Conic sections are formed by using a plane to slice a cone, or a double-napped cone, and can be described by using two fixed points (called foci) or using one fixed point (a focus) and a line (directrix).

A parabola is the set of all points in a plane equidistant from a directrix and a focus not on the directrix. In this course, parabolas open either vertically or horizontally. Given vertex \((h, k)\) and \(c = \text{distance from the focus to the vertex}\), the general equation of a parabola opening vertically is \(y - k = \frac{1}{4c} (x - h)^2\). The general equation of a parabola opening horizontally is \(x - h = \frac{1}{4c} (y - k)^2\). Notice that for a parabola, only one of the variables is squared.

An ellipse is the set of all points in a plane such that the sum of the distances from two foci is constant. The general equation of an ellipse with center \((h, k)\) is \(\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1\). The foci are located on the major axis \(c\) units from the center where \(c^2 = a^2 - b^2\). For an ellipse, both variables are squared and the squared terms are being added.

A circle is a special case of an ellipse where both foci are at the same location, the center of the circle. The general equation for a circle with center \((h, k)\) and radius \(r\) is \((x - h)^2 + (y - k)^2 = r^2\).

A hyperbola is the set of all points in a plane such that the difference of the distances from two foci is constant. The general equation of a hyperbola opening vertically with center \((h, k)\) is \(\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1\), and the equations of the asymptotes are \(y - k = \pm \frac{b}{a} (x - h)\). The general equation of a hyperbola opening horizontally with center \((h, k)\) is \(\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1\), and the equations of the asymptotes are \(y - k = \pm \frac{a}{b} (x - h)\). The foci are located on the major axis \(c\) units from the center where \(c^2 = a^2 + b^2\). For a hyperbola, both variables are squared and the squared terms are being subtracted.

Given a conic section in standard form, \(ax^2 + by^2 + cx + dy + e = 0\), the process of completing the square must be used to rewrite the equation in graphing form.

Review the Math Notes boxes in Section 10.1 for more information on completing the square and graphing conic sections.
**Example 1:** Graph each of the following conic sections. Identify all of the key information for each curve.

a. \((x - 1)^2 + (y - 5)^2 = 25\)  
b. \(\frac{(x-4)^2}{9} + \frac{(y-1)^2}{64} = 1\)  
c. \(\frac{(y-6)^2}{49} - \frac{(x+5)^2}{100} = 1\)  
d. \(x - 2 = -\frac{1}{16}(y + 4)^2\)

**Solution:**

a. \((x - 1)^2 + (y - 5)^2 = 25\) is a circle with center \((1, 5)\) and a radius of 5. See graph at right.

b. \(\frac{(x-4)^2}{9} + \frac{(y-1)^2}{64} = 1\) is an ellipse with a center of \((4, 1)\). The larger denominator is under the \(y\). This is a \(a^2\). \(a^2 = 64\), so \(a = 8\) and the vertices are at \((4, 1 \pm 8) = (4, -7)\) and \((4, 9)\). \(b^2 = 9\), so \(b = 3\) and the co-vertices are at \((4 \pm 3, 1) = (-1, 1)\) and \((7, 1)\). \(c^2 = a^2 - b^2 = 64 - 9 = 55\), so \(c = \sqrt{55}\) . Since the major axis is vertical, the foci are located at \((4, 1 \pm \sqrt{55})\). See graph at right.

c. \(\frac{(y-6)^2}{49} - \frac{(x+5)^2}{100} = 1\) is a hyperbola oriented vertically (since the \(y\)-term is first) with a center of \((-5, 6)\). \(a^2 = 49\), so \(a = 7\) and the vertices are at \((-5, 6 \pm 7) = (-5, -1)\) and \((-5, 13)\). \(b^2 = 100\), so \(b = 10\). To sketch the graph, create a box centered at \((-5, 6)\), draw horizontal lines 7 units above and below, and vertical lines 10 units to the right and left. Then sketch lines through the diagonals of the box, which are the asymptotes. \(c^2 = a^2 + b^2 = 49 + 100 = 149\), so \(c = \sqrt{149}\) . Since the transverse axis is vertical, the foci are located at \((-5, 6 \pm \sqrt{149})\). The slopes of the asymptotes are \(\pm \frac{\sqrt{149}}{10}\). The equations of the asymptotes are \(y - 6 = \pm \frac{7}{10}(x + 5)\). See graph at right.

