Graphing form of a basic rational function is $y = \frac{a}{x-h} + k$. When graphed, these rational functions have a vertical asymptote and a horizontal asymptote. To graph a rational function, locate the asymptotes and the intercepts. It can be helpful to also determine or plot a few points.

The vertical asymptote occurs from division by 0. Determine the value of $x$ that will cause division by 0. If $y = \frac{a}{x-h} + k$, then the vertical asymptote is $x = h$.

The horizontal asymptote comes from the end behavior of the function, so think about the value of the function as $x$ becomes a large positive or large negative number. If $y = \frac{a}{x-h} + k$, then the horizontal asymptote is $y = k$ because as $x$ becomes a large number, the fractional part of the function gets close to 0.

Example 1: Graph $y = \frac{-7}{x-4} + 3$. State the locations of the intercepts and asymptotes.

Solution: $y = \frac{-7}{x-4} + 3$ has a vertical asymptote of $x = 4$ and a horizontal asymptote of $y = 3$. The stretch factor is −7, so the curve is reflected vertically and stretched by a factor of 7.

$x$-intercept: $0 = \frac{-7}{x-4} + 3$

$-3 = \frac{-7}{x-4}$

$-3(x-4) = -7$

$x - 4 = \frac{7}{3}$

$x = \frac{7}{3} + 4 = 6\frac{1}{3}$

$y$-intercept: $y = \frac{-7}{0-4} + 3$

$y = \frac{2}{4} + 3$

$y = \frac{4}{3} + \frac{3}{4}$

$y = \frac{16}{12} + \frac{9}{12}$

$y = \frac{25}{12}$

Example 2: Graph $y = \frac{4x-5}{x-3}$. State the locations of the intercepts and asymptotes.

Solution: Begin by rewriting $y = \frac{4x-5}{x-3}$ in the form $y = \frac{a}{x-h} + k$.

Since the denominator is $x - 3$, that factor is needed in the numerator. Therefore: $4x - 5 = 4(x - 3) + ? \Rightarrow ? = 7$

See the work shown at right.

The vertical asymptote is $x = 3$. The horizontal asymptote is $y = 4$. The curve is vertically stretched by a factor of 7.

$x$-intercept: $0 = \frac{4x-5}{x-3}$

$0(x-3) = 4x - 5$

$0 = 4x - 5$

$5 = 4x$

$x = \frac{5}{4}$

$y$-intercept: $y = \frac{4(0)-5}{(0)-3}$

$y = \frac{-5}{-3}$

$y = \frac{5}{3} = 1\frac{2}{3}$
Now we can go back and solve problem 6-115.

Graph each of the following rational functions. State the locations of any asymptotes.

a. \( r(x) = \frac{1}{x^2} + 5 \)
   - vertical asymptote: \( x = -2 \)
   - horizontal asymptote: \( y = 5 \)
   - \( x \)-intercept: \( 0 = \frac{1}{x^2} + 5 \)
     \[ -5 = \frac{1}{x^2} \]
     \[ -5(x^2 + 2) = 1 \]
     \[ x^2 + 2 = \frac{1}{5} \]
     \[ x = \frac{1}{5} - 2 = -2 \frac{1}{5} \]
   - \( y \)-intercept: \( y = \frac{1}{0^2} + 5 \)

b. \( q(x) = \frac{2x}{x+1} = \frac{2(x+1) - 2}{x+1} = \frac{2(x+1)}{x+1} - \frac{2}{x+1} = -\frac{2}{x+1} + 2 \)
   - vertical asymptote: \( x = -1 \)
   - horizontal asymptote: \( y = 2 \)
   - \( x \)-intercept: \( 0 = \frac{2x}{x+1} \)
     \[ 0(x+1) = 2x \]
     \[ 0 = 2x \]
     \[ x = 0 \]
   - \( y \)-intercept: \( y = \frac{2(0)}{0+1} \)
     \[ y = 0 \]

