + kx + 6 \) is factorable.

**Answers**

1. E  
2. E  
3. B  
4. B  
5. C  
6. C  
7. 112.5 miles  
8. 20  
9. Any of 2, 3, 4, or 5 will work.  
10. 19, 9, 11, –19, –9, –11
SOLVING EQUATIONS WITH LOGARITHMS  7.1.1 and 7.1.4

In this chapter, students turn their attention back to logarithms. Using pattern recognition, and other problem solving strategies, students develop several properties of logarithms that enable them to solve equations that, until now, they have been unable to solve algebraically. These properties are listed in the Math Notes box in Lesson 7.1.4.

Example 1

Solve each of the following equations for \( x \).

a. \( 5^x = 67 \)  
b. \( 3(7^x) + 4 = 124 \)

Each of these equations has the variable as the exponent, which makes them different from other equations that students have been solving. The log property, \( \log(b^x) = x \log(b) \), can be used to solve these equations. As with other equations, however, the variable must be isolated on one side of the equation.

\[
\begin{align*}
\text{a. } 5^x &= 67 \\
\log(5^x) &= \log(67) \\
x \log(5) &= \log(67) \\
x &= \frac{\log(67)}{\log(5)} \\
x &\approx 2.613 \\
\end{align*}
\]

b. \( 3(7^x) + 4 = 124 \)

\[
\begin{align*}
\text{b. } 3(7^x) + 4 &= 124 \\
3(7^x) &= 120 \\
7^x &= 40 \\
\log(7^x) &= \log(40) \\
x \log(7) &= \log(40) \\
x &= \frac{\log(40)}{\log(7)} \\
x &\approx 1.896 \\
\end{align*}
\]

Note: The decimal answer is an approximation. The exact answer is the fraction \( \frac{\log(67)}{\log(5)} \).

Example 2

Using the properties of logs to rewrite each expression.

a. \( \log_3(16x) \)  
b. \( \log_6(32) + \log_6(243) \)

c. \( \log_8\left(\frac{3x}{2}\right) \)  
d. \( \log_{12}(276) - \log_{12}(23) \)

The two properties to use are \( \log_b(mn) = \log_b(m) + \log(n) \) and \( \log_b\left(\frac{m}{n}\right) = \log_b(m) - \log_b(n) \).

\[
\begin{align*}
\text{a. } \log_3(16x) &= \log_3(16) + \log_3(x) \\
\text{b. } \log_6(32) + \log_6(243) &= \log_6(32 \cdot 243) = \log_6(7776) = 5 \\
\text{c. } \log_8\left(\frac{3x}{2}\right) &= \log_8(3x) - \log_8(2) = \log_8(3) + \log_8(x) - \log_8(2) \\
\text{d. } \log_{12}(276) - \log_{12}(23) &= \log_{12}\left(\frac{276}{23}\right) = \log_{12}(12) = 1 \\
\end{align*}
\]
Example 3

Fall came early in Piney Orchard, and the community swimming pool was still full of water when the first frost occurred. The outside temperature hovered at 30°F. Maintenance quickly turned off the heat so that energy would not be wasted heating a pool that nobody would be swimming in for at least six months. As Tess walked by the pool each day on her way to school, she would peer through the fence at the slowly cooling pool. She could just make out the thermometer across the deck that displayed the water’s temperature. On the first day, she noted that the water temperature was 68°F. Four days later, the temperature reading was 58°F. Write an equation that models this situation. If the outside temperature remains at 30°F, and the pool is allowed to continue to cool, how long before it freezes (32°F)?

Heating and cooling problems are typical application problems that use exponential equations. The equation that will model this problem is an exponential equation of the form \( y = ab^x + k \). In the problem description, two data points are given: (0, 68°) and (4, 58°). There is one other piece of important information. The outside temperature is hovering at 30°F. This is the temperature the water will approach, that is, \( y = 30 \) is the horizontal asymptote for this equation. Knowing this fact means that the equation \( y = ab^x + 30 \) can be written. To determine \( a \) and \( b \), substitute values into the equation and solve for \( a \) and \( b \).

\[
(0, 68) \Rightarrow y = ab^x + 30 \Rightarrow 68 = ab^0 + 30 \\
(4, 58) \Rightarrow y = ab^x + 30 \Rightarrow 58 = ab^4 + 30
\]

Doing this gives two equations with two unknowns that can be solved. Simplifying first makes the work a lot easier. The first equation simplifies to \( 38 = a \) since \( b^0 = 1 \). Since \( a = 38 \) this value can be substituted into the second equation to determine \( b \) as shown at left.

\[
58 = 38b^4 + 30 \\
28 = 38b^4 \\
b^4 = \frac{28}{38} = 0.7368 \\
b = 0.9265
\]

Therefore the equation is \( y = 38(0.9265)^x + 30 \). To determine when the pool will freeze, use the equation to determine when the water’s temperature reaches 32°F.

\[
32 = 38(0.9265)^x + 30 \\
2 = 38(0.9265)^x \\
\frac{2}{38} = 0.9265^x \\
\log\left(\frac{2}{38}\right) = \log(0.9265^x) \\
\log\left(\frac{2}{38}\right) = x \log(0.9265) \\
x = \frac{\log\left(\frac{2}{38}\right)}{\log(0.9265)} \approx 38.57
\]

In approximately 39 days, the water in the pool will freeze if the outside temperature remains at 30°F for those days. In reality, the pool would be drained to prevent damage from freezing.
Problems

Solve each of the following equations.

1. \((2.3)^x = 7\)
2. \(12^x = 6\)
3. \(\log_7 49 = x\)
4. \(\log_3 x = 4\)
5. \(5(3.14)^x = 18\)
6. \(7x^8 = 294\)
7. \(\log_x 100 = 4\)
8. \(\log_5 45 = x\)
9. \(2(6.5)^x + 7 = 21\)
10. \(-\frac{1}{2}(14)^x + 6 = -9.1\)

Use the properties of logarithms to rewrite each log expression.

11. \(\log(23 \cdot 3)\)
12. \(\log\left(\frac{3x}{8}\right)\)
13. \(\log_2 \left(\frac{60}{7}\right)\)
14. \(\log_8(12) - \log_8(2)\)
15. \(\log_5(25) + \log_5(25)\)
16. \(\log(10 \cdot 10)\)
17. \(\log_{13}(15x^2)\)
18. \(\log(123) + \log(456)\)
19. \(\log(10^8) - \log(10^3)\)
20. \(\log(5x - 4)\)

Simplify.

21. \(\log_2(64)\)
22. \(\log_{17}(17^{1/8})\)
23. \(8^{\log_8(1.3)}\)
24. \(2.3^{\log_{2.3}(1)}\)

25. Climbing Mt. Everest is not an easy task! Not only is it a difficult hike, but the Earth’s atmosphere decreases exponentially as you climb above the Earth’s surface, and this makes it harder to breathe. The air pressure at the Earth’s surface (sea level) is approximately 14.7 pounds per square inch (or 14.7 psi). In Denver, Colorado, elevation 5280 feet, the air pressure is approximately 12.15 psi. Write an equation to represent this situation expressing air pressure as a function of altitude. What is the air pressure in Mexico City, elevation 7300 feet? At the top of Mt. Everest, elevation 29,000 feet? (Note: You will need to carry out the decimal values several places to get an accurate equation and air pressures.)
Answers

1. \[ x = \frac{\log(7)}{\log(2.3)} \approx 2.336 \]  
2. \[ x = \frac{\log(6)}{\log(12)} \approx 0.721 \]  
3. \[ x = 2 \]  
4. \[ x = 81 \]  
5. \[ x = \frac{\log(3.6)}{\log(3.14)} \approx 1.119 \]  
6. \[ x = 42^{1/8} \approx 1.596 \]  
7. \[ x = 100^{1/4} \approx 3.162 \]  
8. \[ x = \frac{\log(45)}{\log(5)} \approx 2.365 \]  
9. \[ x = \frac{\log(7)}{\log(6.5)} \approx 1.040 \]  
10. \[ x = \frac{\log(30.2)}{\log(14)} \approx 1.291 \]  
11. \[ \log(23) + \log(3) \]  
12. \[ \log(3x) - \log(8) \]  
13. \[ \log_2(60) - \log_2(7) \]  
14. \[ \log_8 \left( \frac{12}{2} \right) = \log_8(6) \]  
15. \[ \log_5(625) \]  
16. \[ \log(10) + \log(10) = 2 \]  
17. \[ \log_{13}(15) + \log_{13}(x^2) \]  
18. \[ \log(56,088) \]  
19. \[ \log \left( \frac{10^8}{10^3} \right) = \log 10^5 = 5 \]  
20. cannot be rewritten  
21. 6  
22. \[ \frac{1}{8} \]  
23. 1.3  
24. 1  
25. The equation is \( y = 14.7(0.999964)^x \) where \( x \) is the elevation in feet, and \( y \) is the number of pounds per square inch (psi). The air pressure in Mexico City is approximately 11.3 psi, and at the top of Mt. Everest, the air pressure is approximately 5.175 psi.
Several tools can be used for calculating parts of right triangles, including the Pythagorean Theorem, the tangent ratio, the sine ratio, and the cosine ratio. These relationships only work, however, with right triangles. What if the triangle is not a right triangle? Can the lengths and angles of any triangle still be calculated with trigonometry from certain pieces of information? Yes, by using two laws, the Law of Sines and the Law of Cosines state that for any triangle:

**Law of Sines**

\[
\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}
\]

**Law of Cosines**

\[
c^2 = a^2 + b^2 - 2ab\cos(C)
\]

See the Math Notes boxes in Lessons 7.2.2 and 7.2.3.

### Example 1

Use the Law of Sines to calculate the value of \(x\) in each triangle.

#### a.

\[\begin{align*}
21 & \text{ is the length of the side opposite the } 35^\circ \text{ angle, while } x \text{ is the length of the side opposite the } 65^\circ \text{ angle. The equation and solution are shown at right.} \\
\text{\quad \quad } x \sin(35^\circ) &= 21 \sin(65^\circ) \\
\text{\quad \quad } x &= \frac{21 \sin(65^\circ)}{\sin(35^\circ)} \\
\text{\quad \quad } x &\approx 33.2
\end{align*}\]

#### b.

\[\begin{align*}
15 & \text{ is the length of the side opposite the } 52^\circ \text{ angle, while } x \text{ is the angle opposite the side of length } 13. \text{ The equation and solution are shown at right.} \\
\text{\quad \quad } 15 \sin(x) &= 13 \sin(52^\circ) \\
\text{\quad \quad } \sin(x) &= \frac{13 \sin(52^\circ)}{15} \\
\text{\quad \quad } \sin^{-1}(\sin(x)) &= \sin^{-1}\left(\frac{13 \sin(52^\circ)}{15}\right) \\
\text{\quad \quad } x &\approx 43^\circ
\end{align*}\]
**Example 2**

Use the Law of Cosines to calculate the value of \( x \) in each triangle.

\[
\begin{align*}
\text{a.} & \quad x^2 = 6^2 + 9^2 - 2(6)(9)\cos(93^\circ) \\
& \quad x^2 \approx 36 + 81 - 108(-0.052) \\
& \quad x^2 \approx 117 + 5.612 \\
& \quad x \approx 122.612 \\
& \quad x \approx 11.07 \\
\text{b.} & \quad 7^2 = 17^2 + 21^2 - 2(17)(21)\cos(x) \\
& \quad 49 = 289 + 441 - 714\cos(x) \\
& \quad 49 = 730 - 714\cos(x) \\
& \quad -681 = -714\cos(x) \\
& \quad x = 17.5^\circ \text{ using } \cos^{-1}(x)
\end{align*}
\]

The Law of Cosines does not use ratios, as the Law of Sines does. Rather, it uses a formula somewhat similar to the Pythagorean Theorem.

**Example 3**

Marisa’s, June’s, and Daniel’s houses form a triangle. The distance between June’s and Daniel’s houses is 1.2 km. Standing at June’s house, the angle formed by looking out to Daniel’s house and then to Marisa’s house is 63°. Standing at Daniel’s house, the angle formed by looking out to June’s house and then to Marisa’s house is 75°. What are the distances between each of the houses?

As with any application, it is helpful to draw a picture of the situation. One distance is already known: the distance from June’s house to Daniel’s house. Label the length of the side from \( D \) to \( J \) as 1.2. The situation also indicates that the \( m\angle J = 63^\circ \) and \( m\angle D = 75^\circ \). From this it can be determined that the \( m\angle M = 42^\circ \). The needed information is the lengths of \( DM \) and \( MJ \). To do this, use the Law of Sines.

\[
\begin{align*}
\text{MJ:} & \quad \frac{\sin(75^\circ)}{MJ} = \frac{\sin(42^\circ)}{1.2} \\
& \quad 1.2\sin(75^\circ) = (MJ)\sin(42^\circ) \\
& \quad \frac{1.2\sin(75^\circ)}{\sin(42^\circ)} = MJ \\
& \quad MJ \approx 1.7 \text{ km} \\
\text{DM:} & \quad \frac{\sin(63^\circ)}{DM} = \frac{\sin(42^\circ)}{1.2} \\
& \quad 1.2\sin(63^\circ) = (DM)\sin(42^\circ) \\
& \quad \frac{1.2\sin(63^\circ)}{\sin(42^\circ)} = DM \\
& \quad DM \approx 1.6 \text{ km}
\end{align*}
\]

Therefore the distances between the homes are: From Marisa’s to Daniel’s: 1.6 km, from Marisa’s to June’s: 1.7 km, and from Daniel’s to June’s: 1.2 km.
Problems

Solve for $x$, $y$, and/or $\theta$ in each diagram. Round all answers to the nearest tenth.

1. \[ \begin{array}{c}
    \text{7} \\
    \text{42°} \\
    \text{x} \\
    \text{9.1} \\
\end{array} \]

2. \[ \begin{array}{c}
    \text{6} \\
    \text{76°} \\
    \text{x} \\
    \text{17} \\
\end{array} \]

3. \[ \begin{array}{c}
    \text{9.4} \\
    \text{20°} \\
    \text{10} \\
    \text{x} \\
\end{array} \]

4. \[ \begin{array}{c}
    \text{y} \\
    \text{94°} \\
    \text{x} \\
    \text{15} \\
    \text{52°} \\
\end{array} \]

5. \[ \begin{array}{c}
    \text{θ} \\
    \text{30} \\
    \text{18°} \\
    \text{27} \\
\end{array} \]

6. \[ \begin{array}{c}
    \text{21} \\
    \text{13} \\
    \text{14} \\
\end{array} \]

7. \[ \begin{array}{c}
    \text{θ} \\
    \text{8} \\
    \text{59°} \\
    \text{5} \\
\end{array} \]

8. \[ \begin{array}{c}
    \text{x} \\
    \text{8.2} \\
    \text{38°} \\
    \text{43°} \\
\end{array} \]

9. \[ \begin{array}{c}
    \text{6.25} \\
    \text{1.75} \\
    \text{θ} \\
    \text{6} \\
\end{array} \]

10. \[ \begin{array}{c}
    \text{6.2} \\
    \text{93°} \\
    \text{x} \\
    \text{8.2} \\
\end{array} \]
11. Draw and label a triangle for each part below. Then use the given information to determine the required part(s).

17. \( \angle A = 40^\circ, \angle B = 88^\circ, a = 15 \)  
   Calculate \( b \).

18. \( \angle B = 75^\circ, a = 13, c = 14 \)  
   Calculate \( b \).

19. \( \angle B = 50^\circ, \angle C = 60^\circ, b = 9 \)  
   Calculate \( a \).

20. \( \angle A = 62^\circ, \angle C = 28^\circ, c = 24 \)  
   Calculate \( a \).

