Solving Logarithmic Equations
Problem 7-133

The following definitions and properties derive from the inverse relationship of exponentials and logarithms. They hold true for all positive numbers \( b, m, \) and \( n, \) and \( b \neq 1. \)

Definition of logarithm: \( \log_b(y) = x \) means \( y = b^x \) for \( y > 0. \)

Basic Properties:
\[ \log_b(b) = 1 \quad \log_b(1) = 0 \quad \log_b(b^x) = x \]

Product Property:
\[ \log_b(m \cdot n) = \log_b(m) + \log_b(n) \]

Quotient Property:
\[ \log_b \left( \frac{m}{n} \right) = \log_b(m) - \log_b(n) \]

Power Property:
\[ \log_b(m^n) = n \log_b(m) \]

**Example 1:** Solve \( \log_3(x) + 2\log_3(x+2) = 2. \)

Solution:
\[
\begin{align*}
\log_3(x) + 2\log_3(x + 2) &= 2 & \text{original equation} \\
\log_3(x) + \log_3((x + 2)^2) &= 2 & \text{Power Property} \\
\log_3(x + 2)^2 &= 2 & \text{Product Property} \\
\log_3(x^2 + 4x + 4) &= 2 & \text{expand the square terms} \\
\log_3(x^2 + 4x + 4) &= 2 & \text{Distributive Property} \\
3^2 &= x^2 + 4x + 4 & \text{definition of logarithm} \\
9 &= x^2 + 4x + 4 - 9 & \text{subtract 9 from both sides} \\
x &= 1 & \text{solve the polynomial equation}
\end{align*}
\]

**Example 2:** Solve \( \log(x - 1) - \log(x^2 - 1) = 2. \)

Solution:
\[
\begin{align*}
\log(x - 1) - \log(x^2 - 1) &= -2 & \text{original equation} \\
\log \left( \frac{x - 1}{x^2 - 1} \right) &= -2 & \text{Quotient Property} \\
10^{-2} &= \frac{x - 1}{x^2 - 1} & \text{definition of a logarithm} \\
\frac{1}{100} &= \frac{x - 1}{(x+1)(x-1)} & \text{solve the rational equation} \\
\frac{1}{100} &= \frac{1}{x+1} \\
100 &= x + 1 \\
99 &= x
\end{align*}
\]
Example 3: Solve $\log_5(x) + \log_5(x + 6) = 2\log_5(x - 8)$.

Solution: 

\[
\begin{align*}
\log_5(x) + \log_5(x + 6) &= 2\log_5(x - 8) & \text{original equation} \\
\log_5((x)(x + 6)) &= \log_5((x - 8)^2) & \text{Product Property and Power Property} \\
(x)(x + 6) &= (x - 8)^2 & \text{Equivalence} \\
x^2 + 6x &= x^2 - 16x + 64 & \text{Distributive Property} \\
22x &= 64 & \text{Subtraction Property of Equality} \\
\frac{x}{22} &= \frac{32}{11} & \text{Division Property of Equality} \\
x &= \frac{32}{11} \\
\text{no solution since } x \leq 8
\end{align*}
\]

Example 4: Solve $8\log_8(3) = \log_7(x + 4)$.

Solution: 

\[
\begin{align*}
8\log_8(3) &= \log_7(x + 4) & \text{original equation} \\
3 &= \log_7(x + 4) & \text{Basic Properties} \\
7^3 &= x + 4 & \text{definition of logarithm} \\
7^3 - 4 &= x & \text{Subtraction Property of Equality} \\
x &= 349 & \text{simplify}
\end{align*}
\]

Now we can go back and solve problem 7-133.

a. $\log_2(x^2 + 2x) = 3$
\[
\begin{align*}
2^3 &= x^2 + 2x \\
0 &= x^2 + 2x - 8 \\
0 &= (x + 4)(x - 2) \\
x &= -4, 2
\end{align*}
\]

b. $\log_5(\sqrt{x}) + \log_5(3\sqrt{x}) = 2$
\[
\begin{align*}
\log_5(3x) &= 2 \\
3x &= 25 \\
x &= \frac{25}{3}
\end{align*}
\]

c. $\log_2(x) = \frac{1}{3} \log_2(9) + \log_2(x - 4)$
\[
\begin{align*}
\log_2(x) &= \log_2((9^{1/2})(x - 4)) \\
x &= 3(x - 4) \\
x &= 3x - 12 \\
-2x &= -12 \\
x &= 6
\end{align*}
\]

d. $17\log_{17}(x-6) = 3$
\[
\begin{align*}
x - 6 &= 3 \\
x &= 9
\end{align*}
\]
Here are some more to try. Solve each of the following equations.

1. \( \log_5(3x + 8) = 2 \) 
2. \( \log_8(x + 2) + \log_8(5) = \log_8(15) \)

3. \( \log_3(x - 7) - \log_3(x - 8) = \log_4(16) \) 
4. \( 2 = \log_{x+9}(x^2 + 3x - 18) \)

5. \( \frac{2\log(x)}{\log(63-2x)} = 1 \) 
6. \( \log_6(x + 5)^7 = 7 \)

7. \( \log_{x+10}(16) + \log_{x+10}(0.25) = 2 \) 
8. \( 12\log_{12}(x^2-2x) = 15 \)

9. \( \log_2(x + 2) + \log_2(x - 5) = 3 \) 
10. \( \log_x(64) - 3 = \log_4(125) \)

11. \( \frac{1+\log_4(x)}{\log_4(16x)-2} = 6 \) 
12. \( \log_6(x^2) - \log_6(x + 5) = \log_6(x + 1) \)

**Answers:**

1. \( x = \frac{17}{3} \) 
2. \( x = 1 \) 
3. \( x = 2 \)

4. \( x = -\frac{33}{5} \) 
5. \( x = 7 \) 
6. \( x = 1 \)

7. \( x = -8 \) 
8. \( x = -3, 5 \) 
9. \( x = 6 \)

10. \( x = \frac{4}{5} \) 
11. \( x = 4^{1/5} = \sqrt[5]{4} \) 
12. \( x = -\frac{5}{6} \)