Since \( (0, 0) \) is the only intercept, it can be helpful to locate another point on the graph. For example, if \( x = -2 \), then \( q(-2) = \frac{2(-2)}{-2+1} = 4 \).

c. \( t(x) = \frac{4x-1}{x-5} = \frac{4(x-1)}{x-5} + \frac{19}{x-5} = \frac{4(x-5) + 19}{x-5} + 4 \)
   - vertical asymptote: \( x = 5 \)
   - horizontal asymptote: \( y = 4 \)
   - \( x \)-intercept: \( 0 = \frac{4x-1}{x-5} \)
     \[ 0(x-5) = 4x - 1 \]
     \[ 0 = 4x - 1 \]
     \[ 1 = 4x \]
     \[ x = \frac{1}{4} \]
   - \( y \)-intercept: \( y = \frac{4(0)-1}{0-5} \)
     \[ y = \frac{-1}{-5} = \frac{1}{5} \]

Since both intercepts are in the same section of the asymptotes, it can be helpful to locate another point on the graph. For example, if \( x = 6 \), then \( t(6) = \frac{4(6)-1}{6-5} = \frac{24-1}{1} = 23 \).
Here are some more to try.

Graph each of the following rational functions. State the locations of the intercepts and asymptotes.

1. \( a(x) = \frac{-2}{x+1} - 5 \)
2. \( b(x) = \frac{1}{x} + 6 \)
3. \( c(x) = \frac{3x+5}{x+5} \)
4. \( d(x) = \frac{4x-4}{x+2} \)
5. \( e(x) = \frac{1}{x+6} - 3 \)
6. \( f(x) = \frac{-7}{x-4} \)
7. \( g(x) = \frac{-3x}{7x+1} \)
8. \( h(x) = \frac{5x-3}{4x+3} \)
9. \( i(x) = \frac{-3}{x-7} - 3 \)
10. \( j(x) = \frac{4x+3}{7x} \)
11. \( k(x) = \frac{4}{-2x-2} + 7 \)
12. \( l(x) = \frac{6x-5}{4x+1} \)

**Answers:**

1. \((-1 \frac{2}{5}, 0)\)
   
   \((0, -7)\)
   
   VA: \(x = -1\)
   
   HA: \(y = -5\)

2. \((-\frac{1}{6}, 0)\)
   
   VA: \(x = 0\)
   
   HA: \(y = 6\)

3. \((-\frac{5}{3}, 0)\)
   
   \((0, 1)\)
   
   VA: \(x = -5\)
   
   HA: \(y = 3\)

4. \((1, 0)\)
   
   \((0, -2)\)
   
   VA: \(x = -2\)
   
   HA: \(y = 4\)

5. \((-5 \frac{2}{3}, 0)\)
   
   \((0, -2 \frac{5}{6})\)
   
   VA: \(x = -6\)
   
   HA: \(y = -3\)

6. \((0, \frac{7}{4})\)
   
   VA: \(x = 4\)
   
   HA: \(y = 0\)
7. \((0, 0)\)
   VA: \(x = \frac{1}{7}\)
   HA: \(y = -\frac{3}{7}\)

8. \((\frac{2}{3}, 0)\)
   \((0, -1)\)
   VA: \(x = -\frac{3}{4}\)
   HA: \(y = \frac{5}{4}\)

9. \((6, 0)\)
   \((0, -2 \frac{4}{7})\)
   VA: \(x = 7\)
   HA: \(y = -3\)

10. \((-\frac{3}{4}, 0)\)
    VA: \(x = 0\)
    HA: \(y = -\frac{4}{7}\)

11. \((-\frac{5}{7}, 0)\)
    \((0, 5)\)
    VA: \(x = -1\)
    HA: \(y = 7\)

12. \((\frac{5}{6}, 0)\)
    \((0, -5)\)
    VA: \(x = -\frac{1}{4}\)
    HA: \(y = \frac{3}{2}\)