21. \( \angle A = 51^\circ, c = 8, b = 12 \)  
   Calculate \( a \).

22. \( \angle B = 34^\circ, \angle C = 98^\circ, b = 3 \)  
   Calculate \( a \).

23. \( a = 9, b = 12, c = 15 \)  
   Calculate \( \angle B \).

24. \( \angle B = 96^\circ, \angle A = 32^\circ, a = 6 \)  
   Calculate \( c \).

25. \( \angle C = 18^\circ, \angle B = 54^\circ, b = 18 \)  
   Calculate \( c \).

26. \( a = 15, b = 12, c = 14 \)  
   Calculate \( \angle C \).

27. \( \angle C = 76^\circ, a = 39, b = 19 \)  
   Calculate \( c \).

28. \( \angle A = 30^\circ, \angle C = 60^\circ, a = 8 \)  
   Calculate \( b \).

29. \( a = 34, b = 38, c = 31 \)  
   Calculate \( \angle B \).

30. \( a = 8, b = 16, c = 7 \)  
   Calculate \( \angle C \).

31. \( \angle C = 84^\circ, \angle B = 23^\circ, c = 11 \)  
   Calculate \( b \).

32. \( \angle A = 36^\circ, \angle B = 68^\circ, b = 8 \)  
   Calculate \( a \) and \( c \).

33. \( \angle B = 40^\circ, b = 4, \) and \( c = 6 \)  
   Calculate \( a \), \( \angle A \), and \( \angle C \).

34. \( a = 2, b = 3, c = 4 \)  
   Calculate \( \angle A \), \( \angle B \), and \( \angle C \).

35. Marco wants to cut a sheet of plywood to fit over the top of his triangular sandbox. One angle measures 38°, and it is between sides with lengths 14 feet and 18 feet. What is the length of the third side?

36. From the planet Xentar, Dweeble can see the stars Quazam and Plibit. The angle between these two stars is 22°. Dweeble knows that Quazam and Plibit are 93,000,000 miles apart. He also knows that when standing on Plibit, the angle made from Quazam to Xentar is 39°. How far is Xentar from Quazam?
Answers

1. \(x \approx 13.0, y = 107^\circ\)
2. \(x \approx 16.6\)
3. \(x \approx 3.4\)
4. \(x \approx 8.4, y = 34^\circ\)
5. \(\theta \approx 16.2^\circ\)
6. \(\theta \approx 37.3^\circ\)
7. \(\theta \approx 32.4^\circ\)
8. \(x \approx 9.1\)
9. \(\theta = 90^\circ\)
10. \(x \approx 10.5\)
11. \(x \approx 24.5\)
12. \(x \approx 22.6\)
13. \(x \approx 4.0\)
14. \(\theta \approx 83.3^\circ\)
15. \(x \approx 17.2\)
16. \(x \approx 11.3\)
17. \(b \approx 23.3\)
18. \(b \approx 16.5\)
19. \(a \approx 11.0\)
20. \(a \approx 45.1\)
21. \(a \approx 9.3\)
22. \(a \approx 4.0\)
23. \(m\angle B \approx 53.1^\circ\)
24. \(c \approx 8.9\)
25. \(c \approx 6.9\)
26. \(m\angle C \approx 61.3^\circ\)
27. \(c \approx 39.0\)
28. \(b = 16\)
29. \(m\angle B \approx 71.4^\circ\)
30. no possible triangle (Triangle Inequality)
31. \(b \approx 4.3\)
32. \(a \approx 5.0, c \approx 8.4\)
33. \(a \approx 5.7, m\angle A \approx 65.4^\circ, m\angle C \approx 74.6^\circ\)
34. \(m\angle A \approx 29.0^\circ, m\angle B \approx 46.6^\circ, m\angle C \approx 104.5^\circ\)
35. \(\approx 11.1\) feet
36. \(\approx 156,235,361\) miles
Sometimes the information known about the sides and angles of a triangle is not enough to determine one unique triangle. Sometimes a triangle may not even exist, as demonstrated by the Triangle Inequality. When a triangle formed is not unique (that is, more than one triangle can be made with the given conditions) this is called the ambiguous case. This can happen when two sides and an angle not between the two sides are given, known as SSA.

Example

In \( \triangle ABC \), \( m \angle A = 50^\circ \), \( AB = 12 \), and \( BC = 10 \). Can you make a unique triangle? If so, calculate all the angle measures and side lengths for \( \triangle ABC \). If not, show more than one triangle that meets these conditions.

As with many problems, it is helpful to first make a sketch of what the problem is describing. Once the diagram is labeled, it can be seen that the information displays the SSA pattern.

Use the Law of Sines to determine \( m \angle C \).

\[
\frac{\sin(50^\circ)}{10} = \frac{\sin(C)}{12}
\]

\[
12 \cdot \frac{\sin(50^\circ)}{10} = \sin(C)
\]

\[
0.919 = \sin(C)
\]

\( 66.8^\circ \approx \angle C \quad \Rightarrow \quad \angle B \approx 63.2^\circ \)

Because this triangle displays the SSA pattern, check to see if there is another possibly for \( \angle C \). See the diagram at right. The triangle formed with the two possible arrangements (the light grey triangle) is isosceles. From that it can be concluded that the two possibilities for \( \angle C \) are supplementary.

\( 180^\circ - 66.8^\circ = 113.2^\circ \), so another possibility is \( \angle C \approx 113.2^\circ \).

In this case, \( \angle B \approx 16.8^\circ \).

Use the Law of Cosines to determine the length of side \( AC \). In this case a quadratic equation needs to be solved, so use the Quadratic Formula.

\[
10^2 = 122 + x^2 - 2(12)(x)\cos(50^\circ)
\]

\[
100 = 144 + x^2 - 24x\cos(50^\circ)
\]

\[
100 \approx 144 + x^2 - 15.43x
\]

\[
x^2 - 15.43x + 44 \approx 0
\]

Both of these answers are positive numbers, and could be lengths of sides of a triangle. In one arrangement, \( \angle C \) is fairly small and \( AC \) is longer, while in the other arrangement, \( \angle C \) is large and \( AC \) is shorter.
Problems

Solve for the remaining parts of each triangle described below, explain why a triangle does not exist, or explain why there is more than one possible triangle.

1. In $\triangle ABC$, $\angle A = 32^\circ$, $AB = 20$, and $BC = 12$.
2. In $\triangle XYZ$, $\angle Z = 84^\circ$, $XZ = 6$, and $YZ = 9$.
3. In $\triangle ABC$, $m\angle A = m\angle B = 45^\circ$, and $AB = 7$.
4. In $\triangle PQR$, $PQ = 15$, $\angle P = 28^\circ$, and $PR = 23$.
5. In $\triangle XYZ$, $\angle X = 59^\circ$, $XY = 18$, and $YZ = 10$.
6. In $\triangle PQR$, $\angle P = 54^\circ$, $\angle R = 36^\circ$, and $PQ = 6$.

Answers

1. Two triangles: $AC \approx 22.6$, $m\angle B \approx 85.3^\circ$, $m\angle C \approx 62.7^\circ$ or $AC \approx 11.4$, $m\angle B \approx 30.0^\circ$, $m\angle C \approx 118.0^\circ$
2. One triangle: $XY \approx 10.3$, $m\angle X \approx 60.6^\circ$, $m\angle Y \approx 35.5^\circ$
3. One triangle: $m\angle C \approx 90^\circ$, $BC = AC = \frac{7\sqrt{2}}{2} \approx 4.9$
4. Two triangles: $QR \approx 30.7$, $m\angle Q \approx 46.0^\circ$, $m\angle P \approx 106.0^\circ$, or $QR \approx 9.90$, $m\angle Q \approx 134.0^\circ$, $m\angle P \approx 18.0^\circ$
5. No triangle exists.
6. One triangle: $m\angle Q = 90^\circ$, $QR \approx 8.3$, $PR \approx 10.2$
SAT Prep

1. If $-1 < t < 0$, which of the following statements must be true?
   a. $t^3 < t < t^2$
   b. $t^2 < t^3 < t$
   c. $t^2 < t < t^3$
   d. $t < t^3 < t^2$
   e. $t < t^2 < t^3$

2. Without taking a single break, Mercedes hiked for 10 hours, up a mountain and back down by the same path. While hiking, she averaged 2 miles per hour on the way up and 3 miles per hour on the way down. How many miles was it from the base of the mountain to the top?
   a. 4  b. 6  c. 9  d. 12  e. 18

3. When a certain rectangle is folded in half, it forms two squares. If the perimeter of one of these squares is 28, what is the perimeter of the original rectangle?
   a. 30  b. 42  c. 49  d. 56  e. Cannot be determined from the information given.

4. A class of 50 girls and 60 boys sponsored a road rally race. If 60% of the girls and 50% of the boys participated in the road rally, what percent of the class participated in the road rally?
   a. 54.5%  b. 55%  c. 57.5%  d. 88%  e. 110%

5. The sum of four consecutive integers is $s$. In terms of $s$, which of the following is the smallest of these four integers?
   a. $\frac{s-6}{4}$  b. $\frac{s-4}{4}$  c. $\frac{s-3}{4}$  d. $\frac{s-2}{4}$  e. $\frac{s}{4}$

6. On a certain map, 30 miles is represented by one-half inch. On the same map, how many miles are represented by 2.5 inches?

7. How many of the first one hundred positive integers contain the digit 9?

8. The sum of $n$ and $n + 1$ is greater than five but less than 15. If $n$ is an integer, what is one possible value of $n$?
9. In the figure at right, ΔABC is a right triangle and \( \frac{y}{6} = \frac{6}{10} \). What is the value of \( y \)?

10. For three numbers \( a, b, \) and \( c \), the average (arithmetic mean) is twice the median. If \( a < b < c, a = 0, \) and \( c = kb \), what is the value of \( k \)?

**Answers**

1. D
2. D
3. B
4. A
5. A
6. 150 miles
7. 19 integers
8. \( n \) can be 3, 4, 5, or 6
9. \( y = 3.6 \)
10. \( k = 5 \)
The chapter explores polynomial functions in greater depth. Students will learn how to sketch polynomial functions without using their graphing tool by using the factored form of the polynomial. In addition, they learn the reverse process: determining the polynomial equation from the graph.

For additional information see the Math Notes box in Lesson 8.1.1.

Example 1

State whether or not each of the following expressions is a polynomial. If it is not, explain why not. If it is a polynomial, state the degree of the polynomial.

a. \(-7x^4 + \frac{2}{3}x^3 + x^2 - 4.1x - 6\)
b. \(8 + 3.2x^2 - \pi x^5 - 61x^{10}\)
c. \(9x^3 + 4x^2 - 6x^{-1} + 7^x\)
d. \(x(x^3 + 2)(x^4 - 4)\)

A polynomial in one variable is an expression that can be written as the sum or difference of terms. The terms are in the form \(ax^n\) where \(a\) is any number called the coefficient of \(x\), and \(n\), the exponent, must be a whole number.

a. This is a polynomial. A coefficient that is a fraction \(\left(\frac{2}{3}\right)\) is acceptable. The degree of the polynomial is the largest exponent on the variable, so in this case the degree is 4.
b. This is also a polynomial, and its degree is 10.
c. This expression is not a polynomial for two reasons. First, the \(x^{-1}\) is not allowed because the exponents of the variable cannot be negative. The second reason is because of the \(7^x\). The variable cannot be a power in a polynomial.
d. Although the expression is not the sum or difference of terms, it can be written as the sum or difference of terms by multiplying the expression and simplifying. Doing so gives \(x^8 + 2x^5 - 4x^4 - 8x\), which is a polynomial of degree 8.
Example 2

Without using your graphing tool, make a sketch of each of the following polynomial functions by using the leading coefficient, the roots, and the degree.

a. \( f(x) = (x + 1)(x - 3)(x - 4) \)

b. \( y = (x - 2)^2(x + 3) \)

c. \( p(x) = x(x + 1)^2(x - 4)^2 \)

d. \( f(x) = -(x + 1)^3(x - 1)^2 \)

The roots of the polynomial are the \( x \)-intercepts, which are easily determined when the polynomial is in factored form, as are all the polynomials above, by using the Zero Product Property. The degree of the polynomial and leading coefficient can be determined by multiplying the first terms of each factor. Note that some factors are repeated. The graphs shown below are possible sketches.

a. The leading term will be \( x^3 \), so this graph will be a cubic function. The roots of this polynomial are \( x = -1, 3, \) and 4. To help sketch the graph, if \( x = 0 \), then \( f(0) = 12 \), so the \( y \)-intercept is (0, 12).

b. The leading term will be \( x^3 \), so this graph will be a cubic function. The distinct roots of this polynomial are \( x = -3 \) and 2. \( x = 2 \) is called a double root, since the expression \( (x - 2) \) is squared and is thus equivalent to \( (x - 2)(x - 2) \). The graph will just touch the \( x \)-axis at \( x = 2 \), and “bounce” off. The \( y \)-intercept is (0, 12).

c. This fifth degree polynomial with a leading term of \( x^5 \) has three distinct roots, \( x = 0, -1, \) and 4. Both \( x = -1 \) and \( x = 4 \) are double roots. The \( y \)-intercept is (0, 0).

d. This fifth degree polynomial with a leading term of \( -x^5 \) has two distinct roots, \( x = -1 \) and 1. \( x = 1 \) is a double root, and \( x = -1 \) is a triple root. The leading coefficient in negative, so this graph will be a vertical reflection of a typical fifth degree polynomial. The \( y \)-intercept is (0, -1).
Example 3

Write the equation of the graph shown at right.

From the graph, a general equation can be written based on the roots and y-intercept of the polynomial. Since the x-intercepts (roots) are \( x = -3, 3, \) and 8, then \((x + 3), (x - 3),\) and \((x - 8)\) are factors. Also, since the graph touches and bounces off at \( x = -3, \) \((x + 3)\) is a double root. Thus this function can be written as \( f(x) = a(x + 3)^2(x - 3)(x - 8). \) The value of \( a \) needs to be determined.

Using the fact that the graph passes through the point \((0, -2),\) write:

\[
\begin{align*}
-2 &= a(0 + 3)^2(0 - 3)(0 - 8) \\
-2 &= a(9)(-3)(-8) \\
-2 &= 216a \\
a &= \frac{-2}{216} = \frac{-1}{108}
\end{align*}
\]

Therefore the equation is \( f(x) = -\frac{1}{108}(x + 3)^2(x - 3)(x - 8). \)

Problems

State whether or not each of the following functions is a polynomial function. If it is, give the degree. If it is not, explain why not.

1. \( y = \frac{1}{8} x^7 + 4.23x^6 - x^4 - \pi x^2 + \sqrt{2}x - 0.1 \)

2. \( f(x) = 45x^3 - 0.75x^2 - \frac{3}{100} x + \frac{5}{x} + 15 \)

3. \( y = x(x + 2)\left(6 + \frac{1}{x}\right) \)

Sketch the graph of each of the following polynomial functions.

4. \( y = (x + 5)(x - 1)^2(x - 7) \)

5. \( y = -(x + 3)(x^2 + 2)(x + 5)^2 \)

6. \( f(x) = -x(x + 8)(x + 1) \)

7. \( y = x(x + 4)(x^2 - 1)(x - 4) \)
Below are the complete graphs of some polynomial functions. Based on the shape and location of the graph, describe all the roots of the polynomial function and state its minimum degree. Be sure to include information such as whether or not a root is a double or triple root.

8. 

9. 

10. 

Using the graphs below and the given information, write the specific equation for each polynomial function.

