SOLVING INEQUALITIES

When an equation has a solution, depending on the type of equation, the solution can be represented as a point on a line or a point, line, or curve in the coordinate plane. Dividing points, lines, and curves are used to solve inequalities. Also see the textbook, pages 376-77, 386, 393, and 432.

If the inequality has one variable, the solution can be represented on a line. To solve any type of inequality, first solve it as you would if it were an equation. Use the solution(s) as dividing point(s) of the line. Then test a value from each region on the number line in the inequality. If the test value makes the inequality true, then that region is part of the solution. If it is false then the value and thus that region is not part of the solution. In addition, if the inequality is \( \geq \) or \( \leq \) then the dividing point is part of the solution and is indicated by a solid dot. If the inequality is \( > \) or \( < \), then the dividing point is not part of the solution and is indicated by an open dot.

Example 1

Solve \(-2x - 3 \geq x + 6\)

Solve the equation
\[
-2x - 3 = x + 6 \\
-2x - x = 9 \\
x = -3
\]

Draw a number line and put a solid dot at \(x = -3\), which is the dividing point.

Test a value from each region. Here we test \(-4\) and 0. Be sure to use the original inequality.

\[
\begin{align*}
\text{True} & \quad \text{False} \\
\text{False} & \quad \text{True} \\
\end{align*}
\]

\[
\begin{align*}
x = -4 & \quad x = 0 \\
-2(-4) - 3 \geq -4 + 6 & \quad -2(0) - 3 \geq 0 + 6 \\
5 > 2 & \quad -3 > 6 \\
\end{align*}
\]

The region(s) that are true represent the solution. The solution is \(-3\) and all numbers in the left region, written: \(x \leq -3\).

Example 2

Solve \(x^2 - 2x + 2 < 5\)

Solve the equation
\[
\begin{align*}
x^2 - 2x + 2 & = 5 \\
x^2 - 2x - 3 & = 0 \\
(x - 3)(x + 1) & = 0 \\
x = 3 & \quad \text{or} \quad x = -1
\end{align*}
\]

Draw a number line and put open dots at \(x = 3\) and \(x = -1\), the dividing points.

Test a value from each region in the original inequality. Here we test \(-3, 0,\) and \(4\).

\[
\begin{align*}
\text{False} & \quad \text{True} & \quad \text{False} \\
\text{True} & \quad \text{False} & \quad \text{True} \\
\text{False} & \quad \text{True} & \quad \text{False} \\
\end{align*}
\]

\[
\begin{align*}
x = -3 & \quad x = 0 & \quad x = 4 \\
(-3)^2 - 2(-3) + 2 < 5 & \quad (0)^2 - 2(0) + 2 < 5 & \quad (4)^2 - 2(4) + 2 < 5 \\
17 < 5 & \quad 2 < 5 & \quad 10 < 5
\end{align*}
\]

The region(s) that are true represent the solution. The solution is the set of all numbers greater than \(-1\) but less than 3, written: \(-1 < x < 3\).
If the inequality has two variables, then the solution is represented by a graph in the xy-coordinate plane. The graph of the inequality written as an equation (a line or curve) divides the coordinate plane into regions which are tested in the same manner described above using an ordered pair for a point on a side of the dividing line or curve. If the inequality is > or <, then the boundary line or curve is dashed. If the inequality is ≥ or ≤, then the boundary line or curve is solid.

**Example 3**

Shade the solution to this system of inequalities:

\[
\begin{align*}
  y &\leq \frac{2}{5}x \\
  y &> 5 - x
\end{align*}
\]

Graph each equation. For \( y = \frac{2}{5}x \) the slope of the solid line is \( \frac{2}{5} \) and y-intercept is 0. For \( y = 5 - x \) the slope of the dashed line is -1 and the y-intercept is 5.

Test a point from each region in both of the original inequalities.

<table>
<thead>
<tr>
<th>Region</th>
<th>Test Points</th>
<th>Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 2)</td>
<td>False in both</td>
<td>True in first, False in second</td>
</tr>
<tr>
<td>(2, 0)</td>
<td>True in first, False in second</td>
<td>True in both</td>
</tr>
<tr>
<td>(4, 5)</td>
<td>False in first, False in second</td>
<td>True in both</td>
</tr>
<tr>
<td>(5, 1)</td>
<td>False in both</td>
<td>True in first, False in second</td>
</tr>
</tbody>
</table>

The region that makes both statements (inequalities) true is the solution.

The solution is the region below the solid line \( y = \frac{2}{5}x \) and above the dashed line \( y = 5 - x \) as shown at right.

Solve and graph each inequality.

1. \( x + 12 \geq 2x - 5 \)
2. \( -16 + 4x > 10 - x \)
3. \( 7x - 2x - x \geq 24 + 3x \)
4. \( 3(x - 4) - 9x \geq 2x - 4 \)
5. \( |x - 1| < 5 \)
6. \( |x + 10| > 5 \)
7. \( ||2x| \geq 24 \)
8. \( |\frac{x}{3}| < 8 \)
9. \( x^2 + 3x - 10 \leq 0 \)
10. \( x^2 - 7x + 6 > 0 \)
11. \( x^2 + 2x - 8 \leq 7 \)
12. \( x^2 - 5x - 16 > -2 \)
13. \( y < 2x + 1 \)
14. \( y \leq -\frac{2}{3}x + 3 \)
15. \( y \geq \frac{1}{4}x - 2 \)
16. \( 2x - 3y \leq 5 \)
17. \( y \geq -2 \)
18. \( -3x - 4y > 4 \)
19. \( y \leq \frac{1}{2}x + 2 \) and \( y > -\frac{2}{3}x - 1 \)
20. \( y \leq -\frac{3}{5}x + 4 \) and \( y \leq \frac{1}{3}x + 3 \)
21. \( y < 3 \) and \( y \leq -\frac{1}{2}x + 2 \)
22. \( x \leq 3 \) and \( y \leq \frac{3}{4}x - 4 \)
23. \( y \leq x^2 + 4x + 3 \)
24. \( y > x^2 - x - 2 \)
Answers

1. $x \leq 17$

2. $x > 5 \frac{1}{5}$

3. $x \geq 24$

4. $x \leq -1$

5. $-4 < x < 6$

6. $x > -5 \text{ or } x < -15$

7. $x \geq 2 \text{ or } x \leq -2$

8. $-24 < x < 24$

9. $-5 \leq x \leq 2$

10. $x < 1 \text{ or } x > 6$

11. $-5 \leq x \leq 3$

12. $x < -2 \text{ or } x > 7$

13. [Graph of inequality]

14. [Graph of inequality]

15. [Graph of inequality]

16. [Graph of inequality]

17. [Graph of inequality]

18. [Graph of inequality]

19. [Graph of inequality]

20. [Graph of inequality]

21. [Graph of inequality]