d. \(x - 2 = -\frac{1}{16}(y + 4)^2\) is a parabola opening to the left. The vertex is \((2, -4)\). \(-\frac{1}{16} = -\frac{1}{4c}\), so \(c = -4\). The focus is at \((2 - 4, -4) = (-2, -2)\) and the directrix is \(y = -4 + 4 = 0\). See graph at right.
Example 2: Complete the square to rewrite each equation in graphing form.

a. \(7x^2 - 7x - 8y^2 + 16y = 15\)

\[
7(x^2 - x) - 8(y^2 - 2y) = 15
\]
\[
7(x^2 - x + \frac{1}{4}) - 8(y^2 - 2y + 1) = 15 + 7\left(\frac{1}{4}\right) - 8(1)
\]
\[
7\left(x - \frac{1}{2}\right)^2 - 8(y - 1)^2 = \frac{35}{4}
\]
\[
\frac{(x-\frac{1}{2})^2}{\frac{35}{48}} - \frac{(y-1)^2}{\frac{35}{32}} = 1
\]

b. \(8x^2 + 9y^2 - 80x + 36y - 4 = 0\)

\[
8(x^2 - 10x) + 9(y^2 + 4y) = 4
\]
\[
8(x^2 - 10x + 25) + 9(y^2 + 4y + 4) = 4 + 8(25) + 9(4)
\]
\[
8(x - 5)^2 + 9(y + 2)^2 = 240
\]
\[
\frac{(x-5)^2}{30} + \frac{(y+2)^2}{\frac{80}{3}} = 1
\]

c. \(15x - 2y^2 = -141 + 6y\)

\[
2y^2 + 6y = 15x + 141
\]
\[
2(y^2 + 3y) = 15x + 141
\]
\[
2(y^2 + 3y + \frac{9}{4}) = 15x + 141 + 2\left(\frac{9}{4}\right)
\]
\[
2\left(y + \frac{3}{2}\right)^2 = 15x + \frac{291}{2}
\]
\[
2\left(y + \frac{3}{2}\right)^2 = 15\left(x + \frac{291}{30}\right)
\]
\[
\frac{2}{15}\left(y + \frac{3}{2}\right)^2 = x + \frac{291}{30}
\]

d. \(3x^2 + 18x - 4 = -3y^2 + 12y\)

\[
3x^2 + 18x + 3y^2 - 12y = 4
\]
\[
3(x^2 + 6x) + 3(y^2 - 4y) = 4
\]
\[
3(x^2 + 6x + 9) + 3(y^2 - 4y + 4) = 4 + 3(9) + 3(4)
\]
\[
3(x + 3)^2 + 3(y - 2)^2 = 43
\]
\[
(x + 3)^2 + (y - 2)^2 = \frac{43}{3}
\]
Now we can go back and solve problem 12-122.

Identify each conic and sketch the graph. Also include all of the key information, such as the locations of the foci and/or the equations of any asymptotes.

a. \( \frac{(x-4)^2}{50} + \frac{(y-3)^2}{1} = 1 \) or \( \frac{(x-4)^2}{50} + \frac{(y-3)^2}{1} = 1 \)

Since both variables are squared and the squared terms are being added, this is an ellipse. The center is (4, 3). The denominators are 50 and 1, the greater of which is 50. Therefore the distance from the center to the vertices is \( \sqrt{50} = 5\sqrt{2} \). Since 50 is the denominator of the x term, the vertices are located at \( (4 \pm 5\sqrt{2}, 3) \). The distance from the center to the co-vertices is \( \sqrt{1} = 1 \). The co-vertices are located at (4, 3 ± 1) = (4, 2) and (4, 4). To determine the focal distance, \( c \), for an ellipse, subtract the denominators and then take the square root of the result, \( c = \sqrt{50 - 1} = 7 \). Since the foci lie along the axis that passes through the vertices, the foci are located at \( (4 \pm 7, 3) = (-3, 3) \) and \( (11, 3) \). See graph at right.

b. \( \frac{(y+2)^2}{9} - \frac{(x-3)^2}{16} = 1 \)