11. y-intercept: (0, 12) 

12. y-intercept: (0, −15) 

13. y-intercept: (0, 3)
**Answers**

1. Yes, degree 7.

2. No. You cannot have $x$ in the denominator.

3. No. When multiplied out, there is an $x$ in the denominator.

4. The roots are $x = -5, 1, 7$, with $x = 1$ being a double root.

5. The roots are $x = -3$ and $x = -5$, which is a double root. The $(x^2 + 2)$ factor does not produce any real roots since this expression cannot equal zero. The graph crosses the y-axis at $y = -150$.

6. This graph has roots $x = -8, -1, 0$.

7. $x^2 - 1$ gives us two roots. Since it factors to $(x + 1)(x - 1)$, the five roots are $x = -4, -1, 0, 1, 4$.

8. A third degree polynomial (cubic) with one root at $x = 0$, and one double root at $x = -4$.

9. A fourth degree polynomial with real roots at $x = -5$ and $-3$, and a double root at $x = 5$.

10. A fifth degree polynomial with five real roots at $x = -5, -1, 2, 4, 6$.

11. $y = (x + 3)(x - 1)(x - 4)$

12. $y = -0.1(x + 5)(x + 2)(x - 3)(x - 5)$

13. $y = \frac{1}{12} (x + 3)^2(x - 1)(x - 4)$
Complex numbers arise naturally when trying to solve some equations such as \( x^2 + 1 = 0 \). The solutions to this equation are \( x = \pm \sqrt{-1} \), or \( x = \pm i \).

Sometimes polynomials have complex roots. Complex roots always come in pairs called complex conjugates. For example, if \( x = 3 + 2i \) is a root, then \( x = 3 - 2i \) is also a root.

**Example 1**

Simplify each of the following expressions.

a. \( 3 + \sqrt{-16} \)  
b. \( (3 + 4i) + (-2 - 6i) \)

c. \( (4i)(-5i) \)  
d. \( (8 - 3i)(8 + 3i) \)

Remember that \( i = \sqrt{-1} \), or \( i^2 = -1 \).

a. \( 3 + \sqrt{-16} = 3 + 4\sqrt{-1} = 3 + 4i \)

b. Real parts can be combined with real parts, and imaginary parts with imaginary parts:  
\( (3 + 4i) + (-2 - 6i) = 1 - 2i \)

c. \( (4i)(-5i) = (4 \cdot -5)(i \cdot i) = -20i^2 = -20(-1) = 20 \)

d. The Distributive Property or an area model can be used to compute this product.

\[
(8 - 3i)(8 + 3i) = 8(8) + 8(3i) - 3i(8) - 3i(3i) \\
= 64 + 24i - 24i + 9 \\
= 73
\]

The two factors are called complex conjugates, and they are useful when working with complex numbers. 
Multiplying a complex number by its conjugate produces a real number! This will always happen. Also, whenever a function with real coefficients has a complex root, it always has the conjugate as a root as well.
**Example 2**

Determine the roots of the function below using the Quadratic Formula. Explain what the roots tell you about the graph of the function.

\[ f(x) = 2x^2 - 20x + 53 \]

Quadratic Formula: If \( ax^2 + bx + c = 0 \) then \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \).

The roots of the function occur when \( f(x) = 0 \). Here, \( a = 2 \), \( b = -20 \), and \( c = 53 \). The solution is shown at right.

\[
x = \frac{-(20) \pm \sqrt{(-20)^2 - 4(2)(53)}}{2(2)}
\]

\[
x = \frac{20 \pm \sqrt{400 - 424}}{4}
\]

\[
x = \frac{20 \pm \sqrt{-4}}{4}
\]

\[
x = \frac{20 \pm 2i\sqrt{6}}{4}
\]

\[
x = \frac{2(10 \pm i\sqrt{6})}{4}
\]

\[
x = \frac{10 \pm i\sqrt{6}}{2}
\]

This creates an expression with a negative under the radical. This equation has no real solutions, but it does have complex solutions.

In mathematics, \( i = \sqrt{-1} \) is defined as an imaginary number. When an imaginary number is combined with a real number, then it is called a complex number. Complex numbers are written in the form \( a + bi \).

Using \( i = \sqrt{-1} \), the answer above can be simplified.

Therefore, the graph of the equation \( y = 2x^2 - 20x + 53 \) has no \( x \)-intercepts, but it does have two complex roots, \( x = \frac{10 \pm i\sqrt{6}}{2} \). Recall that the degree of a polynomial function indicates the maximum number of roots. In fact the degree indicates the exact number of roots; some (or all) which might be complex.
Example 3

Make a sketch of a graph of a polynomial function \( y = p(x) \) so that \( p(x) \) has exactly four real roots. Then change the graph so that \( p(x) \) has two real roots and two complex roots.

If \( p(x) \) has four real roots, then this will be a fourth degree polynomial that crosses the \( x \)-axis at exactly four different places. One such graph is shown at right.

In order for the graph to have only two real and two complex roots, it must be changed so one of the “dips” does not reach the \( x \)-axis. One example is shown at right.

Problems

Simplify the following expressions.

1. \((6 + 4i) - (2 - i)\) \hspace{1cm} 2. \(8i - \sqrt{-16}\) \hspace{1cm} 3. \((-3)(4i)(7i)\)

4. \((5 - 7i)(-2 + 3i)\) \hspace{1cm} 5. \((3 + 2i)(3 - 2i)\) \hspace{1cm} 6. \((\sqrt{3} - 5i)(\sqrt{3} + 5i)\)

Below are the graphs of some polynomial functions. Based on the shape and location of the graph, describe all the roots. Be sure to include information such as whether roots are double or triple, real or complex, etc.

7.

8.

9. Write the specific equation for the polynomial function passing through the point \((0, 5)\), and with roots \(x = 5, x = -2, \) and \(x = 3i\).
Answers

1. $4 + 5i$

2. $4i$

3. 84

4. $11 + 29i$

5. 13

6. 28

7. A third degree polynomial with one real root at $x = 5$ and two complex roots.

8. A fifth degree polynomial with one real root at $x = -4$ and four complex roots.

9. $y = -\frac{1}{18} (x - 5)(x + 2)(x - 3i)(x + 3i) = -\frac{1}{18} (x^2 - 3x - 10)(x^2 + 9)$
Students learn to divide polynomials as one method for factoring polynomials of degrees higher than 2. Through division, polynomials can be rewritten in a form more suitable for graphing. A polynomial’s roots, both real and complex, can be determined from the factored form.

For additional information see the Math Notes boxes in Lessons 8.3.2 and 8.3.3.

Example 1

Divide \( x^3 + 4x^2 - 7x - 10 \) by \( x + 1 \).

Using an area model is a method that works for polynomial division.

For division, start with the area and one dimension of the rectangle, then use the area model to determine the other dimension.

Set up a rectangle that has a width of \( x + 1 \) and an area of \( x^3 + 4x^2 - 7x - 10 \). The length is unknown at this time, however. So add information to the diagram gradually, adjusting as needed. The top left section has an area equal to the highest-powered term: \( x^3 \).

Now work backwards: If the area of this section is \( x^3 \) and one side has a length of \( x \), what does the length of the other side have to be? It has to be \( x^2 \). If the missing side length is \( x^2 \), then the area of the lower section is \( 1(x^2) \).

The total area is \( x^3 + 4x^2 - 7x - 10 \), but there are only portions of the rectangle equal to \( 1x^3 \) and \( 1x^2 \) so far. So \( 3x^2 \) more needs to be added to the total area for the second term of the expression.

Once the remaining “\( x^2 \)” area is filled in then the length of the top side can be determined. Remember that part of the left side has a length of \( x \). This means that part of the top must have a length of \( 3x \).

Use this new piece of information to compute the area of the rectangle that is to the right of the \( x^3 \) rectangle, and then the small rectangle below that result.
The total area has a total of \(-7x\), but there is only \(3x\) so far. This means \(-10x\) more needs to be added. Place this amount of area in the rectangle to the right of \(3x^2\).

With this new piece of area added, the top piece’s length can be computed and used to calculate the area of the rectangle below the \(-10x\). Note that the constant term in the total area is \(-10\), which is what the rectangle has as well.

Therefore, \(\frac{x^3+4x^2-7x-10}{x+1} = x^2 + 3x - 10\), or \(x^3 + 4x^2 - 7x - 10 = (x + 1)(x^2 + 3x - 10)\).

**Example 2**

Determine all of the roots of \(P(x) = x^4 + x^2 - 14x - 48\).

The roots of a polynomial can be determined from its factors. If this polynomial is factored, it will look like \((x + a)(x + b)(x + c)(x + d)\) where \((a)(b)(c)(d) = -48\). This means the possible real roots of this polynomial are \(\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 16, \pm 24, \) or \(\pm 48\). In this case there are 20 possible roots to check! These can be checked in a number of ways. One method is to divide the polynomial by the corresponding binomial expression (for instance, if \(x = -1\) is a root, divide the polynomial by \((x + 1)\) to see if it is a factor. Another method is to substitute each zero into the polynomial to see which of them, if any, make the polynomial equal to zero.

Use a graphing calculator to check which of the possible roots might actually be roots. In this case the graph shows that \(x = -2\) and \(x = 3\) might be roots. Verify by using substitution.

\[
P(-2) = (-2)^4 + (-2)^2 - 14(-2) - 48 = 16 + 4 + 28 - 48 = 0
\]
\[
P(3) = (3)^4 + (3)^2 - 14(3) - 48 = 81 + 9 - 42 - 48 = 0
\]

We will start with \(x = -2\). Since there is a root at \(x = -2\), \((x + 2)\) is a factor of the polynomial. Now divide the polynomial by this factor to determine the other factors.
Now, \( P(x) \) can be written as \( P(x) = (x + 2)(x^3 - 2x^2 + 5x - 24) \).
The cubic factor still needs to be factored. We know that \( x = 3 \) is also a root and therefore \((x + 3)\) is a factor.
Therefore \( x^3 - 2x^2 + 5x - 24 \) should be divided by \((x + 3)\) as shown in the diagram at right.

This means that \( P(x) = (x + 2)(x - 3)(x^2 + x + 8) \). The last factor is a quadratic (degree 2) so factor or use the Quadratic Formula to determine the corresponding roots (zeros). In this case, the expression cannot be factored, so use the Quadratic Formula to calculate the roots as shown at right.

Therefore, the original polynomial factors as:

\[
P(x) = x^4 + x^2 - 14x - 48 = (x + 2)(x - 3) \left( x - \frac{-1 + i\sqrt{31}}{2} \right) \left( x - \frac{-1 - i\sqrt{31}}{2} \right)
\]

The roots of the polynomial are \( x = -2, 3, \frac{-1 + i\sqrt{31}}{2}, \frac{-1 - i\sqrt{31}}{2} \).

**Problems**

1. Divide \( 3x^3 - 5x^2 - 34x + 24 \) by \( 3x - 2 \).
2. Divide \( x^3 + x^2 - 5x + 3 \) by \( x - 1 \).
3. Divide \( 6x^3 - 5x^2 + 5x - 2 \) by \( 2x - 1 \).

Determine all of the roots of each of the following polynomials.

4. \( f(x) = 2x^3 + x^2 - 19x + 36 \)
5. \( g(x) = x^4 - x^3 - 11x^2 - 5x + 4 \)
6. \( P(x) = x^4 - 2x^3 + x^2 - 8x - 12 \)
7. \( Q(x) = x^3 - 14x^2 + 65x - 102 \)

**Answers**

1. \( x^2 - x - 12 \)
2. \( x^2 + 2x - 3 \)
3. \( 3x^2 - x + 2 \)
4. \( x = -4, \frac{7 + i\sqrt{23}}{4}, \frac{7 - i\sqrt{23}}{4} \)
5. \( x = -1, 4, -1 + \sqrt{2}, -1 - \sqrt{2} \)
6. \( x = -1, 3, 2i, -2i \)
7. \( x = 6, 4 + i, 4 - i \)
SAT PREP

1. \((5 + 6)^2 = ?\)
   a. \((2 \cdot 5) + (2 \cdot 6)\)  b. \(5^2 + 6^2\)  c. \(11^2\)  d. 61  e. \(5^2 \times 6^2\)

2. If \(6x – 7y = 12\), what is the value of \(-2(6x – 7y)\)?
   a. −24  b. 13  c. −1  d. 420  e. −42

3. The average (arithmetic mean) of three numbers is 25. If two numbers are 25 and 30, what is the third number?
   a. 35  b. 30  c. 25  d. 20  e. 15

4. People from the country of Turpa measure with different units. Each curd is 7 garlongs long and each garlong is made up of 15 bleebs. How many complete curds are there in 510 bleebs?
   a. 105  b. 15  c. 5  d. 4  e. 2

5. If \(x^2 – y^2 = 12\) and \(x – y = 2\), what is the value of \(x + y\)?

6. Five consecutive integers sum to 25. What is the largest of these consecutive numbers?

7. For all positive integers \(m\) and \(n\), we define \(m \nearrow n\) to be the whole number remainder when \(m\) is divided by \(n\). If \(11 \nearrow k = 3\), what does \(k\) equal?

8. At Pies R Us, each pie is cut into a slice as shown in the figure at right. Each slice of pie has a central angle of 30°. They sell the pies by the slice. If the weight of each pie is uniformly distributed, weighing 108 grams, how much does each slice weigh in grams?

9. In the figure at right, what is the area of the shaded region if that region is a square?

10. What is the sixth term in the sequence beginning 432, 72, 12, … ?

Answers

6. 7  7. 8  8. 9 g  9. 20 un^2  10. \(\frac{1}{18}\)
This chapter introduces the unit circle and students explore how the unit circle can be used to calculate exact values of trigonometric functions. In addition, they learn how to graph and transform \( y = \sin(x) \), \( y = \cos(x) \), and \( y = \tan(x) \).

For additional information see the Math Notes boxes in Lessons 9.1.6 and 9.1.7.

Example 1

As Daring Davis stands in line waiting to ride the huge Ferris wheel, he notices that this Ferris wheel is not like any of the others he has ridden. First, this Ferris wheel does not board the passengers at the lowest point of the ride; rather, they board after climbing several flights of stairs, at the level of that wheel’s horizontal axis. Also, if Davis thinks of the boarding point as a height of zero above that axis, then the maximum height above the boarding point that a person rides is 25 feet, and the minimum height below the boarding point is \(-25\) feet. Use this information to create a graph that shows how a passenger’s height on the Ferris wheel depends on the number of degrees of rotation from the boarding point of the Ferris wheel.

As the Ferris wheel rotates counterclockwise, a passenger’s height above the horizontal axis increases, and reaches its maximum of 25 feet above the axis after 90° of rotation. Then the passenger’s height decreases as measured from the horizontal axis, reaching 0 feet after 180° of rotation, and continues to decrease as measured from the horizontal axis. The minimum height, \(-25\) feet, occurs when the passenger has rotated 270°. After rotating 360°, the passenger is back where he started, and the ride continues.

To create this graph, calculate the height of the passenger at various points along the rotation. These heights are shown using the grey line segments drawn from the passenger’s location on the wheel perpendicular to the horizontal axis of the Ferris wheel.

<table>
<thead>
<tr>
<th>Angle of rotation (degrees)</th>
<th>0°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
<th>135°</th>
<th>180°</th>
<th>210°</th>
<th>225°</th>
<th>270°</th>
<th>315°</th>
<th>360°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (feet)</td>
<td>0</td>
<td>25</td>
<td></td>
<td></td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

At 0°, the height above the horizontal axis is 0 feet. At 90°, the height is 25 feet. To complete the rest of the table, calculate the heights using right triangle trigonometry. Three of these values are demonstrated as follows, 30°, 135°, and 225°.
Each of these calculations involves focusing on the portion of the picture that makes a right triangle. For the 30° angle, use a right triangle with a hypotenuse of 25 feet. (The radius of the circle is 25 feet because it is the maximum and minimum height the passenger reaches.) For this right triangle, use the sine function:

\[
\sin(30°) = \frac{h}{25}
\]

\[
25 \sin(30°) = h
\]

\[
h = 12.5 \text{ feet}
\]

For the 135° angle, use the right triangle on the “outside” of the curve. Since the angles are supplementary, the angle used measures 45°.

\[
\sin(45°) = \frac{h}{25}
\]

\[
25 \sin(45°) = h
\]

\[
h = 17.68 \text{ feet}
\]

For the 225° angle, the triangle used drops below the horizontal axis. Use the 45° angle that is within the right triangle, so \( h = -17.68 \), using the previous calculation and changing the sign to represent that the rider is below the starting point. Now fill in the rest of the values of the table.

