Parametrically-Defined Functions Checkpoint
Problem 12-136

In a set of parametric equations, the parameter, generally \( t \), is the independent variable. Instead of \( y \) being a direct function of \( x \), in a set of parametric equations, \( x \) and \( y \) are both functions of the parameter, \( t \). Therefore the notation \( x(t) \) and \( y(t) \) is often used (where \( x(t) \) represents “\( x \) of \( t \),” not multiplication). Typically, parametric equations allow two related motions (generally horizontal and vertical) to be expressed separately with respect to time and then combined to create a single graph. In order to make the symbols appear less complicated while working with parametric equations, \( x(t) \) can be expressed as \( x \) and \( y(t) \) as \( y \).

Example 1: Graph \( x(t) = 3t - 1 \), \( y(t) = t + 5 \) for \( 0 \leq t \leq 5 \). Label the \( t \)-values at each integer-valued point.

Solution: Begin by making a table with columns for \( t \), \( x \), and \( y \). Remember that in this table, \( t \) is the independent variable. Each \( x \)-value is calculated by substituting the given value of \( t \) into \( x = 3t - 1 \), and each \( y \)-value is calculated by substituting into \( y = t + 5 \).

Despite the \( x \)- and \( y \)-coordinates being calculated separately, they are combined to create the points on the graph. Plot and connect the points (-1, 5), (2, 6), (5, 7), (8, 8), (11, 9), and (14, 10), and label each point with its corresponding \( t \)-value. See graph at right.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3(0) - 1 = -1</td>
<td>(0) + 5 = 5</td>
</tr>
<tr>
<td>1</td>
<td>3(1) - 1 = 2</td>
<td>(1) + 5 = 6</td>
</tr>
<tr>
<td>2</td>
<td>3(2) - 1 = 5</td>
<td>(2) + 5 = 7</td>
</tr>
<tr>
<td>3</td>
<td>3(3) - 1 = 8</td>
<td>(3) + 5 = 8</td>
</tr>
<tr>
<td>4</td>
<td>3(4) - 1 = 11</td>
<td>(4) + 5 = 9</td>
</tr>
<tr>
<td>5</td>
<td>3(5) - 1 = 14</td>
<td>(5) + 5 = 10</td>
</tr>
</tbody>
</table>

Example 2: Rewrite \( x(t) = 3t - 1 \), \( y(t) = 6t + 5 \) in rectangular form.

Solution: To write as set of parametric equations as a single equation containing only \( x \) and \( y \), first solve one of the parametric equations for \( t \). In order to make the symbols appear less complicated during the solving process, it is okay to write \( x(t) \) as \( x \) and \( y(t) \) as \( y \).

Solving \( x = 3t - 1 \) for \( t \) gives the equation \( t = \frac{x + 1}{3} \). The expression \( \frac{x + 1}{3} \) can be substituted for \( t \) in the equation \( y = 6t + 5 \). The resulting equation is \( y = 6 \left( \frac{x + 1}{3} \right) + 5 \), which simplifies as \( y = 2x + 7 \).

Example 3: Rewrite \( x(t) = \log(t) \), \( y(t) = (\log(t))^3 \) in rectangular form.

Solution: In this example, it is not necessary to first solve one of the equations for \( t \) and then substitute. A more efficient method for this problem is to substitute \( x = \log(t) \) into the \( y \) equation.

\[
\begin{align*}
y &= (\log(t))^3 \\
y &= x^3
\end{align*}
\]
Now we can go back and solve problem 12-136.

a. Graph \( x(t) = t^2 + 3 \), \( y(t) = 2t - 1 \) for \(-3 \leq t \leq 3\).

Solution: Create a table like the one at right.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x(t) = t^2 + 3 )</th>
<th>( y(t) = 2t - 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>((-3)^2 + 3 = 12)</td>
<td>(2(-3) - 1 = -7)</td>
</tr>
<tr>
<td>-2</td>
<td>((-2)^2 + 3 = 7)</td>
<td>(2(-2) - 1 = -5)</td>
</tr>
<tr>
<td>-1</td>
<td>((-1)^2 + 3 = 4)</td>
<td>(2(-1) - 1 = -3)</td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>((1)^2 + 3 = 4)</td>
<td>(2(1) - 1 = 1)</td>
</tr>
<tr>
<td>2</td>
<td>((2)^2 + 3 = 7)</td>
<td>(2(2) - 1 = 3)</td>
</tr>
<tr>
<td>3</td>
<td>((3)^2 + 3 = 12)</td>
<td>(2(3) - 1 = 5)</td>
</tr>
</tbody>
</table>

Plotting the points \((12, -7), (7, -5), (4, -3)\), etc. creates the curve shown at right.

b. Rewrite the parametrically-defined function in part (a) in rectangular form.

\( y(t) = 2t - 1 \) can be written as \( y = 2t - 1 \).

Solving for \( t \) gives the equation \( t = \frac{y+1}{2} \). Then substitute \( \frac{y+1}{2} \) for \( t \) in the equation \( x = t^2 + 3 \).

\[
x = \left( \frac{y+1}{2} \right)^2 + 3
\]

\[
x = \frac{(y+1)^2}{4} + 3
\]

\[
x - 3 = \frac{1}{4} (y + 1)^2
\]
Here are some more to try.

Graph the following parametrically-defined functions.

1. \( x(t) = \frac{t}{2}, \ y(t) = t^2 - 5 \) for \(-4 \leq t \leq 4\)
2. \( x(t) = t^2 + 2, \ y(t) = t^2 - 3 \) for \(-3 \leq t \leq 3\)

Rewrite the parametrically-defined functions in rectangular form.

3. \( x(t) = t + 7, \ y(t) = 4t - 1 \)
4. \( x(t) = \frac{t - 5}{3}, \ y(t) = t^2 \)
5. \( x(t) = t^7, \ y(t) = 8t^7 + 3 \)
6. \( x(t) = t^5, \ y(t) = -t^{10} \)
7. \( x(t) = t^3, \ y(t) = t - 4 \)
8. \( x(t) = e^t, \ y(t) = 6e^{4t} \)
9. \( x(t) = t^7, \ y(t) = 0.5t^{35} + 1 \)
10. \( x(t) = t^2 - 3, \ y(t) = t^2 + 9 \)

Answers:

1. \[
\begin{align*}
  &\begin{array}{c}
    t = -4 \\
    t = -3 \\
    t = -2 \\
    t = -1 \\
    t = 0 \\
    t = 1 \\
    t = 2 \\
    t = 3 \\
  \end{array} \\
  &\begin{array}{c}
    y = 12 \\
    y = 8 \\
    y = 4 \\
    y = 2 \\
  \end{array} \\
\end{align*}
\]

3. \( y = 4x - 29 \)
4. \( y = (3x + 5)^2 \) or \( y = 9x^2 + 30x + 25 \)
5. \( y = 8x + 3 \)
6. \( y = -x^2 \)
7. \( x = (y + 4)^3 \) or \( y = \sqrt[3]{x} - 4 \)
8. \( y = 6x^4 \)
9. \( y = 0.5x^5 + 1 \)
10. \( y = x + 12 \)