Since both variables are squared and the squared terms are being subtracted, this is a hyperbola. Since the y term is positive, this hyperbola opens vertically. The center is \( (3, -2) \). The distance from the center to the vertices is \( \sqrt{9} = 3 \). The vertices are located at \( (3, -2 \pm 3) = (3, -5) \) and \( (3, 1) \). The slopes of the asymptotes are \( \pm \frac{\sqrt{9}}{\sqrt{16}} = \pm \frac{3}{4} \). Since the asymptotes pass through the center the equations for the asymptotes are \( y + 2 = \pm \frac{3}{4} (x - 3) \). To determine the focal distance, \( c \), for a hyperbola, add the denominators and then take the square root of the sum. \( c = \sqrt{9 + 16} = 5 \). Since the foci lie along the axis that passes through the vertices, the foci are located at \( (3, -2 \pm 5) = (3, -7) \) and \( (3, 3) \). See graph at right.

c. \( 2y^2 - x + 4y + 2 = 0 \)

Only one of the variables is squared, so this is a parabola. Since \( y \) is squared, the parabola opens horizontally. The equation needs to be rewritten in graphing form.

\[
\begin{align*}
2y^2 + 4y &= x - 2 \\
2(y^2 + 2y) &= x - 2 \\
2(y^2 + 2y + 1) &= x - 2 + 2(1) \\
2(y + 1)^2 &= x
\end{align*}
\]

The vertex is at \( (0, -1) \).

\[
\frac{1}{4} = 2 \quad \Rightarrow \quad c = \frac{1}{8} \quad \text{Since } c \text{ is positive, the parabola opens to the right.}
\]

The distance from the vertex to the focus/directrix is \( \frac{1}{8} \). Therefore the focus is located at \( (0 + \frac{1}{8}, -1) = (\frac{1}{8}, -1) \) and the directrix is \( x = 0 - \frac{1}{8} \) or \( x = \frac{1}{8} \).

See graph at right.
d.  \(9x^2 - 16y^2 + 54x - 32y + 29 = 0\)

Since both variables are squared and the squared terms are being subtracted, this is a hyperbola. The equation needs to be rewritten in graphing form.

\[
\begin{align*}
9x^2 - 16y^2 + 54x - 32y &= -29 \\
9(x^2 + 6x) - 16(y^2 + 2y + 1) &= -29 + 9(9) - 16(1) \\
9(x + 3)^2 - 16(y + 1)^2 &= 36 \\
\frac{(x+3)^2}{\frac{36}{9}} - \frac{(y+1)^2}{\frac{36}{16}} &= 1
\end{align*}
\]

Since the \(x\) term is positive, the hyperbola opens horizontally.

The center is located at \((-3, -1)\). The distance from the center to the vertices is \(\sqrt{4} = 2\).

The vertices are located at \((-3 \pm 2, -1) = (-5, -1)\) and \((-1, -1)\).

The slopes of the asymptotes are \(\pm \frac{\sqrt{2.25}}{4} = \pm \frac{1.5}{2} = \pm \frac{3}{4}\). Therefore the equations of the asymptotes are \(y + 1 = \pm \frac{3}{4} (x + 3)\).

\(c = \sqrt{4 + 2.25} = 2.5\); Therefore the foci are located at \((-3 \pm 2.5, -1) = (-5.5, -1)\) and \((-0.5, -1)\).

See graph above right.

Here are some more to try.

Identify and graph each of the following conic sections and state all key features.

1. \(\frac{(x-2)^2}{4} + \frac{y^2}{25} = 1\)
2. \(y - 5 = -\frac{1}{25} (x - 4)^2\)
3. \((x + 8)^2 + (y - 2)^2 = 25\)
4. \(\frac{(y-2)^2}{25} - \frac{(x-6)^2}{36} = 1\)
5. \(x + 5 = 4(y - 6)^2\)
6. \(\frac{x^2}{25} - \frac{(y-3)^2}{4} = 1\)
7. \(25x^2 + 300x + 49y^2 - 686y = -2076\)
8. \(4x^2 - 16x - y + 19 = 0\)
9. \(25x^2 + 150x - 16y^2 - 192y = 751\)
10. \(x^2 - 6x + y^2 - 10y + 25 = 0\)
11. \(x = 2y^2 - 3y - 5\)
12. \(9y^2 - 144y - 1114 = 169x^2 + 338x\)
Write an equation for each of the following curves. The open dots are foci.