<table>
<thead>
<tr>
<th>Angle of rotation (degrees)</th>
<th>0°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
<th>135°</th>
<th>180°</th>
<th>210°</th>
<th>225°</th>
<th>270°</th>
<th>315°</th>
<th>360°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (feet)</td>
<td>0</td>
<td>12.5</td>
<td>17.68</td>
<td>21.65</td>
<td>25</td>
<td>17.68</td>
<td>0</td>
<td>−12.5</td>
<td>−17.68</td>
<td>−25</td>
<td>−17.68</td>
<td>0</td>
</tr>
</tbody>
</table>

Plot these points and connect them with a smooth curve; the graph should look like the one at right. Note: This curve shows two revolutions of the Ferris wheel. This curve continues, repeating the cycle for each revolution of the Ferris wheel. It also represents a particular sine function: \( y = 25 \sin(\theta) \).
Example 2

On a unit circle, represent and then calculate $\cos(60^\circ)$, $\cos(150^\circ)$, and $\cos(315^\circ)$. Then graph $y = \cos(\theta)$.

Trigonometric functions ("trig" functions) arise naturally in circles as seen with the first example. A circle of radius 1 unit is called a unit circle and this circle that is most often used with trig functions.

On the unit circle at right, several points are labeled. Point $P$ corresponds to a $60^\circ$ rotation, point $Q$ corresponds to $150^\circ$, and point $R$ corresponds to $315^\circ$. Rotations are measured from the point $(1, 0)$ counter-clockwise to determine the angle. If right triangles are created at each of these points, then right triangle trig can be used to determine the lengths of the legs of the triangle. In the previous example, the height of the triangle was found by using sine. Here, cosine gives the length of the other leg of the triangle.

To fully understand why the length of the horizontal leg is labeled with "cosine," consider the triangle below. In the first triangle, if the short leg is labeled $x$, then:

$$\cos(60^\circ) = \frac{x}{1}$$

Therefore the length of the horizontal leg of the first triangle is $\cos(60^\circ)$. Note: The second triangle representing $150^\circ$, lies in the second quadrant where the $x$-values are negative. Therefore $\cos(150^\circ) = -\cos(30^\circ)$. Check this using your calculator.

On a unit circle, point $P$ corresponds to a rotation of $\theta$ degrees. If a right triangle is created by dropping a height from point $P$ to the $x$-axis, the length of this height is always $\sin(\theta)$. The length of the horizontal leg is always $\cos(\theta)$. Additionally, this means that the coordinates of point $P$ are $(\cos(\theta), \sin(\theta))$. This is the power of using a unit circle: the coordinates of any point on the circle are found by determining the sine and cosine of the angle. The graph at right shows the cosine curve for two rotations around the unit circle.
Example 3

On a unit circle, determine the points that correspond to the following radians. Then convert each angle given in radians to degrees.

a. \( \frac{\pi}{6} \)  
b. \( \frac{11\pi}{12} \)  
c. \( \frac{5\pi}{4} \)  
d. \( \frac{5\pi}{3} \)

Think of the unit circle, and remember that one rotation is the same as traveling around the unit circle one time, which is the circumference. The circumference of the unit circle is \( C = 2\pi r = 2\pi(1) = 2\pi \). Therefore, one rotation around the circle, 360°, is the same as traveling 2\( \pi \) radians around the circle. Radians do not just apply to unit circles. A circle with any size radius still has 2\( \pi \) radians in a 360° rotation.

Place these points around the unit circle in appropriate places without converting them. If 2\( \pi \) radians is the same as a 360°, then half of that, 180°, corresponds to \( \pi \) radians. Half of that, 90°, is \( \frac{\pi}{2} \) radians. Using similar reasoning, 270° corresponds to \( \frac{3\pi}{2} \) radians. The other radian measures around the circle can be placed with a simple knowledge of fractions. For example, \( \frac{\pi}{6} \) is one-sixth the distance to \( \pi \).

It is helpful to be able to convert from radians to degrees and back. To do so, use a ratio of \( \frac{\text{radians}}{\text{degrees}} \). To convert \( \frac{\pi}{6} \) radians to degrees, create a ratio and solve as shown at right. Use \( \frac{\pi}{180} \) as a simpler form of \( \frac{2\pi}{360} \). Therefore \( \frac{\pi}{6} \) radians is equivalent to 30°. Similarly, convert the other angles above to degrees:

a. \( \frac{\pi}{180} = \frac{\pi/6}{x} \)  
   \( x\pi = 180^\circ \left( \frac{\pi}{6} \right) \)  
   \( x = 30^\circ \)
b. \( \frac{\pi}{180} = \frac{11\pi/12}{x} \)  
   \( x\pi = 180^\circ \left( \frac{11\pi}{12} \right) \)  
   \( x = 165^\circ \)
c. \( \frac{\pi}{180} = \frac{5\pi/4}{x} \)  
   \( x\pi = 180^\circ \left( \frac{5\pi}{4} \right) \)  
   \( x = 225^\circ \)
d. \( \frac{\pi}{180} = \frac{5\pi/3}{x} \)  
   \( x\pi = 180^\circ \left( \frac{5\pi}{3} \right) \)  
   \( x = 300^\circ \)
Example 4

Graph \( y = \tan(\theta) \). Explain what happens at the points \( \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots \).
Why does this happen?

As with the graphs of \( y = \sin(\theta) \) and \( y = \cos(\theta) \), \( y = \tan(\theta) \) repeats, that is, it is periodic. The graph does not, however, have the familiar hills and valleys the other two trig functions display. At \( \theta = \frac{\pi}{2} \), the graph approaches a vertical asymptote. This also occurs at \( \theta = -\frac{\pi}{2} \), and because the graph is periodic, it happens repeatedly at \( \theta = \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots \). In fact, it happens at all values of \( \theta \) of the form \( \frac{(2k-1)\pi}{2} \) for all integer values of \( k \).

The real question is, why does this asymptote occur at these points? Recall that \( \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \).
At every point where \( \cos(\theta) = 0 \), \( y = \tan(\theta) \) is undefined (zero cannot be in the denominator).

Problems

Graph each of the following trigonometric functions.

1. \( y = \sin(\theta) \) 
2. \( y = \cos(\theta) \) 
3. \( y = \tan(\theta) \)

Determine each of the following values by using what you know about right triangle trigonometry, the unit circle, and special right triangles.

4. \( \cos(180^\circ) \) 
5. \( \sin(360^\circ) \) 
6. \( \tan(45^\circ) \)

7. \( \cos(-90^\circ) \) 
8. \( \sin(150^\circ) \) 
9. \( \tan(240^\circ) \)

Convert each of the angle measures.

10. \( 60^\circ \) to radians 
11. \( 170^\circ \) to radians 
12. \( 315^\circ \) to radians

13. \( \frac{\pi}{15} \) radians to degrees 
14. \( \frac{13\pi}{8} \) radians to degrees 
15. \( -\frac{3\pi}{4} \) radians to degrees
Answers

1. 

2. 

3. 

4. $-1$

5. $0$

6. $1$

7. $0$

8. $\frac{1}{2}$

9. $\sqrt{3}$

10. $\frac{\pi}{3}$ radians

11. $\frac{17\pi}{18}$ radians

12. $\frac{7\pi}{4}$ radians

13. $12^\circ$

14. $292.5^\circ$

15. $-135^\circ$
Students apply their knowledge of transforming parent graphs to the graphs of trigonometric functions. They generate general equations for the family of sine, cosine, and tangent functions, and learn about a property of trigonometric functions called the period.

See the Math Notes box in Lesson 9.2.4 for additional information.

Example 1

For each of the following functions, state the amplitude, number of cycles in $2\pi$, horizontal shift, and vertical shift of the graph. Then graph each function.

a. $y = 3 \cos\left(2(x - \frac{\pi}{3})\right) - 2$

b. $y = -\sin\left(\frac{1}{4}(x + \pi)\right) + 1$

The general form of a sine function is $y = a \sin[b(x - h)] + k$. The value of $a$ determines the amplitude of the function: half of the distance the function stretches from the maximum and minimum points vertically. $h$ is the horizontal shift, and $k$ is the vertical shift. $b$ gives information about the period of the function. The graphs of $y = \sin(x)$ and $y = \cos(x)$ each have a period of $2\pi$, which means that one cycle (before it repeats) has a length of $2\pi$. $b$ indicates the number of cycles that occur in the length $2\pi$.

a. The function has an amplitude of 3, and since this is positive, it is not reflected across the $x$-axis. The graph is shifted to the right $\frac{\pi}{3}$ units, and shifted down 2 units. Since $b = 2$, we know that there are two cycles in $2\pi$ units. If the graph has two cycles in $2\pi$ units, then the length of the period is $\pi$ units. The graph is shown at right.

b. This function has an amplitude of 1, but it is reflected across the $x$-axis. It is shifted to the left $\pi$ units, and shifted up 1 unit. Here, within a $2\pi$ span, only one fourth of a cycle appears. This means the period is four times as long as normal, or is $8\pi$. The graph is shown at right.
Example 2

For the Fourth of July parade, Vicki decorated her tricycle with streamers and balloons. She stuck one balloon on the outside rim of one of her back tires. The balloon starts at ground level. As she rides, the height of balloon rises up and down, sinusoidally (that is, a sine curve). The diameter of her tire is 10 inches.

a. Sketch a graph showing the height of the balloon above the ground as Vicki rolls along.

b. What is the period of this graph?

c. Write the equation of this function.

d. Use your equation to predict the height of the balloon after Vicki has traveled 42 inches.

This problem is similar to the Ferris wheel example at the beginning of this chapter. The balloon is rising up and down just as a sine or cosine curve rises up and down. A simple sketch is shown at right.

The balloon begins next to the ground and as the tricycle wheel rolls, the balloon rises to the top of the wheel, then comes back down. If the x-axis represents the ground, the balloon is at its highest point when it is at the top of the wheel, a distance of one wheel's height or diameter, 10 inches. So the distance from the highest point to the lowest point is 10. The amplitude is half of this distance, or 5.

To determine the period, think about the situation. The balloon starts at ground level, rises as the wheel rolls and comes down again to the ground. How far has the wheel traveled in one complete revolution? It has traveled the distance of one circumference. The circumference of a circle with diameter 10 inches is \(10\pi\) inches. Therefore the period of this graph is \(10\pi\).

To write the equation for this situation some decisions need to be made. The graphs of sine and cosine are similar. In fact, one is just the other shifted \(90^\circ\) (or \(\frac{\pi}{2}\) radians). At this point, decide whether to use sine or cosine to model this situation. Either one will work, but the equations will be different. Since the graph starts at the lowest point and not in the middle, this suggests that cosine might be easier. Using cosine, the amplitude is 5, the graph is reflected vertically, and there is no horizontal shift. All of this information can be written in the equation as \(y = -5\cos(bx) + k\). Determine \(k\) by remembering that the x-axis represents the ground. This implies the graph is shifted up 5 units. To determine the number of cycles in \(2\pi\) (that is, \(b\)), recall that the period of this graph is \(10\pi\). Therefore \(\frac{2\pi}{10} = \frac{1}{5}\) of the curve appears within the \(2\pi\) span. Finally, the equation can be written as \(y = -5\cos\left(\frac{1}{5} x\right) + 5\). This is shown in the graph at right.
Chapter 9

Note: If you decided to use the sine function for this situation, you must realize that the graph is shifted to the right $\frac{10\pi}{4}$ units. One equation that gives this graph is $y = 5 \sin \left( \frac{1}{4} (x - \frac{10\pi}{4}) \right) + 5$. There are other equations that work, so if you do not get the same equation as shown here, graph yours and compare.

To calculate the height of the balloon after Vicki rides 42 inches, substitute 42 for $x$ in the equation.

If you do not get this answer, make sure your calculator is in radian mode!

Problems

State the amplitude and period of each function graphed below.

1. \[ y = 2\cos(3x) + 7 \]
2. \[ y = \frac{1}{2} \sin(x) - 6 \]
3. \[ f(x) = -3\sin(4x) \]
4. \[ y = \sin \left( \frac{1}{3} x \right) + 3.5 \]
5. \[ f(x) = -\cos(x) + 2\pi \]
6. \[ f(x) = 5 \cos(x - 1) - \frac{1}{4} \]

Note: If you decided to use the sine function for this situation, you must realize that the graph is shifted to the right $\frac{10\pi}{4}$ units. One equation that gives this graph is $y = 5 \sin \left( \frac{1}{4} (x - \frac{10\pi}{4}) \right) + 5$. There are other equations that work, so if you do not get the same equation as shown here, graph yours and compare.

To calculate the height of the balloon after Vicki rides 42 inches, substitute 42 for $x$ in the equation.

If you do not get this answer, make sure your calculator is in radian mode!
Sketch the graphs of each of the following functions by hand. Use a graphing calculator to check your answer.

11. \( y = -2\sin(x + \pi) \)

12. \( f(x) = \frac{1}{2} \sin(3x) \)

13. \( f(x) = \cos\left(4 \left(x - \frac{\pi}{4}\right)\right) \)

14. \( y = 3\cos\left(x + \frac{\pi}{4}\right) + 3 \)

15. \( f(x) = 7\sin\left(\frac{1}{4}x\right) - 3 \)

16. A wooden water wheel makes ten revolutions every minute. At its lowest point it dips down 2 feet below the surface of the water and at its highest point it is 18 feet above the water. A snail attaches to the edge of the wheel when the wheel is at its lowest point and rides the wheel as it goes around. Use this information to write an equation that gives the height of the snail over time.

17. To keep baby Cristina entertained, her mother often puts her in a Baby Jump Up. It is a seat on the end of a strong spring that attaches in a doorway. When her mom puts Cristina in, she notices that the seat drops to just 8 inches above the floor. Cristina starts to jump and 1.5 seconds later, the seat reaches its highest point of 20 inches above the ground. The seat continues to bounce up and down as time passes. Use this information to write the equation that gives the height of Cristina’s Baby Jump Up seat over time. (Note: You can start the graph at the point where the seat is at its lowest point.)
Answers

1. amplitude: 2, period: $\pi$
2. amplitude: 0.5, period: $2\pi$
3. amplitude: 3, period: $4\pi$
4. amplitude: 2.5, period: $\frac{\pi}{3}$
5. amplitude: 2, period: $\frac{2\pi}{3}$
6. amplitude: $\frac{1}{2}$, period: $2\pi$
7. amplitude: 3, period: $\frac{\pi}{2}$
8. amplitude: 1, period: $6\pi$
9. amplitude: 1, period: $2\pi$
10. amplitude: 5, period: $2\pi$

11. 

12. 

Surprised? The negative reflects the graph vertically, but the “+ $\pi$” shifts it right back to how it looks originally!

13. 

14. 

15. 

16. $y = -10 \cos\left(\frac{1}{10} x\right) + 8$; Other equations are possible.

17. If the $x$-axis represents the ground, then a possible equation is $y = -6 \cos\left(\frac{2\pi}{3} x\right) + 14$.
1. If one “pentaminute” is the same as five minutes of time, how many pentaminutes are equivalent to four hours of time?
   a. 1200 b. 240 c. 60 d. 48 e. 20

2. If \(a = 12\) and \(b = -4\), what is the value of \(4a - 3b\)?
   a. 60 b. 36 c. 16 d. 9 e. -52

3. The average (arithmetic mean) of 4 and \(s\) is equal to the average of 3, 8, and \(s\). What does \(s\) equal?
   a. 3 b. 5.5 c. 9 d. 10 e. No such \(s\) exists.

4. In the figure at right, \(AB = CD\). What does \(k\) equal?
   a. -6 b. -5 c. -4
d. -3 e. -2

5. The initial term of a sequence is 36. Each term after that is half of the term before it if that term is even. If the preceding term is odd, the next term is half that term plus 0.5. What is the sixth term of this sequence?
   a. 1 b. 2.25 c. 2 d. 3.5 e. 4

6. At a spa, the customer is offered a choice of five different massages and eight different pedicures. How many different combinations are there of one massage and one pedicure?
   a. 3 b. 13 c. 16 d. 28 e. 40

7. A rectangular box is 12 cm long, 20 cm wide, and 15 cm high. If exactly 60 smaller identical rectangular boxes can be stored perfectly in this larger box, which of the following could be the dimensions, in cm, of these smaller boxes?
   a. 5 by 6 by 12 b. 4 by 5 by 6 c. 3 by 5 by 6
d. 3 by 4 by 6 e. 2 by 5 by 6
8. When Harry returned his book to the library, Madame Pince told him he owed a fine of $6.45. This included $3.00 for three weeks, plus a fine of $0.15 per day for every day he was late in returning the book. How many overdue days did Harry have the book?

9. What is the slope of the line that passes through the points (0, 2) and (–10, –2)?

10. At right is the complete graph of the function $f$. For how many positive values of $x$ does $f(x) = 3$?

Answers

1. D
2. A
3. D
4. C
5. C
6. E
7. E
8. 23 days
9. $\frac{2}{5}$
10. 2
ARITHMETIC SERIES

10.1.1 – 10.1.4

This chapter revisits sequences (from Core Connections Integrated I)—arithmetic and geometric—and extends the topic to series, and how they arise in situations. A sequence is a list of numbers, whereas a series is the sum of those numbers. Methods for determining those sums and a compact way to write the sums, known as summation notation, are developed. These ideas are related to Pascal’s Triangle, the Binomial theorem, and natural logarithms.

For additional information, see the Math Notes boxes in Lessons 10.1.3 and 10.1.4.

Example 1

Calculate the sum of each of the following series.

a. \( 4 + 9 + 14 + 19 + 24 + \ldots + 59 \)

b. Twelve terms with \( t(1) = 3 \) and \( t(12) = 69 \).

c. \( t(n) = -3n + 10 \), for integer values starting at 1 and ending at 15.

Each of these problems represents an arithmetic series because there is a constant difference between each consecutive term. They are series because they are the sums of the terms.

a. One method is to simply add each term, filling in the terms represented by the “…,” but that takes some time and there is a good chance for an arithmetic error. Instead, use one of the methods developed in the chapter.

First, determine the formula for the \( n \)th term of this sequence, as well as which term the number 59 is. The formula for the \( n \)th term is \( t(n) = 5(n - 1) + 4 \) (verify this!) and by setting this equal to 59, this means that 59 is the 12th term. To calculate the sum, write out the sum labeling it \( S_{12} \). Repeat this by writing it in reverse order. Adding these two equations gives a new equation that makes it easier to solve for \( S \). The sum of these twelve terms is 378.

b. Determine the formula for the \( n \)th term by using the given information to write an equation and solve. This gives the formula \( t(n) = 6(n - 1) + 3 \). Knowing the first and last terms is enough information to use the method shown in part (a) to determine the sum. The method generalizes as follows: add the first and last terms, multiply the result by the number of terms in the series, then divide by 2. \( S_{12} = \frac{12(3+69)}{2} = 432 \)

\[ t(n) = d(n - 1) + t(1) \]
\[ t(12) = d(12 - 1) + t(1) \]
\[ 69 = 11d + 3 \]
\[ d = 6 \]
c. The formula for the \( n^{\text{th}} \) term is known and there are 15 terms. Calculate the first and the last term, and then calculate the sum by using \( S(n) = \frac{n(t(1) + t(n))}{2} \).

\[
\begin{align*}
t(n) &= -3n + 10 \\
t(1) &= -3(1) + 10 \\
t(1) &= 7 \\
t(15) &= -3(15) + 10 \\
t(15) &= -45 + 10 \\
t(15) &= -35 \\
S(15) &= \frac{15(7 - 35)}{2} \\
S(15) &= -210
\end{align*}
\]

**Example 2**

Expand the series \( \sum_{k=1}^{15} (5k - 7) \) and calculate the sum.

The expression above is written using summation notation, and it is a shorthand way to write a series. \( \Sigma \) is the Greek letter sigma and stands for “sum.” \( k \) (called the index when using summation notation) starts at 1 and represents each integer up to 15 in the expression \( 5k - 7 \). This is written out as:

\[
\sum_{k=1}^{15} (5k - 7) = [5(1) - 7] + [5(2) - 7] + [5(3) - 7] + [5(4) - 7] + \ldots + [5(15) - 7]
\]

\[
= -2 + 3 + 8 + 13 + \ldots + 68
\]

To calculate the sum, use the same method used in Example 1 with \( t(1) = -2 \) and \( t(15) = 68 \) to determine \( S(15) = 495 \).

**Example 3**

Write the series \( 5 + 3 + 1 + -1 + -3 + \ldots + -29 \) using summation notation.

First, determine the formula for the \( n^{\text{th}} \) term. Here, \( t(1) = 5 \) and the common difference is \(-2\). Therefore \( t(n) = -2(n - 1) + 5 = -2n + 7 \).

We need to know the number of terms in the series, so determine the term number that corresponds with \(-29\).

\[
\begin{align*}
-29 &= -2n + 7 \\
-36 &= -2n \\
n &= 18
\end{align*}
\]

Since there are 18 terms in the series, it can be written using summation notation as:

\[
\sum_{k=1}^{18} (-2n + 7)
\]
Problems

Write the following series using summation notation and calculate each sum.

1.  \[ 5 + 7 + 9 + \ldots + 25 \]
2.  \[ 1 + 11 + 21 + \ldots + 161 \]
3.  \[ -5 + -11 + -17 + \ldots + -47 \]
4.  \[ 1 + 2 + 3 + \ldots + 100 \]
5.  \[ 2 + 5 + 8 + \ldots + 89 \]
6.  \[ 5 + 4 + 3 + \ldots + -5 \]
7.  Sum the positive even integers less than or equal to 100.

Expand the following series and calculate each sum.

8.  \[ \sum_{k=1}^{15} (3k + 1) \]
9.  \[ \sum_{j=1}^{68} (11j - 8) \]
10.  \[ \sum_{k=1}^{22} (50 - 4k) \]
11.  \[ \sum_{i=1}^{17} (4i + 9) \]
12.  \[ \sum_{k=1}^{9} (1 - 6k) \]
13.  \[ \sum_{j=1}^{4} j^2 \]

Answers

1.  \[ \sum_{k=1}^{11} (2k + 3), S(11) = 165 \]
2.  \[ \sum_{k=1}^{17} (10k - 9), S(17) = 1377 \]
3.  \[ \sum_{k=1}^{8} (-6k + 1), S(8) = -208 \]
4.  \[ \sum_{k=1}^{100} k, S(100) = 5050 \]
5.  \[ \sum_{k=1}^{30} (3k - 1), S(30) = 1365 \]
6.  \[ \sum_{k=1}^{11} (6 - k), S(11) = 0 \]
7.  \[ \sum_{k=1}^{50} (2k), S(50) = 2550 \]
8.  \[ 4 + 7 + 10 + \ldots + 46 = 375 \]
9.  \[ 3 + 14 + 25 + \ldots + 740 = 25,262 \]
10.  \[ 46 + 42 + 38 + \ldots + -38 = 88 \]
11.  \[ 13 + 17 + 21 + \ldots + 77 = 765 \]
12.  \[ -5 + -11 + -17 + \ldots + -53 = -261 \]
13.  \[ 1 + 4 + 9 + 16 = 30 \] (this series is not arithmetic)
GEOMETRIC SERIES  10.2.1 and 10.2.2

Just as arithmetic sequences lead to arithmetic series (the sum of the terms of the arithmetic sequence), geometric sequences lead to geometric series as well. In this section students develop a formula to determine the sum of geometric series, write geometric series using summation notation, and explore infinite geometric series.

See the Math Notes box in Lesson 10.3.1 for additional information.

Example 1

Write the series $32 + 16 + 8 + \ldots + \frac{1}{16}$ using summation notation and calculate the sum.

Since each term of this series is half the preceding term, this is a geometric series. The formula for the $n^{th}$ term of this series is $t(n) = 32 \left(\frac{1}{2}\right)^{n-1}$. To write the series using summation notation, determine how many terms are in the series. One way is to write out all of the terms and count them, but a more efficient method is to use the formula for the $n^{th}$ term, and solve for $n$. This is shown at right.

Therefore: $\sum_{k=1}^{10} 32 \left(\frac{1}{2}\right)^{k-1}$

To calculate the sum, use the formula(s) for the sum of the first $n$ terms of a geometric series. Note: The two given formulas are equivalent ($a = t(1)$).

$$S(n) = \frac{r \cdot t(n) - t(1)}{r - 1} \quad \text{or} \quad S(n) = \frac{a(1 - r^n)}{1 - r}$$

$$S(10) = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{16}\right) - 32}{\frac{1}{2} - 1} = \frac{-\frac{1}{2} - 32}{-\frac{1}{2}} = \frac{1023}{32} = 32 \left(\frac{1023}{1024}\right) = \frac{1023}{32} = \frac{1023}{16} = 63.9375$$

$$\frac{1}{16} = 32 \left(\frac{1}{2}\right)^{n-1} \quad \frac{1}{512} = \left(\frac{1}{2}\right)^{n-1} \quad 512 = 2^{n-1} \quad 2^9 = 2^{n-1} \quad 9 = n - 1 \quad n = 10$$

$$\frac{1}{16} = 32 \left(\frac{1}{2}\right)^{10} = 32 \left(\frac{1}{2}ight) = \frac{1023}{32} = 32 \left(\frac{1023}{1024}\right) = \frac{1023}{32} = \frac{1023}{16} = 63.3975$$
Example 2

Expand and determine the sum of the geometric series: \( \sum_{k=1}^{12} 3(4)^{k-1} \).

Expanding the summation notation gives the series \( 3 + 12 + 48 + 192 + \ldots + 12,582,912 \).

Use the formula from the previous example to calculate the sum.

\[
S(12) = \frac{4(12582912) - 3}{4 - 1} = \frac{50331645}{3} = 16,777,215
\]

Example 3

Calculate the sum of each infinite geometric series.

a. \( 81 + 27 + 9 + 3 + \ldots \)

b. \( \sum_{k=1}^{\infty} 25 \left( \frac{1}{5} \right)^{k-1} \)

If the common ratio in an infinite geometric series is between \(-1\) and \(1\), the series has a finite sum, even though the series goes on forever. The sum is given by the formula:

\[
S = \frac{t(1)}{1-r} = \frac{a}{1-r}
\]

a. The common ratio is \( \frac{1}{3} \), therefore:

\[
S = \frac{81}{1 - \frac{1}{3}} = 243
\]

b. The first term, \( t(1) = 25 \) (or \( a = 25 \)), and the common ratio, \( r \), is \( \frac{1}{5} \).

Therefore:

\[
S = \frac{25}{1 - \frac{1}{5}} = 31.25
\]
Problems

Expand and determine the sum of each geometric series.

1. \( \sum_{k=1}^{6} 2 \cdot 3^{k-1} \)
2. \( \sum_{k=1}^{8} 1.07^k \)
3. \( \sum_{k=1}^{4} 5 \cdot 3^k \)
4. \( \sum_{k=1}^{\infty} \left( \frac{2}{3} \right)^k \)

Write each series using summation notation and calculate the sum.

5. \( 2 + 10 + 50 + \ldots + 19,531,250 \)
6. \( 1 + 3 + 9 + 27 + \ldots + 59,049 \)
7. \( 500 + 100 + 20 + \ldots + 0.0000512 \)
8. \( 88 + 44 + 22 + \ldots \)

Answers

1. \( 2 + 6 + 18 + 54 + 162 + 486 = 728 \)  
2. \( 1.07 + 1.1449 + 1.2250 + 1.3101 + 1.4026 + 1.5007 + 1.6068 + 1.7182 \approx 10.9783 \)
3. \( 15 + 45+ 135 + 405 = 600 \)  
4. \( \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \ldots = \frac{2}{3} \)
5. \( \sum_{k=1}^{11} 2 \cdot 5^{k-1} = 24,414,062 \)  
6. \( \sum_{k=1}^{11} 3^{k-1} = 88,573 \)
7. \( \sum_{k=1}^{11} 500 \cdot \left( \frac{1}{5} \right)^{k-1} = 625 \)  
8. \( \sum_{k=1}^{\infty} 88 \cdot \left( \frac{1}{2} \right)^{k-1} = 176 \)
After exploring patterns in Pascal’s Triangle, students see how the numbers in the triangle relate to the counting principles. The connection to the Binomial Theorem allows students to quickly expand binomials raised to various powers as well as determine probabilities in situations involving two outcomes.

See the Math Notes boxes in Lessons 10.1.1 and 10.3.3 for additional information.

Example 1

Use Pascal’s Triangle to expand \((a + b)^4\), then apply the same pattern to expand \((2x – 3y)^4\).

The fourth line of Pascal’s Triangle gives the coefficients of the terms in the expansion of this binomial since its degree is 4.
The fourth line is 1 4 6 4 1. Therefore the expansion is:

\[
(a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.
\]

Notice that the powers of \(a\) decrease while the powers of \(b\) increase, but the sum of the two exponents in each term is always 4.

Apply this pattern to the second binomial, replacing each \(a\) with \(2x\) and each \(b\) with \(-3y\):

\[
(2x – 3y)^4 = 1(2x)^4 + 4(2x)^3(-3y) + 6(2x)^2(-3y)^2 + 4(2x)(-3y)^3 + (-3y)^4
\]

\[
= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4
\]

Example 2

Use the Binomial Theorem to determine the eighth term in the expansion of \((p + 3q)^{14}\).

The Math Notes box in Lesson 10.1.1 provides the formula for using combinations to generate the numbers in Pascal’s Triangle. The Math Notes box in Lesson 10.3.3 shows how to use the Binomial Theorem in combinations form to determine the expansion of any binomial raised to any positive integer value. In particular, the \(k\)th term of the expansion \((a + b)^n\) is given by \(\binom{n}{k}a^{n-k}b^k\). In this case the term is:

\[
\binom{14}{14-8}a^{14-(8-1)}b^{8-1} = \binom{14}{7}a^7b^7
\]

\[
= \frac{14!}{7!7!}a^7b^7
\]

\[
= 3432a^7b^7
\]
Problems

Expand each of the following binomials.

1. \((5x - 6y)^6\)
2. \((8 + 3q)^5\)
3. Determine the fifth term in the expansion of \((x + y)^{12}\).
4. Determine the seventh term in the expansion of \((x + y)^{12}\).
5. Determine the 15th term in the expansion of \((3p - 2q)^{32}\).
6. Eight coins are tossed. What is the probability that exactly five are heads?