13.

14.

15.

16.

17.

18.

Answers:

1. center: (2, 0),
   vertices: (2, –5) and (2, 5)
   co-vertices: (0, 0) and (4, 0)
   foci: (2, ±√21)

2. vertex: (4, 5)
   focus: (4, –1)
   directrix: y = 11
3. center: \((-8, 2)\)
   radius: 5
   ![Graph of a circle with center at \((-8, 2)\) and radius 5]

4. center: \((6, 2)\)
   vertices: \((6, -3)\) and \((6, 7)\)
   foci: \((6, 2 \pm \sqrt{61})\)
   asymptotes: \(y - 2 = \pm \frac{5}{6}(x - 6)\)
   ![Graph of a hyperbola with center at \((6, 2)\) and foci \((6, 2 \pm \sqrt{61})\)]

5. vertex: \((-5, 6)\)
   focus: \((-\frac{79}{16}, 6)\)
   directrix: \(x = -\frac{81}{16}\)
   ![Graph of a parabola with vertex at \((-5, 6)\) and focus \((-\frac{79}{16}, 6)\)]

6. center: \((0, 3)\)
   vertices: \((-5, 3)\) and \((5, 3)\)
   foci: \((5 \pm \sqrt{29}, 3)\)
   asymptotes: \(y = \pm \frac{2}{3}x + 3\)
   ![Graph of a hyperbola with center at \((0, 3)\) and foci \((5 \pm \sqrt{29}, 3)\)]

7. \(\frac{(x+6)^2}{49} + \frac{(y-7)^2}{25} = 1\)
   center: \((-6, 7)\)
   vertices: \((-13, 7)\) and \((1, 7)\)
   co-vertices: \((-6, 2)\) and \((-6, 12)\)
   foci: \((-6 \pm 2\sqrt{6}, 6)\)
   ![Graph of an ellipse with center at \((-6, 7)\) and foci \((-6 \pm 2\sqrt{6}, 6)\)]

8. \(y - 3 = 4(x - 2)^2\)
   vertex: \((2, 3)\)
   focus: \(\left(2, \frac{49}{16}\right)\)
   directrix: \(y = \frac{47}{16}\)
   ![Graph of a parabola with vertex at \((2, 3)\) and focus \(\left(2, \frac{49}{16}\right)\)]
9. \[ \frac{(x+3)^2}{16} - \frac{(y+6)^2}{25} = 1 \]
   center: \((-3, -6)\)
   vertices: \((-7, -6)\) and \((1, 6)\)
   foci: \((-7 \pm \sqrt{41}, -6)\)
   asymptotes: \(y + 6 = \pm \frac{5}{4} (x + 3)\)

10. \( (x - 3)^2 + (y - 5)^2 = 9 \)
    center: \((3, 5)\)
    radius: 3

11. \( x + \frac{49}{8} = 2 \left( y - \frac{3}{4} \right)^2 \)
    vertex: \(\left( -\frac{49}{8}, \frac{3}{4} \right) \)
    focus: \(\left( -6, \frac{3}{4} \right) \)
    directrix: \(x = -\frac{25}{4} \)

12. \[ \frac{(y-8)^2}{169} - \frac{(x+1)^2}{9} = 1 \]
    center: \((-1, 8)\)
    vertices: \((-1, -5)\) and \((-1, 21)\)
    foci: \((-1, 8 \pm 4\sqrt{10})\)
    asymptotes: \(y - 8 = \pm \frac{13}{3} (x + 1)\)

13. \[ \frac{(y+5)^2}{4} - (x-8)^2 = 1 \]

14. \( x - 3 = -\frac{1}{16} (y + 7)^2 \)

15. \( (x + 6)^2 + (y - 6)^2 = 9 \)

16. \[ \frac{(x-7)^2}{4} - \frac{(y-5)^2}{25} = 1 \]

17. \[ \frac{(x+6)^2}{36} + \frac{(y+2)^2}{25} = 1 \]

18. \( y - 8 = -2(x - 5)^2 \)