Answers

1. \[15625x^6 - 112500x^5y + 337500x^4y^2 - 540000x^3y^3 + 486000x^2y^4 - 233280xy^5 + 46656y^6\]
2. \[32768 + 61440q + 46080q^2 + 17280q^3 + 3240q^4 + 243q^5\]
3. \(495x^8y^4\)
4. \(924x^6y^6\)
5. \[\binom{32}{18}(3p)^{18}(-2q)^{14} = 471435600(387420489)p^{18}(16384)q^{14}\]
6. \[\binom{8}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^3 = \frac{7}{32}\]
SAT Prep

1. If \( y \) is directly proportional to \( x \) and if \( y = 24 \) when \( x = 4 \), what is the value of \( y \) when \( x = 7 \)?
   a. \( \frac{6}{7} \)  b. \( \frac{24}{7} \)  c. 27  d. 28  e. 42

2. For all positive integers \( a \) and \( b \), let the operation \( \{ \} \) be defined by \( a\{\}b = a^{b+2} - 6a^b + 14 \). For how many positive integers \( a \) is \( a\{\}2 \) equal to 9?
   a. 0  b. 1  c. 2  d. 3  e. 4

3. In the figure at right, the line \( l \) has a slope of \(-2\). What is the \( y \)-intercept of \( l \)?
   a. 5  b. 8  c. 11  d. 13  e. 15

4. Which of the following numbers is divisible by 7 and 11 but not divisible by 10?
   a. 49  b. 66  c. 70  d. 308  e. 504

5. In the figure below right, the intersection of ray \( AC \) with ray \( BA \) is:
   a. Segment \( AC \)
   b. Segment \( AB \)
   c. Ray \( AC \)
   d. Ray \( BA \)
   e. Line \( AC \)

6. What is the greatest integer value of \( x \) for which \( 4x - 28 < 0 \)?

7. The mean of a list of 87 consecutive integers is 33. What is the greatest integer in the list?

8. When a positive integer \( p \) is divided by 5 the remainder is 4. What is the remainder when \( 12p \) is divided by 10?

9. When a certain number is divided by \( \frac{1}{2} \) and the quotient is then multiplied by 9 the result is 108. What is the number?

10. The diameter of a circle is 5. If the circle is cut in half, what is the total perimeter of the two pieces?
Answers
1. E
2. B
3. C
4. D
5. B
6. 6
7. 77
8. 8
9. 6
10. $5\pi + 10$
To **simplify rational expressions**, determine factors in the numerator and denominator that are the same and then write them as fractions equal to 1. For example:

\[
\frac{6}{6} = 1 \quad \frac{x^2}{x^2} = 1 \quad \frac{x + 2}{x + 2} = 1 \quad \frac{3x - 2}{3x - 2} = 1
\]

The rational expressions below cannot be simplified to equal 1 (or anything else):

\[
\frac{6 + 5}{6} \quad \frac{x^3 + y}{x^3} \quad \frac{x}{x + 2} \quad \frac{3x - 2}{2}
\]

As shown in the examples below, most problems that involve simplifying rational expressions will require factoring the numerator and denominator.

Note that in all cases it is assumed that the denominator does not equal zero, so in Example 4 below the simplification is only valid provided \(x \neq -6\) or 2. For additional information, see examples 1 and 2 in the Math Notes box in Lesson 11.1.3.

One other special situation is shown in the following examples:

\[
\frac{-2}{2} = -1 \quad \frac{-x}{x} = -1 \quad \frac{-x - 2}{x + 2} \Rightarrow -(x + 2) \Rightarrow -1 \quad \frac{5 - x}{x - 5} \Rightarrow -\frac{(x - 5)}{x - 5} \Rightarrow -1
\]

Again, it is assumed that the denominator does not equal zero.

### Example 1

\[
\frac{12}{54} \Rightarrow \frac{2 \cdot 2 \cdot 3}{2 \cdot 3 \cdot 3 \cdot 3} \Rightarrow \frac{2}{9} \text{ since } \frac{2}{3} = \frac{3}{3} = 1
\]

### Example 2

\[
\frac{6x^3y^2}{15x^2y^4} \Rightarrow \frac{2 \cdot 3 \cdot x^2 \cdot x \cdot y^2}{5 \cdot 3 \cdot x^2 \cdot y^2 \cdot y^2} \Rightarrow \frac{2x}{5y^2}
\]

### Example 3

\[
\frac{12(x - 1)^3(x + 2)}{3(x - 1)^2(x + 2)^2} \Rightarrow \frac{4 \cdot 3(x - 1)^2(x - 1)(x + 2)}{3(x - 1)^2(x + 2)(x + 2)}
\]

\[
\Rightarrow \frac{4(x - 1)}{(x + 2)} \text{ since } \frac{3}{3}, \frac{(x - 1)^2}{(x - 1)^2}, \text{ and } \frac{x + 2}{x + 2} = 1
\]

### Example 4

\[
\frac{x^2 - 6x + 8}{x^2 + 4x - 12} \Rightarrow \frac{(x - 4)(x - 2)}{(x + 6)(x - 2)}
\]

\[
\Rightarrow \frac{(x - 4)}{(x + 6)} \text{ since } \frac{(x - 2)}{(x - 2)} = 1
\]
Problems

Simplify each of the following expressions completely. Assume the denominator does not equal zero.

1. \( \frac{2(x + 3)}{4(x - 2)} \)
2. \( \frac{2(x - 3)}{6(x + 2)} \)
3. \( \frac{2(x + 3)(x - 2)}{6(x - 2)(x + 2)} \)
4. \( \frac{4(x - 3)(x - 5)}{6(x - 3)(x + 2)} \)
5. \( \frac{3(x - 3)(4 - x)}{15(x + 3)(x - 4)} \)
6. \( \frac{15(x - 1)(7 - x)}{25(x + 1)(x - 7)} \)
7. \( \frac{24(y - 4)(y - 6)}{16(y + 6)(6 - y)} \)
8. \( \frac{36(y + 4)(y - 16)}{32(y + 16)(16 - y)} \)
9. \( \frac{(x + 3)^3(x - 2)^4}{(x + 3)^4(x - 2)^3} \)
10. \( \frac{(5 - x)^2(x - 2)^2}{(x + 5)^4(x - 2)^3} \)
11. \( \frac{(5 - x)^4(3x - 1)^2}{(x - 5)^4(3x - 2)^3} \)
12. \( \frac{12(x - 7)(x + 2)^4}{20(x - 7)^2(x + 2)^5} \)
13. \( \frac{x^2 + 5x + 6}{x^2 + x - 6} \)
14. \( \frac{2x^2 + x - 3}{x^2 + 4x - 5} \)
15. \( \frac{x^2 + 4x}{2x + 8} \)
16. \( \frac{24(3x - 7)(x + 1)^6}{20(3x - 7)^3(x + 1)^5} \)
17. \( \frac{x^2 - 1}{(x + 1)(x - 2)} \)
18. \( \frac{x^2 - 4}{(x + 1)^2(x - 2)} \)
19. \( \frac{x^2 - 4}{x^2 + x - 6} \)
20. \( \frac{x^2 - 16}{x^3 + 9x^2 + 20x} \)
21. \( \frac{2x^2 - x - 10}{3x^2 + 7x + 2} \)

Answers

1. \( \frac{(x + 3)}{2(x - 2)} \)
2. \( \frac{(x - 3)}{3(x + 2)} \)
3. \( \frac{(x + 3)}{3(x + 2)} \)
4. \( \frac{2(x - 5)}{3(x + 2)} \)
5. \( -\frac{(x - 3)}{5(x + 3)} \)
6. \( -\frac{3(x - 1)}{5(x + 1)} \)
7. \( -\frac{3(y - 4)}{2(y + 6)} \)
8. \( -\frac{9(y + 4)}{8(y + 16)} \)
9. \( \frac{(x - 2)}{(x + 3)^2} \)
10. \( \frac{(5 - x)^2}{(x + 5)^4(x - 2)} \)
11. \( \frac{(3x - 1)^2}{(3x - 2)^3} \)
12. \( \frac{3}{5(x - 7)(x + 2)} \)
13. \( \frac{x + 2}{x - 2} \)
14. \( \frac{2x + 3}{x + 5} \)
15. \( \frac{x}{2} \)
16. \( \frac{6(x + 1)}{5(3x - 7)^2} \)
17. \( \frac{x - 1}{x + 2} \)
18. \( \frac{x + 2}{(x + 1)^2} \)
19. \( \frac{x + 2}{x + 3} \)
20. \( \frac{x - 4}{x(x + 5)} \)
21. \( \frac{2x - 5}{3x + 1} \)
MULTIPLICATION AND DIVISION
OF RATIONAL EXPRESSIONS

Multiplication or division of rational expressions follows the same procedures used with numerical fractions. However, it is often necessary to factor the polynomials in order to simplify them. As in the previous section, remember that simplification assumes that the denominator is not equal to zero. For additional information, see examples 3 and 4 in the Math Notes box in Lesson 11.1.3.

Example 1
Multiply \( \frac{x^2 + 6x}{(x + 6)^2} \cdot \frac{x^2 + 7x + 6}{x^2 - 1} \) and simplify the result.

After factoring, the expression becomes:
\[
\frac{x(x + 6)}{(x + 6)(x + 6)} \cdot \frac{(x + 1)(x + 6)}{(x + 1)(x - 1)}
\]

After multiplying, reorder the factors:
\[
\frac{(x + 6)}{(x + 6)} \cdot \frac{(x + 6)}{(x + 6)} \cdot \frac{x}{(x - 1)} \cdot \frac{(x + 1)}{(x + 1)}
\]

Since \( \frac{(x + 6)}{(x + 6)} = 1 \) and \( \frac{(x + 1)}{(x + 1)} = 1 \), simplify:
\[
1 \cdot 1 \cdot \frac{x}{x - 1} \Rightarrow \frac{x}{x - 1}
\]

Example 2
Divide \( \frac{x^2 - 4x - 5}{x^2 - 4x + 4} \div \frac{x^2 - 2x - 15}{x^2 + 4x - 12} \) and simplify the result.

First change to a multiplication expression by inverting (flipping) the second fraction:
\[
\frac{x^2 - 4x - 5}{x^2 - 4x + 4} \cdot \frac{x^2 + 4x - 12}{x^2 - 2x - 15}
\]

After factoring, the expression is:
\[
\frac{(x - 5)(x + 1)}{(x - 2)(x - 2)} \cdot \frac{(x + 6)(x - 2)}{(x - 5)(x + 3)}
\]

Reorder the factors (if you need to):
\[
\frac{(x - 5)}{(x - 5)} \cdot \frac{(x - 2)}{(x - 2)} \cdot \frac{(x + 1)}{(x + 1)} \cdot \frac{(x + 6)}{(x + 6)}
\]

Since \( \frac{(x - 5)}{(x - 5)} = 1 \) and \( \frac{(x - 2)}{(x - 2)} = 1 \), simplify:
\[
\frac{(x + 1)}{(x + 3)} \cdot \frac{(x + 6)}{(x - 2)} \text{ or } \frac{(x + 1)(x + 6)}{(x - 2)(x + 3)}
\]
Problems

Multiply or divide each pair of rational expressions. Simplify the result. Assume the denominator does not equal to zero.

1. \( \frac{x^2 + 5x + 6}{x^2 - 4x} \cdot \frac{4x}{x + 2} \)
2. \( \frac{x^2 - 2x}{x^2 - 4x + 4} + \frac{4x^2}{x - 2} \)
3. \( \frac{x^2 - 16}{(x - 4)^2} \cdot \frac{x^2 - 3x - 18}{x^2 - 2x - 24} \)
4. \( \frac{x^2 - x - 6}{x^2 + 3x - 10} \cdot \frac{x^2 + 2x - 15}{x^2 - 6x + 9} \)
5. \( \frac{x^2 - x - 6}{x^2 - x - 20} \cdot \frac{x^2 + 6x + 8}{x^2 - x - 6} \)
6. \( \frac{x^2 - x - 30}{x^2 + 13x + 40} \cdot \frac{x^2 + 11x + 24}{x^2 - 9x + 18} \)
7. \( \frac{15 - 5x}{x^2 - x - 6} + \frac{5x}{x^2 + 6x + 8} \)
8. \( \frac{17x + 119}{x^2 + 5x - 14} + \frac{9x - 1}{x^2 - 3x + 2} \)
9. \( \frac{2x^2 - 5x - 3}{3x^2 - 10x + 3} - \frac{9x^2 - 1}{4x^2 + 4x + 1} \)
10. \( \frac{x^2 - 1}{x^2 - 6x - 7} - \frac{x^3 + x^2 - 2x}{x - 7} \)
11. \( \frac{3x - 21}{x^2 - 49} + \frac{3x}{x^2 + 7x} \)
12. \( \frac{x^2 - y^2}{x + y} \cdot \frac{1}{x - y} \)
13. \( \frac{y^2 - y}{w^2 - y^2} + \frac{y^2 - 2y + 1}{1 - y} \)
14. \( \frac{y^2 - y - 12}{y + 2} + \frac{y - 4}{y^2 - 4y - 12} \)
15. \( \frac{x^2 + 7x + 10}{x + 2} \div \frac{x^2 + 2x - 15}{x + 2} \)

Answers

1. \( \frac{4(x + 3)}{x - 4} \)
2. \( \frac{1}{4x} \)
3. \( \frac{x + 3}{x - 4} \)
4. \( \frac{x + 2}{x - 2} \)
5. \( \frac{x + 2}{x - 5} \)
6. \( \frac{x + 3}{x - 3} \)
7. \( -\frac{x - 4}{x} \)
8. \( \frac{17(x - 1)}{9x - 1} \)
9. \( \frac{3x + 1}{2x + 1} \)
10. \( \frac{1}{x(x + 2)} \)
11. \( 1 \)
12. \( 1 \)
13. \( -\frac{y}{w^2 - y^2} \)
14. \( (y + 3)(y - 6) \)
15. \( \frac{x + 2}{x - 3} \)
Addition or subtraction of rational expressions follows the same procedures used with numerical fractions. First, determine a common denominator (if necessary). Next, rewrite the original fractions as equivalent ones with the common denominator. Then, add (or subtract) the new numerators, leaving the common denominator. Finally, factor the numerator and denominator and reduce (if possible). For additional information, see the Math Notes box in Lesson 11.1.4. Note that these steps are only valid provided that the denominator does not equal zero.

Example 1

The least common multiple of $2(n + 2)$ and $n(n + 2)$ is $2n(n + 2)$.

To get a common denominator, in the first fraction, multiply the fraction by $\frac{n}{n}$. Multiply the second fraction by $\frac{2}{2}$.

Rewrite each term with the common denominator.

Add, factor, and simplify the result.

Example 2

$$\frac{2 - x}{x + 4} + \frac{3x + 6}{x + 4} \Rightarrow \frac{(2 - x) + (3x + 6)}{x + 4} \Rightarrow \frac{2x + 8}{x + 4} \Rightarrow \frac{2(x + 4)}{x + 4} \Rightarrow 2$$

Example 3

$$\frac{3}{x - 1} - \frac{2}{x - 2} \Rightarrow \frac{3 \cdot (x - 2) - 2 \cdot (x - 1)}{(x - 1)(x - 2)} \Rightarrow \frac{(3x - 6) - (2x - 2)}{(x - 1)(x - 2)} \Rightarrow \frac{x - 4}{(x - 1)(x - 2)}$$
### Problems

Add or subtract each expression and simplify the result. In each case assume the denominator does not equal zero.

1. \( \frac{x}{(x+2)(x+3)} + \frac{2}{(x+2)(x+3)} \)
2. \( \frac{x}{x^2 + 6x + 8} + \frac{4}{x^2 + 6x + 8} \)
3. \( \frac{b^2}{b^2 + 2b - 3} + \frac{9}{b^2 + 2b - 3} \)
4. \( \frac{2a}{a^2 + 2a + 1} + \frac{2}{a^2 + 2a + 1} \)
5. \( \frac{x + 10}{x + 2} + \frac{x - 6}{x + 2} \)
6. \( \frac{a + 2b}{a + b} + \frac{2a + b}{a + b} \)
7. \( \frac{3x - 4}{3x + 3} - \frac{2x - 5}{3x + 3} \)
8. \( \frac{3x}{4x - 12} - \frac{9}{4x - 12} \)
9. \( \frac{6a}{5a^2 + a} - \frac{a - 1}{5a^2 + a} \)
10. \( \frac{x^2 + 3x - 5}{10} - \frac{x^2 - 2x + 10}{10} \)
11. \( \frac{2x}{x(x + 3)} + \frac{6}{x(x + 3)} \)
12. \( \frac{5}{x - 7} + \frac{3}{4(x - 7)} \)
13. \( \frac{5x + 6}{x^2} - \frac{5}{x} \)
14. \( \frac{2}{x + 4} - \frac{x - 4}{x^2 - 16} \)
15. \( \frac{10a}{a^2 + 6a} - \frac{3}{3a + 18} \)
16. \( \frac{3x}{2x^2 - 8x} + \frac{2}{x - 4} \)
17. \( \frac{5x + 9}{x^2 - 2x - 3} + \frac{6}{x^2 - 7x + 12} \)
18. \( \frac{x + 4}{x^2 - 3x - 28} - \frac{x - 5}{x^2 + 2x - 35} \)
19. \( \frac{3x + 1}{x^2 - 16} - \frac{3x + 5}{x^2 + 8x + 16} \)
20. \( \frac{7x - 1}{x^2 - 2x - 3} - \frac{6x}{x^2 - x - 2} \)

### Answers

1. \( \frac{1}{x + 3} \)
2. \( \frac{1}{x + 2} \)
3. \( \frac{b - 3}{b - 1} \)
4. \( \frac{2}{a + 1} \)
5. \( 2 \)
6. \( 3 \)
7. \( \frac{1}{3} \)
8. \( \frac{3}{4} \)
9. \( \frac{1}{a} \)
10. \( \frac{x - 3}{2} \)
11. \( \frac{2}{x} \)
12. \( \frac{23}{4(x - 7)} = \frac{23}{4x - 28} \)
13. \( \frac{6}{x^2} \)
14. \( \frac{1}{x + 4} \)
15. \( \frac{9}{a + 6} \)
16. \( \frac{7}{2(x - 4)} = \frac{7}{2x - 8} \)
17. \( \frac{5(x + 2)}{(x - 4)(x + 1)} = \frac{5x + 10}{x^2 - 3x - 4} \)
18. \( \frac{14}{(x - 7)(x + 7)} = \frac{14}{x^2 - 49} \)
19. \( \frac{4(5x + 6)}{(x - 4)(x + 4)^2} \)
20. \( \frac{x + 2}{(x - 3)(x - 2)} = \frac{x + 2}{x^2 - 5x + 6} \)
This section begins with students using technology to explore graphing in three dimensions. By using strategies that they used for graphing in two dimensions, students extend their skills to plotting points in three dimensions as well as graphing planes represented by equations with three variables. This leads to intersecting planes, and using algebra to determine the point of intersection for the system of three equations with three variables. For additional information, see the Math Notes boxes in Lessons 11.2.3 and 11.2.4.

Example 1

Graph the 3-D point and equation below.

\[(3, 4, 5) \quad 2x + 3y + 4z = 24\]

Although we live in a three-dimensional world, visualizing three-dimensional objects on a two-dimensional piece of paper can be difficult. In class, students used the computer to help them visualize the graphs. (You can access this eTool at technology.cpm.org/general/3dgraph/) Students begin by plotting points on axes as shown at right. As with plotting points in two dimensions, each number in the coordinates indicates how far to move in each direction. In the diagram at right only the positive direction for each axis is shown; these axes extend in the negative direction as well.

To plot the point \((3, 4, 5)\), move 3 units along the \(x\)-axis, 4 units in the direction of the \(y\)-axis, and then 5 units in the direction of the \(z\)-axis. It might help to think of the point as the corner on a box, farthest from the origin. This imaginary box is lightly shaded in above.

To graph the equation with three variables on the three-dimensional graph, it is helpful to determine the intercepts. Do this by letting the different variables equal zero, which then allows you to determine the \(x\)-, \(y\)-, and \(z\)-intercepts.

\[
\begin{align*}
2x + 3y + 4z &= 12 \\
x &= 0, \quad y = 0 \Rightarrow 0 + 0 + 4z, \quad z = 3 \\
x &= 0, \quad z = 0 \Rightarrow 0 + 3y + 0 = 12, \quad y = 4 \\
y &= 0, \quad z = 0 \Rightarrow 2x + 0 + 0 = 12, \quad x = 6
\end{align*}
\]

By plotting these intercepts it can be seen how the plane slices through this quadrant of space. The shaded plane continues; it does not stop at the edges of the triangle.
Example 2

Solve the following system of three equations and three unknowns.

\[
\begin{align*}
2x + y - 3z &= 13 \\
x - 3y + z &= -21 \\
-2x + y + 4z &= -7 \\
\end{align*}
\]

Explain what the solution says about the graphs of each equation.

Before beginning, it is helpful to recall how to solve two equations with two unknowns. If an equation was written in \( y = \) form, the expression for \( y \) can be substituted into the other equation. Other times the equations can be added or subtracted to eliminate a variable. Sometimes multiplying an equation by a number before adding or subtracting is necessary. In either procedure, the goal is the same: to eliminate a variable.

By adding the first and third equation in the system, \( x \) is eliminated. The problem is, there are still two variables remaining. Now use another pair of equations to eliminate \( x \). There are different ways to do this. One way is to multiply the second equation by 2, then add the result to the third equation.

\[
\begin{align*}
2(x - 3y + z &= -21) &\quad \Rightarrow &\quad 2x - 6y + 2z &= -42 \\
-2x + y + 4z &= -7 &\quad \Rightarrow &\quad -5y + 6z &= -49 \\
\end{align*}
\]

Now there are two equations with two variables. Use this simpler system to solve for \( y \) and \( z \).

\[
\begin{align*}
-6 \cdot (2y + z &= 6) &\quad \Rightarrow &\quad -12y - 6z &= -36 \\
\quad -5y + 6z &= -49 &\quad \Rightarrow &\quad 2(5) + z &= 6 \\
\quad -17y &= -85 &\quad 10 + z &= 6 \\
\quad y &= 5 &\quad z &= -4 \\

y = 5, \quad z = -4 &\quad \Rightarrow &\quad x - 3y + z &= -21 \\
\quad x - 15 - 4 &= -21 &\quad x &= -2
\end{align*}
\]

The solution to this system is \((-2, 5, -4)\) which indicates that all three planes intersect in one point. This solution can be checked by substituting the values for \( x, y, \) and \( z \) into the original equations.
Example 3

Pizza Planet sells three sizes of combination pizzas.

- Small (8” diameter) $8.50
- Medium (10” diameter) $11.50
- Large (13” diameter) $17.50

Assume that the price of the pizza can be modeled with a quadratic function, where the price depends on the diameter of the pizza. If Pizza Planet is considering selling an Extra Large combination pizza, with an 18” diameter, what should they charge? If they wanted to sell a combination pizza for $50.00, what diameter should it have?

Let \( x \) represent the diameter of the pizza in inches and \( y \) represent the cost of the pizza in dollars. Thus the three data points are (8, 8.50), (10, 11.50), and (13, 17.50). Use these three points in the standard form equation of a quadratic functions, \( y = ax^2 + bx + c \).

First, substitute the data points into the equation.

\[
\begin{align*}
(8, 8.50) & : 8.50 = a(8)^2 + b(8) + c \\
(10, 11.50) & : 11.50 = a(10)^2 + b(10) + c \\
(13, 17.50) & : 17.50 = a(13)^2 + b(13) + c
\end{align*}
\]

This gives the three equations shown at right. (For reference, the equations are numbered.) This is now similar to the previous example; solve three equations with three unknowns. The unknowns here are \( a, b, \) and \( c \) (rather than \( x, y, \) and \( z \)).

To begin, multiply equation (1) by \(-1\) so that \( c \) is eliminated when adding pairs of equations. New equation (2) combined with equation (1) gives \( 3 = 36a + 2b \).

Equation (3) combined with new equation (2) gives \( 6 = 69a + 3b \).

To solve the new system of equations for \( a \) and \( b \), multiply the first equation by \(-3\) and the second by 2 then add the results.

\[
\begin{align*}
3 &= 36a + 2b \\
6 &= 69a + 3b
\end{align*}
\]

Once the value of \( a \) is known, substitute it back into one of the equations to determine the value of \( b \).

\[
a = \frac{1}{10} \Rightarrow 3 = 36a + 2b
\]

\[
3 = 36 \left( \frac{1}{10} \right) + 2b
\]

\[
3 = 3.6 + 2b
\]

\[
-0.6 = 2b
\]

\[
b = -0.3 = -\frac{3}{10}
\]

Example continues on next page
Example continued from previous page.

Lastly, use the values for $a$ and $b$ to determine the value of $c$. Use any of the three original equations to do so. Using equation (1):

$$a = \frac{1}{10},\ b = -\frac{3}{10} \quad \Rightarrow \quad 8.50 = 64a + 8b + c$$

$$8.50 = 64\left(\frac{1}{10}\right) + 8\left(-\frac{3}{10}\right) + c$$

$$8.50 = 6.4 - 2.4 + c$$

$$c = 4.50$$

Now that the values of $a$, $b$, and $c$ are known, the equation that models this data is:

$$y = \frac{1}{10}x^2 - \frac{3}{10}x + 4.50$$

Use this equation to determine the cost of a combination pizza with an 18" diameter.

$$x = 18 \quad \Rightarrow \quad y = \frac{1}{10}x^2 - \frac{3}{10}x + 4.50$$

$$y = \frac{1}{10}(18)^2 - \frac{3}{10}(18) + 4.50$$

$$y = 32.4 - 5.4 + 4.50$$

$$y = \$31.50$$

Therefore an 18" combination pizza should cost $31.50.

How large should a $50.00 pizza be to fit with this data? To answer this question, let $y = 50$ and solve for $x$. The solution will require solving a quadratic equation. Although there are several ways to solve a quadratic equation, the best approach here is to use the Quadratic Formula. To begin, multiply by 10 to eliminate the fractions and the decimals.

$$50 = \frac{1}{10}x^2 - \frac{3}{10}x + 4.50$$

$$500 = x^2 - 3x + 45$$

$$x^2 - 3x - 455 = 0$$

$$x = \frac{3\pm\sqrt{(-3)^2-4(1)(-455)}}{2(1)}$$

$$x = \frac{3\pm\sqrt{9+1820}}{2}$$

$$x = \frac{3\pm42.77}{2}$$

$$x \approx 22.89$$

Therefore, to sell a pizza for $50.00, Pizza Planet should make the diameter of the pizza approximately 22.89 inches. A 23" diameter would surely suffice!
Problems

Solve each of the following systems of equations for $x$, $y$, and $z$. Explain what the answer tells you about the graphs of the equations.

1. \[3x - 2y + z = 3\]
   \[5x + y + 2z = 8\]
   \[-3x - y + 3z = -22\]
2. \[-4x - 6y + 5z = 21\]
   \[3x + 4y - 2z = -15\]
   \[-7x - 5y + 3z = 15\]
3. \[3x + 4z = 19\]
   \[3y + 2z = 8\]
   \[4x - 5y = 7\]
4. \[9x + 6y - 12z = 14\]
   \[3x + 2y - 4z = -11\]
   \[x + y + z = 1\]
5. \[\frac{x}{2} + \frac{y}{4} + \frac{z}{2} = 24\]
   \[\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 29\]
   \[\frac{x}{4} + \frac{y}{2} + \frac{z}{3} = 25\]
6. \[21x - 7y + 14z = 70\]
   \[15x - 5y + 10z = 50\]
   \[-3x + y - 2z = -10\]
7. Write the equation of the parabola passing through the points $(-2, -32)$, $(0, -10)$, and $(2, -12)$.
8. Write the equation of the parabola passing through the points $(2, 81)$, $(7, 6)$, and $(10, 33)$.
9. A recent study counted the number of times in a 24-hour period that the people misunderstood or misinterpreted a statement, comment, or question. The study reported the data shown in the table below.

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Misunderstandings or Misinterpretations</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>44</td>
</tr>
<tr>
<td>30</td>
<td>28</td>
</tr>
<tr>
<td>40</td>
<td>20</td>
</tr>
</tbody>
</table>

Write an equation that best fits this data. Use your equation to predict how many times an 80-year old person will misunderstand or misinterpret a statement, comment, or question in a 24-hour period. What about a 1-year old? What age person has the least amount of misunderstandings?

10. In archery, the arrow appears to travel in a straight line when it is released. However, the arrow actually travels upward slightly before curving back down toward the Earth. For a particular archer, the arrow starts at 5.4 feet above the ground. After 0.3 seconds, the arrow is 5.5 feet above the ground. The arrow hits the target after a total of 2.0 seconds at a height of 5.0 feet above the ground. Write an equation that models this situation.
Answers

1. \((3, 1, -4)\), these three planes intersect in a point.

2. \((1, -5, -1)\), these three planes intersect in a point.

3. \((3, 1, 2.5)\), these three planes intersect in a point.

4. No solution. (Two of these planes are parallel.)

5. \((36, 24, 12)\), these three planes intersect in a point.

6. Infinitely many solutions. All three of these equations represent the same plane.

7. \(y = -3x^2 + 5x - 10\)

8. \(y = 3x^2 - 42x + 153\)

9. The equation that fits this data is \(y = 0.04x^2 - 3.6x + 100\). According to this model, an 80-year old person would have 68 misunderstandings in a 24-hour period. A 1-year old would have approximately 96. The age with the least misunderstandings occurs at vertex of this function. The vertex is at \((45, 19)\) so 45 year olds have the lowest number with only 19 misunderstandings in a 24-hour period.

10. \(y \approx -0.31x^2 + 0.43x + 5.4\)
1. If \( b + 4 = 11 \), then \((b - 2)^2 = ?\)
   a. 16       b. 25       c. 36       d. 49       e. 64

2. Let \( P \) and \( Q \) represent digits in the addition problem shown at right.
   \[
   \begin{array}{c}
   \phantom{2} \phantom{5} \phantom{P}
   \\
   \phantom{2} \phantom{5} \phantom{P}
   \\
   \phantom{2} \phantom{5} \phantom{P}
   \\
   \phantom{2} \phantom{5} \phantom{P}
   \end{array}
   \]
   What must the digit \( Q \) be?
   \[
   \begin{array}{c}
   \phantom{2} \phantom{5} \phantom{+}
   \\
   \phantom{2} \phantom{5} \phantom{P}
   \\
   \phantom{2} \phantom{5} \phantom{P}
   \\
   \phantom{2} \phantom{5} \phantom{Q}
   \end{array}
   \]
   a. 0       b. 1       c. 2       d. 3       e. 4

3. If \( 3^4 = 9^x \), then \( x = ?\)
   a. 2       b. 3       c. 5       d. 8       e. 10

4. When a positive number \( n \) is divided by 7 the remainder is 6. Which of the following expressions will yield a remainder of 1 when it is divided by 7?
   a. \( n + 1 \)   b. \( n + 2 \)   c. \( n + 3 \)   d. \( n + 4 \)   e. \( n + 5 \)

5. How many 4-digit numbers have the thousands digit equal to 2 and the units digit equal to 7?
   a. 100       b. 199       c. 200       d. 500       e. 10005

6. In the figure at right, where \( x < 6 \), what is the value of \( x^2 + 36 \)?
   a. 10       b. 50       c. 100
   d. 600       e. 1296

7. The measures of the angles of a triangle in degrees can be expressed by the ratio 5:6:7. What is the sum of the measures of the two larger angles?
   a. 110       b. 120       c. 130       d. 160       e. 180

8. If \( \frac{r}{3} = \frac{7}{10} \), what is the value of \( r \)?
9. If \( p \) and \( q \) are two different prime numbers greater than 2, and \( n = pq \), how many positive factors, including 1 and \( n \), does \( n \) have?

10. If \( \frac{1}{3}(30x^2 + 20x^2 + 10x + 1) = ax^3 + bx^2 + cx + d \) for all values of \( x \), what is the value of \( a + b + c + d \)?

**Answers**

1. B
2. A
3. A
4. B
5. A
6. B
7. C
8. \( r = 2.1 \)
9. 4
10. 30.5
ANALYTIC TRIGONOMETRY

12.1.1 – 12.1.4

The study of trigonometric functions began in Chapter 9 with radians and transformations of trigonometric functions. This chapter focuses on solving equations involving trigonometric functions by operating on the variable. This work will review the inverse trigonometric functions as well as introduce the reciprocal trigonometric functions.

See the Math Notes box in Lesson 12.2.1 for additional information.

Example 1

For what values of \( \theta \) are the following equations true?

a. \( \cos(\theta) = \frac{\sqrt{3}}{2} \)

b. \( 2 \sin(\theta) = \sqrt{2} \)

c. \( \cos(\theta) = 5 \)

a. The graph of \( y = \cos(x) \) is a periodic function, and the graph of \( y = \frac{\sqrt{3}}{2} \) is a horizontal line. By graphing both equations on the same set of axes, it can be seen that they intersect infinitely many times. How do you determine all solutions?

Solving by using inverse cosine gives one solution.

\[
\cos^{-1}(\cos(\theta)) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)
\]

\[
\theta = \frac{\pi}{6} \text{ radians}
\]

It helps to remember the unit circle. For what values of \( \theta \) does \( \cos(\theta) = \frac{\sqrt{3}}{2} \)? Two points are easily found: \( \frac{\pi}{6} \) and \( -\frac{\pi}{6} \).

How do you determine the rest? On the unit circle, \( \frac{\pi}{6} \) and \( -\frac{\pi}{6} \) are revisited at each rotation of \( 2\pi \). Therefore, not only does \( \frac{\pi}{6} \) make the equation true, but so do \( \frac{\pi}{6} \pm 2\pi \), \( \frac{\pi}{6} \pm 4\pi \), \( \frac{\pi}{6} \pm 6\pi \), etc. Similarly, \( -\frac{\pi}{6} \pm 2\pi \), \( -\frac{\pi}{6} \pm 4\pi \), \( -\frac{\pi}{6} \pm 6\pi \), … will also make the equation true. Consolidate this information as \( \theta = \pm \frac{\pi}{6} \pm 2\pi n \), for all integers \( n \). Note: There are other ways to write this solution that are equivalent to this expression.
b. Solving using inverse sine gives: $2 \sin(\theta) = \sqrt{2}$

$$\sin(\theta) = \frac{\sqrt{2}}{2}$$

$$\theta = \sin^{-1} \left( \frac{\sqrt{2}}{2} \right)$$

$$\theta = \frac{\pi}{4}$$

There are usually two solutions within the unit circle, so in which two quadrants is sine positive? Since the value of sine depends on $y$, sine is positive in Quadrants I and II. Therefore

$$\theta = \frac{\pi}{4} \pm 2\pi n \text{ or } \theta = \frac{3\pi}{4} \pm 2\pi n$$

c. Since the range of $y = \cos(x)$ is $-1 \leq y \leq 1$, this equation has no solution.

Example 2

Let $f(x) = \sin(x)$, $g(x) = \cos(x)$, and $h(x) = \tan(x)$. Graph each of the following functions on separate axes.

$$y = \frac{1}{f(x)}$$

$$y = \frac{1}{g(x)}$$

$$y = \frac{1}{h(x)}$$

Compare your graphs to the graphs of these functions:

$$y = \sin^{-1}(x)$$

$$y = \cos^{-1}(x)$$

$$y = \tan^{-1}(x)$$

The first three functions are the reciprocal functions of sine, cosine and tangent. However, rather than writing them as reciprocals $\left( \frac{1}{f(x)} \right)$, they are given new names: $\frac{1}{\sin(x)} = \text{cosecant} x$

$$\frac{1}{\cos(x)} = \text{secant} x$$

$$\frac{1}{\tan(x)} = \text{cotangent} x$$

The abbreviation for cosecant is csc, for secant it is sec, and for cotangent it is cot. Their graphs are:

Since these are reciprocal functions, everywhere the first functions were zero, the corresponding reciprocal functions will be undefined. Check that this is the case.
In comparing these functions to the inverse trig functions, it is important to note that

\[ \frac{1}{\sin(x)} \neq \sin^{-1}(x) \] (and similarly for the other corresponding functions). This is very clear by examining the graphs. The exponent of “-1” indicates that the function is the inverse, not the reciprocal.

Problems

For each of the following equations, determine all the solutions. You may use your calculator but remember that your calculator only gives one answer.

1. \( 2 \cos(x) = \sqrt{2} \)
2. \( 5 \tan(x) - 5 = 0 \)
3. \( 4 \cos^2(x) - 1 = 0 \)
4. \( 4 \sin^2(x) = 3 \)
5. \( \sin(x) + 2 = 3 \sin(x) \)
6. \( \tan^2(x) + \tan(x) = 0 \)

Graph each of the following equations on a separate set of axes. Label all the important points.

7. \( y = 3 \csc(x) \)
8. \( y = 4 + \sec(x) \)
9. \( y = \cot(x - \pi) \)
Answers

1. \( x = \pm \frac{\pi}{4} \pm 2\pi n \) for all integers \( n \)

2. \( x = \frac{\pi}{4} \pm \pi n \) for all integers \( n \)

3. \( x = \pm \frac{\pi}{4} n \) for all integers \( n \)

4. \( x = \frac{\pi}{3} \pm \pi n \) or \( x = \frac{2\pi}{3} \pm \pi n \) for all integers \( n \)

5. \( x = \frac{\pi}{2} \pm 2\pi n \) for all integers \( n \)

6. \( x = \pm \pi n \) or \( x = \frac{3\pi}{4} \pm \pi n \) for all integers \( n \)

7.  

![Graph](image1)

8.  

![Graph](image2)

9.  

![Graph](image3)
TRIGONOMETRIC IDENTITIES

Before the widespread availability of calculators, tables were used to look up the trig values of various angle measures. Knowing that \( \sin(2\theta) = 2\sin(\theta)\cos(\theta) \), for instance, meant that a trig table of values did not need to extend to 120° angles. \( \sin(120^\circ) \) can be written as \( 2\sin(60^\circ)\cos(60^\circ) \), and only the trig values for 60° need to be known.

Another practice common before the advent of calculators was the proving of trig identities. These proofs usually employ algebraic steps and previously proven identities to show that one side of the equation equals the other. Equivalent trigonometric expressions are known as trig identities. These identities allow trigonometric equations to be rewritten and/or solved.

See the Math Notes box in Lesson 12.2.3 for additional information.

Example 1

Graph the function \( f(x) = \frac{1}{\cos^2(x)} - \tan^2(x) \). Based on the graph, what can you conclude about the expression \( \frac{1}{\cos^2(x)} - \tan^2(x) \)? (That is, what trig identity can you write?) What substitution can you make in the identity so that you no longer have a fraction?

By graphing the given function above, it can readily be seen that the function is a constant, that is, a horizontal line. This graph is equivalent to the graph of \( y = 1 \). Because their graphs are equivalent for all values of \( x \), the expressions are also equivalent. Therefore you can write \( \frac{1}{\cos^2(x)} - \tan^2(x) = 1 \).

How can the given expression be rewritten so it no longer has a fraction?

Since \( \frac{1}{\cos(x)} = \sec(x) \), write \( \sec^2(x) - \tan^2(x) = 1 \).

This trig identity is more commonly written as \( 1 + \tan^2(x) = \sec^2(x) \).
Example 2

Prove the following trig identity: \( \frac{\sin(x)}{1 - \cos(x)} + \frac{1 - \cos(x)}{\sin(x)} = 2 \csc(x) \)

For the identity above, start with the left side of the equation, get common denominators so that the fractions can be added, and see where this goes. It is also important to be aware of the right hand side of the equation, which is the goal. Remember that \( \csc(x) = \frac{1}{\sin(x)} \).

\[
\begin{align*}
& \frac{\sin(x)}{1 - \cos(x)} + \frac{1 - \cos(x)}{\sin(x)} = 2 \csc(x) \\
& \frac{\sin(x)}{\sin(x)} \left( \frac{\sin(x)}{1 - \cos(x)} \right) + \left( \frac{1 - \cos(x)}{\sin(x)} \right) \left( \frac{1 - \cos(x)}{\sin(x)} \right) = \\
& \frac{\sin^2(x)}{(\sin(x))(1 - \cos(x))} + \frac{(1 - \cos(x))^2}{(\sin(x))(1 - \cos(x))} = \\
& \frac{\sin^2(x) + (1 - \cos(x))^2}{(\sin(x))(1 - \cos(x))} = \\
& \frac{\sin^2(x) + 1 - 2 \cos(x) + \cos^2(x)}{(\sin(x))(1 - \cos(x))} = \\
& \frac{\sin^2(x) + \cos^2(x) + 1 - 2 \cos(x)}{(\sin(x))(1 - \cos(x))} = \\
& \frac{1 + 1 - 2 \cos(x)}{(\sin(x))(1 - \cos(x))} = \\
& \frac{2 - 2 \cos(x)}{(\sin(x))(1 - \cos(x))} = \\
& \frac{2 (1 - \cos(x))}{(\sin(x))(1 - \cos(x))} = \\
& \frac{2}{\sin(x)} = 2 \csc(x)
\end{align*}
\]

This proves that this identity is true.

### Problems

1. Show graphically that \( \sin(x + y) \) does not equal \( \sin(x) + \sin(y) \).
2. Graphically, determine what \( \cos(x + 90^\circ) \) equals.
3. Graphically, determine what \( \sin(180^\circ - x) \) equals.

Prove the following identities.

4. \( \frac{\sin(2x)}{2 \sin^2(x)} = \cot(x) \)
5. \( \sin^2(x) - \cos^2(x) = \frac{\tan(x) - \cot(x)}{\tan(x) + \cot(x)} \)
6. \( \frac{\sin^2(x)}{1 + \cos(x)} = 1 - \frac{1}{\sec(x)} \)
7. \( \cos^4(x) - \sin^4(x) = 2 \cos^2(x) - 1 \)
8. \( \frac{1}{1 - \sin(x)} + \frac{1}{1 + \sin(x)} = 2 \sec^2(x) \)
Answers

1. The graphs are not the same.

2. \(\cos(x + 90^\circ) = -\sin(x)\)

3. \(\sin(180^\circ - x) = \sin(x)\)

4. \[
\frac{\sin(2x)}{2\sin^2(x)} = \cot(x)
\]
\[
\frac{2\sin(x)\cos(x)}{2\sin(x)\sin(x)} = \frac{\cos(x)}{\sin(x)} = \cot(x)
\]

5. \[
\sin^2(x) - \cos^2(x) = \frac{\tan(x) - \cot(x)}{\tan(x) + \cot(x)}
\]
\[
= \frac{\sin(x) \cos(x)}{\sin(x) \sin(x)}
\]
\[
= \frac{\sin(x) \cos(x)}{\cos(x) \sin(x)}
\]
\[
= \frac{\sin^2(x) - \cos^2(x)}{\sin(x) \cos(x)}
\]
\[
= \frac{\sin^2(x) - \cos^2(x)}{\sin(x) \cos(x)}
\]
\[
= \sin^2(x) - \cos^2(x)
\]

6. \[
\frac{\sin^2(x)}{1 + \cos(x)} = 1 - \frac{1}{\sec(x)}
\]
\[
= 1 - \cos(x)
\]
\[
= (1 - \cos(x)) \left( \frac{1 + \cos(x)}{1 + \cos(x)} \right)
\]
\[
= \frac{(1 - \cos(x))(1 + \cos(x))}{1 + \cos(x)}
\]
\[
= \frac{1 - \cos^2(x)}{1 + \cos(x)}
\]
\[
= \frac{\sin^2(x)}{1 + \cos(x)}
\]

7. \[
\cos^4(x) - \sin^4(x) = 2\cos^2(x) - 1
\]
\[
= (\cos^2(x) + \sin^2(x)) \cdot (\cos^2(x) - \sin^2(x))
\]
\[
= 1 \cdot (\cos^2(x) - \sin^2(x))
\]
\[
= \cos^2(x) - (1 - \cos^2(x))
\]
\[
= \cos^2(x) - 1 + \cos^2(x)
\]
\[
= 2\cos^2(x) - 1
\]

8. \[
\frac{1}{1 - \sin(x)} + \frac{1}{1 + \sin(x)} = 2\sec^2(x)
\]
\[
\left( \frac{1 + \sin(x)}{1 + \sin(x)} \right) \cdot \left( \frac{1}{1 - \sin(x)} \right) + \left( \frac{1 - \sin(x)}{1 - \sin(x)} \right) \cdot \left( \frac{1}{1 + \sin(x)} \right) =
\]
\[
\frac{1 + \sin(x) + 1 - \sin(x)}{(1 + \sin(x))(1 - \sin(x))} =
\]
\[
\frac{2}{1 - \sin^2(x)} =
\]
\[
\frac{2}{\cos^2(x)} = 2\sec^2(x)
\]
SAT Prep

1. If $7x < 2y$ and $2y < 9z$, which of the following statements is true?
   a. $7x < 9z$  b. $9z < 7x$  c. $z < x$  d. $7x = 9z$  e. $7x + 1 = 9z$

2. If $f(t) = 5t - 15$, then at what value of $t$ does the graph of $y = f(t)$ cross the $x$-axis?
   a. $-15$  b. $-5$  c. $0$  d. $2$  e. $3$

3. If $p^5 + 3 = p^5 + w$, then $w =$?
   a. $-3$  b. $-\sqrt[5]{3}$  c. $\sqrt[5]{3}$  d. $3$  e. $3^5$

4. For all positive numbers $j$ and $k$, let $j \triangledown k$ be defined as $\frac{j + 4k}{j - 4k}$. What is the value of $1018 \triangledown 4.5$?
   a. $1.036$  b. $10.36$  c. $103.6$  d. $1036$  e. $10360$

5. If a number is rounded to 26.7, which of the following values could have been the original number?
   a. 26  b. 26.605  c. 26.655  d. 26.776  e. 27

6. On a coordinate plane, the center of a circle is at $(9, -2)$. If the circle touches the $y$-axis in only one point, what is the radius of the circle?

7. The figure at right shows three squares with sides of length 6, 8, and $k$, respectively. If points $A$, $B$, and $C$ lie on line $l$, what is the value of $k$?

8. Exactly 875 out of 7000 seniors at college are majoring in mathematics. What percent of seniors are NOT majoring in mathematics?

9. Five SnookerBars cost as much as 2 Sodiepop Swirls. If the cost of one Sodiepop Swirl and one SnookerBar is $1.75, what is the cost in dollars of 1 Sodiepop Swirl?

10. The highest score possible on Professor Snape’s test is 100 and the lowest is 0. Harry, Ron, Hermione, and Neville’s tests had an average of 86. If Neville got the lowest score, what is the lowest possible score he could have gotten?

Answers


6. 9  7. $\frac{32}{3}$  8. 87.5%  9. $1.25$  10. 4